... and mathematical physics...

Contents. V.8

5e

From the Author.

- 1 -

ON

THE REAL AND IMAGINARY ROOTS

OF

ALGEBRAICAL EQUATIONS:

A TRILOGY.

BY

J. J. SYLVESTER, M.A., F.R.S.,

PROFESSOR OF MATHEMATICS AT THE ROYAL MILITARY ACADEMY, WOOLWICH.

"Turns them to shapes and gives to airy nothing
A local habitation and a name."

From the PHILOSOPHICAL TRANSACTIONS.—PART III. 1864.

LONDON:

PRINTED BY TAYLOR AND FRANCIS, RED LION COURT, FLEET STREET.

MDCCCLXV.

TO

THE RIGHT HONOURABLE WILLIAM MONSELL, M.P.,

WHO, AS THE ORIGINATOR AND STRENUOUS UPHOLDER

OF THE PRINCIPLE OF FREE COMPETITION FOR ADMISSION INTO THE PUBLIC SERVICE,

HAS CONFERRED A LASTING BENEFIT ON THE CAUSE OF EDUCATION,

AND IMMEASURABLY AUGMENTED THE EFFICIENCY

OF THE FUTURE SERVANTS AND DEFENDERS OF THE STATE,

THESE PAGES,

THE FIRST FRUITS OF RECONQUERED LABORIOUS LEISURE, ARE

GRATEFULLY DEDICATED BY

THE AUTHOR.

XVI. *Algebraical Researches, containing a disquisition on* NEWTON'S *Rule for the Discovery of Imaginary Roots, and an allied Rule applicable to a particular class of Equations, together with a complete invariantive determination of the character of the Roots of the General Equation of the fifth Degree, &c.*

By J. J. SYLVESTER, *M.A.*, *F.R.S.*, *Correspondent of the Institute of France, Foreign Member of the Royal Society of Naples, etc. etc., Professor of Mathematics at the Royal Military Academy, Woolwich.*

Received April 6,—Read April 7, 1864.

(1) THIS memoir in its present form is of the nature of a trilogy; it is divided into three parts, of which each has its action complete within itself, but the same general cycle of ideas pervades all three, and weaves them into a sort of complex unity. In the first is established the validity of NEWTON'S rule for finding an inferior limit to the number of imaginary roots of algebraical equations as far as the fifth degree inclusive. In the second is obtained a rule for assigning a like limit applicable to equations of the form $\Sigma(ax+b)^m=0$, m being any positive integer, and the coefficients a, b real. In the third are determined the absolute invariantive criteria for fixing unequivocally the character of the roots of an equation of the fifth degree, that is to say, for ascertaining the exact number of real and imaginary roots which it contains. This last part has been added since the original paper was presented to the Society. It has grown out of a foot-note appended to the second, itself an independent offshoot from the first part, but may be studied in a great measure independently of what precedes, and constitutes, in the author's opinion, by far the most valuable portion of the memoir, containing as it does a complete solution of one of the most interesting and fruitful algebraical questions which has ever yet engaged the attention of mathematicians (1). I propose in a subsequent addition to the memoir to resume and extend some of the investigations which incidentally arise in this part. The foot-notes are numbered and lettered for facility of reference, and will be found in many instances of equal value with the matter in the text, to which they serve as a kind of free running accompaniment and commentary.

(1) I owe my thanks to my eminent friend Professor DE MORGAN for bringing under my notice, in a marked manner, the original question from which all the rest has proceeded. As all roads are said to lead to Rome, so I find, in my own case at least, that all algebraical inquiries sooner or later end at that Capitol of Modern Algebra over whose shining portal is inscribed "Theory of Invariants."

PART I.—ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS.

(2) In the 'Arithmetica Universalis,' in the first chapter on equations, NEWTON has given a rule for discovering an inferior limit to the number of imaginary roots in an equation of any degree, without proof or indication of the method by which he arrived at it, or the evidence upon which it rests([2]). MACLAURIN, in vol. xxxiv. p. 104, and vol. xxxvi. p. 59 of the Philosophical Transactions, CAMPBELL([3]) in vol. xxxviii. p. 515 of the same, and other authors of reputation have sought in vain for a demonstration of this marvellous and mysterious rule([4]). Unwilling to rest my belief in it on mere empirical evidence, I

([2]) It appears to be the prevalent belief among mathematicians who have considered the question, that NEWTON was not in possession of other than empirical evidence in support of his rule.

([3]) CAMPBELL's memoir is rather on an analogous rule to NEWTON's than on the rule itself, to which he refers only by way of comparison with his own. In it the same singular error of reasoning is committed as in the notes of the French edition of the 'Arithmetica,' viz. of assuming, without a shadow of proof, that if each of a set of criteria indicates the existence of some imaginary roots, a succession of sets of such criteria must indicate the existence of at least as many distinct imaginary pairs of roots as there are such sets (see par. at foot of p. 528, Phil. Trans., vol. xxxv.)—much as if, supposing a number of dogs to be making a point in the same field, the existence could be assumed of as many birds as pointers.

([4]) Mr. ARCHIBALD SMITH has obligingly called my attention to WARING's treatment of the question of NEWTON's rule in the 'Meditationes Analyticæ.' On superficial examination the reader might be induced to suppose that in part 9, p. 68, ed. 1782, WARING had deduced a proof of the rule from the preceding propositions; but on looking into the case will find that there is not the slightest vestige of proof, the rule being stated, but without any demonstration whatever being either adduced or alleged. In fact, on turning to the preface of this (the last) edition of the 'Meditationes,' the reader will find at p. 11 an explicit avowal of the demonstration being wanting. After referring in order to CAMPBELL's, MACLAURIN's, and NEWTON's rules, as well as his own, for discovering the existence of impossible roots, he adds these words:

"At omnes hæ regulæ prædictæ perraro invenerunt verum numerum impossibilium radicum in æquationibus multarum dimensionum *et adhuc demonstratione egent*; vulgares enim demonstrationes solummodo probant impossibiles radices in data æquatione contineri, non vero quod *saltem tot sunt quot invenit regula*."

"Vera resolutio problematis est perdifficilis et valde laboriosa; cognitum est radices ex possibilitate per æqualitatem transire ad impossibilitatem; ergo in generali resolutione hujusce problematis necesse est invenire casum in quo radices datæ æquationis evadunt æquales; resolutio autem hujus casus valde laboriosa est; et consequenter resolutio generalis prædicti problematis magis *erit* laboriosa."

Written in Latin, and when the proper language of algebra was yet unformed, it is frequently a work of much labour to follow WARING's demonstrations and deductions, and to distinguish his assertions from his proofs. I find he agrees with the opinion expressed by myself, that NEWTON's rule will *not* "pene," as stated by NEWTON, but only "perraro," give the true number of imaginary roots. Like myself, too, in the body of the memoir WARING has given theorems of probability in connexion with rules of this kind, but without any clue to his method of arriving at them. Their correctness may legitimately be doubted.

[Since the above was sent to press, I have been enabled to ascertain that the great name of EULER is to be added to the long list of those who have fallen into error in their treatment of this question: see Institutiones Calculi Differentialis, vol. ii. cap. xiii. He says (p. 555, edition of Prony), "videndum est utrum hæc duo criteria (meaning NEWTON's criteria of imaginariness) sint contigua necne; priori casu numerus radicum imaginarium non augebitur; posteriori vero *quia criteria litteras prorsus diversas involvunt*, unumquodque binas radices imaginarias monstrabit."

The force of the supposed argument is contained in the words in italics. It is sufficiently met by the question, why or how the conclusion follows from them? Moreover the letters of two non-contiguous criteria are *not* necessarily *prorsus diversæ*; for two criteria with but a single other intervening between them will contain one letter in common.]

have investigated and obtained a demonstration of its truth as far as the fifth degree inclusive, which, although presenting only a small instalment of the desired result, I am induced to offer for insertion in the Transactions in the hope of exciting renewed attention to a subject so intimately bound up with the fundamental principles of algebra.

Before commencing the inquiry I ought to state that, in addition to the rule for detecting the existence of a certain number of imaginary roots, NEWTON has given a remarkable subsidiary method for dividing this number into two parts, representing respectively how many of the positive and how many of the negative roots indicated by DESCARTES's rule are, so to say, absorbed, and thereby obtains two distinct limits to the number of positive and the number of negative roots separately: of the grounds of this method, as far as I am aware, no one has even attempted an explanation, nor do I propose here to enter upon it; the rule, as I treat it, may be stated, not in NEWTON's own words, but most simply as follows:—

If the literal parts of the coefficients of an equation affected with the usual binomial coefficients be a, b, c, d, e ... h, k, l, *and if we form the successive criteria* b^2-ac; c^2-bd; d^2-ce; ...; k^2-hl, *or, which is the same thing differently expressed, if we write down the determinants*(5) *of all the successive quadratic derivatives of the given equation, then as many sequences as there are of negative signs in the arithmetical values of these criteria, so many pairs of imaginary roots at least there will be in the given equation.* If we choose to consider a^2 and l^2 also as criteria, appearing at the beginning and end of the series, then we may vary the expression of the rule by saying that there will be at least as many imaginary roots as there are variations of sign in the complete series so formed.

It will, however, be found more convenient for our present purpose to confine the designation of criteria to the determinants above alluded to.

(3) I shall deal with the homogeneous equation $f(x, y)=0$ so that the question of the reality of the roots is that of the reality of the ratios $\frac{x}{y}$ or $\frac{y}{x}$. It is obvious, from known principles, that f cannot have fewer imaginary roots than exist in $\frac{d}{dx}f$ or $\frac{d}{dy}f$(6), or, more generally, than in $\left(\frac{d}{dx}+\lambda\frac{d}{dy}\right)f$; from which it immediately follows(7) that if f have all its roots real, and the quadratic derivatives of f be called $Q_1, Q_2, \dots Q_{n-1}$, and the coeffi-

(5) To avoid the possibility of misapprehension, I state here once for all, that in the *discriminant* of a form of any degree I suppose the sign to be so taken as to render *positive* the term which is a power of the product of the first and last coefficients; and it may be well to remember that with this definition the number of real roots in any equation $\equiv 0$ or 1 to modulus 4 when the discriminant is positive, and $\equiv 2$ or 3 when the discriminant is negative; whereas the Determinant of a Quadratic form is to be taken in the same sense as that in which it is used by GAUSS, and is the same for such form as the Discriminant with the sign changed.

(6) This rule I find merges in the following more general and symmetrical one. Let f, φ be any two quantics in x, y; call the Jacobian of f, φ J; then the difference between the number of real roots in f and the like number in φ, taken positively and augmented by unity, cannot exceed the number of real roots in J. When φ is made equal to y, this theorem recurs to the familiar one alluded to in the text.

(7) By operating upon f successively with *any* $(n-2)$ distinct factors each of the form $\left(\frac{d}{dx}+\lambda_i\frac{d}{dy}\right)$.

cients of any function F of two degrees lower than f, whose roots are also *all* real, be $p_1, p_2, \ldots. p_{n-1}$, the quadratic function $p_1Q_1+p_2Q_2+\ldots+p_{n-1}Q_{n-1}$ must have its roots real, *i. e.* its discriminant must be positive: a particular consequence of this is, that by causing F to consist successively of the single terms $x^{n-2}, x^{n-3}y, \ldots. xy^{n-3}, y^{n-2}$, we see that the determinants of $Q_1, Q_2, \ldots Q_{n-1}$ must each of them be positive; or, in other words, if any of the Newtonian criteria of an equation are negative, it must have *some* imaginary roots, which is all that MACLAURIN, CAMPBELL, and others have succeeded in proving.

(4) The labour of proof of the cases hereinafter considered will be much lightened by the following rule of induction, viz., granting NEWTON'S rule to be true for the degree $n-1$, it must be true for all those cases appertaining to the degree n in which the series of the signs of the criteria does not commence with $-+$ and end with $+-$: to prove this, we have only to remember that f must have at least as many imaginary roots as $\frac{df}{dx}$ or $\frac{df}{dy}$, and that the criterion-series corresponding to $\frac{df}{dx}$ and to $\frac{df}{dy}$ will be found by cutting off from the series of f one term to the right and left respectively(8). If, now, the series for f begins with $++$ or $--$ or $+-$, the number of negative *sequences* is the same as when the left-hand sign is removed; so that it is only necessary to prove that the number of imaginary roots in f is not less than the number of negative sequences in $\frac{df}{dx}$; but this, by hypothesis, is not greater than the number of pairs of imaginary roots in $\frac{df}{dx}$, and, *à fortiori*, not greater than the number of such in f. In like manner, if the two *last* criteria of f are not $+-$, it may be shown that the truth of the rule for such form of f is implied in what is supposed to be known to be true for $\frac{df}{dy}$.

We may therefore limit our attention, as we ascend in the scale of proof, to those forms of f in which the criterion-series begins with $-+$ and ends with $+-$. Accordingly, since the rule is a truism for $n=2$, it is at once proved, by virtue of the above considerations, for $n=3$(9).

(8) For
$$\frac{d}{dx}(a, b, \ldots k, l \between x, y)^n = n(a, b, \ldots k \;\between x, y)^{n-1},$$
and
$$\frac{d}{dy}(a, b, \ldots k, l \between x, y)^n = n(\;\; b, \ldots k, l \between x, y)^{n-1}.$$

(9) The theorem for the case of cubic equations may be also proved directly as follows:

Writing the equation $ax^3+3bx^2y+3cxy^2+dy^3=0$, the two criteria are $L=b^2-ac$, $M=c^2-bd$; and the discriminant is $a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd=\Delta$.

1. Let L and M be of opposite signs, so that one and only one of them is negative. Then
$$\Delta=(ad-bc)^2-4(b^2-ac)(c^2-bd)=(ad-bc)^2-4LM,$$
and is therefore positive.

2. Let L and M be both negative. The equation may evidently, by writing x and y for $a^{\frac{1}{3}}x$, $d^{\frac{1}{3}}y$, be brought under the form
$$x^3+3\varepsilon x^2y+3\eta xy^2+y^3=0,$$
with the conditions $\varepsilon^2<\eta$, $\eta^2<\varepsilon$; from which we may deduce that ε and η are both positive, and $\varepsilon\eta<1$ and >0.

If all the criteria are zero, it is evident that, whatever n may be, all the roots are real. In every other case we shall find that *zero* may be made positive or negative at will. Thus in the case before us, if the two criteria are $0+$ or $0-$, there will be a pair of imaginary roots, as the first may be read as $-+$ and the second as $+-$.

To prove this, we have only to observe that in either case $\frac{df}{dx}$ will have two equal roots; so that f will be of the form $(ax+by)^3+cy^3$, which obviously, for any real values of a, b, c, has only one real root.

(5) We may now pass to the case of $n=4$, and excluding for the moment the consideration of *zeros*, limit our attention to the criterion series $-+-$.

Let $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=0$ be the equation for which the signs of the criteria b^2-ac, c^2-bd, d^2-ce are $-+-$. Call these criteria L, M, N respectively. It has to be proved that all four roots are imaginary, since there are two distinct negative sequences, each sequence consisting of a single $-$. Let x become $x+\epsilon y$ [10], where ϵ is an infinitesimal quantity, and transformed into one between u and y; then we have obviously,

$$\delta a=0,\quad \delta b=a\epsilon,\quad \delta c=2b\epsilon,\quad \delta d=3c\epsilon,\quad \delta e=4d\epsilon,$$
$$\delta \mathrm{L}=2b\delta b-a\delta c=0,\quad \delta \mathrm{M}=2c\delta c-b\delta d-d\delta b=(bc-ad)\epsilon,$$
$$\delta^2\mathrm{M}=(b\delta c+c\delta b-a\delta d)\epsilon=2(b^2-ac)\epsilon^2=2\mathrm{L}\epsilon^2;$$

so that $\delta^2\mathrm{M}$ is essentially negative, since L is so.

Hence, by continually augmenting x by an infinitesimal variation, we may, leaving L unaltered, so choose the sign of ϵ as to decrease M: nor can this process stop when $bc-ad$ becomes zero, by reason that $\delta^2\mathrm{M}$ is *negative*. Hence we may reduce M to zero. Now,

---

Also we have

$$\Delta=1+4(\epsilon^3+\eta^3)-6\epsilon\eta-3\epsilon^2\eta^2$$
$$>1+4(\epsilon+\eta)\epsilon\eta-6\epsilon\eta-3\epsilon^2\eta^2$$
$$>1-6\epsilon\eta+8(\epsilon\eta)^{\frac{3}{2}}-3\epsilon^2\eta^2;$$

or, writing $\epsilon\eta=q^2$,

$$\Delta>1-6q^2+8q^3-3q^4,$$
$$>(1-q)^3(1+3q);$$

but $1>q>0$. Hence Δ is positive.

Hence in either case two of the roots of the cubic are impossible. Or the same thing may be shown more immediately from the identities

$$a^2\Delta=(a^2d+2b^3-3abc)^2+4(ac-b^2)^3,$$
$$d^2\Delta=(ad^2+2c^3-3bcd)^2+4(bd-c^2)^3,$$

so that Δ must be positive, and therefore two roots imaginary, if either $bd>c^2$ or $ca>b^2$. It may be noticed that the square and cube in these identities are semi-invariants, being in the first of them unaffected by the change of x into $x+hy$, and in the second by the change of y into $y+hx$.

[10] This method of infinitesimal substitution is that which I applied in my memoir "On the Theory of Forms," in the Cambridge and Dublin Mathematical Journal, to obtain the partial differential equations to every possible species of invariants (including covariants and contravariants) of forms, or systems of forms, with a single set or various sets of variables, proceeding upon the pregnant principle that every finite linear substitution may be regarded as the result of an indefinite number of *simple* and *separate* infinitesimal variations impressed upon the variables. M. Aronhold has erroneously ascribed to others the priority of the publication of these equations.

in the course of this reduction, either N retains its sign or changes it; and if the latter is the case, N must have passed through zero. If when M becomes zero N is still negative, the criteria of the linearly transformed equation become $-0-$; and it may be noticed that its first, middle, and last coefficients must have the same sign, by virtue of the negativity of the two last criteria, and the second and fourth the same signs, by virtue of the zero middle criterion; consequently the equation will take the form

$$(\lambda^2+e^4)x^4 \pm 4e^3\varepsilon x^3y + be^2\varepsilon^2x^2y^2 \pm 4e\varepsilon^3xy^3 + (\mu^2+\varepsilon^4)y^4=0,$$

or

$$\lambda^2x^4+\mu^2y^4+(ex\pm\varepsilon y)^4=0,$$

which obviously has all its roots impossible. This being true of the transformed equation, will also, on the suppositions made, be equally so of the original equation.

Let us next suppose that N changes its sign either at the instant when, or before M becomes zero. If M and N both become zero together, so that the criteria of the transformed equation bear the signs $-\ 0\ 0$, calling the transformed equation $F=0$, $\frac{dF}{dy}$ will have all its roots equal, and F will therefore be of the form $(ax+by)^4+kx^4$, with the condition $(a^3b)^2-(a^4+k)(a^2b^2)<0$.

Hence k is positive, and consequently $F=0$ has all its roots imaginary; and the same, as before, must hold good of the original equation $f=0$.

It remains then only to consider the case when N becomes zero before M vanishes. When this is the case, as soon as N is reduced to zero, in lieu of the substitution of $x+\varepsilon y$ for x, we must leave x unaltered, and continue substituting $y+\varepsilon x$ for y. We thus start from the sequence $-+0$; N will then always remain zero, and we must either come to the series $-\ 0\ 0$, which we know, from what has been shown above, corresponds to four imaginary roots, or to the sequence $0+0$, which I shall proceed to consider.

Since the first and last coefficients must have the same sign, we may, by giving either variable a proper multiple (11), make these two coefficients alike, and with the first,

(11) (a) The form $(1, e, e^2, e, 1)\!\!\chi(x, y)^4$ may be regarded as a new and, for many purposes, useful canonical form of a binary quartic. It may be made to comprise within its sphere of representation all forms corresponding to two or four imaginary factors, but excludes the case of four real factors. The ordinary canonical form $(1, 0, 6m, 0, 1)\!\!\chi(x, y)^4$ comprises within its spheres of representation those forms for which the factors are all real or all imaginary, but, so far as real transformations are concerned, excludes the case of two real and two imaginary factors [that case is met by the form $1, 0, 6m, 0, -1)\!\!\chi(x, y)^4$], as may easily be established either by decomposing the form first named into its factors, or by the consideration that its discriminant Δ is $(1-9m^2)^2$, and is therefore always positive; whereas if a form which it is used to represent have two real and two unreal factors, its discriminant is negative. If now the determinant of transformation be D, and the discriminant corresponding thereto be called Δ', we have $\Delta'=D^6\Delta$, showing that D^2 is negative, and the transformation therefore unreal.

(b) The *reality* of m for each of these cases (usually assumed without proof) may be demonstrated as follows: Calling the cubic invariant and the discriminant of any cubic form T, D, we shall have, using the *ordinary* canonical form, $\frac{(m-m^3)^2}{(1-9m^2)^2}=\frac{T^2}{D}$, showing that when D is positive, which is the case of four real or unreal factors, there will

second, and third, as well as the third, fourth, and fifth coefficients form geometrical series; hence it is obvious that the transformed equation may be reduced to one or the other of the two following forms, viz.

$$x^4+4ex^3y+6e^2x^2y^2-4exy^3+y^4=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

or

$$x^4+4ex^3y+6e^2x^2y^2+4exy^3+y^4=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

with the condition in the latter case that e^4-e^2 is positive, *i. e.* $e^2>1$.

---

be one real value of m, and when D is negative, a real value of im. The former case possesses over the latter a striking distinction, which is that *all* the roots of m will be real; for, as I have shown elsewhere, if m is one root the complete system of roots will be $\pm m$, $\pm\frac{1-2m}{1+3m}$, $\pm\frac{1+2m}{1-3m}$: in the latter case the reality of the two values $\pm im$ does not seem necessarily to imply the reality of the other 4 values of the system.

([c]) Analogy suggests the establishment of an analogous canonical form or forms for ternary cubics, of which, as is well known and is even dimly foreshadowed in NEWTON'S Enumeration of Lines of the Third Order, the theory runs closely parallel to that of binary quartics. This will be effected by assuming the form

$$F(x, y, z)=\Sigma x^3+3e\Sigma x^2y+6gxyz,$$

and assuming g so as to make the discriminants of

$$\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}$$

all zero. This gives rise to a quadratic equation in g, of which the roots are $g=e$, $g=2e^2-e$. When $g=e$, I find

$$S=e(1-e)^3, \quad T=(1-e)^4(1+4e-8e^2), \quad \Delta=T^2+64\,S^3=(1+8e)(1-e)^8.$$

When $g=2e^2-e$, I find $\Delta=(1-e)^i(1-4e)^j(1+2e)^k$, where i, j, k are integers to be determined. These forms will, I think, be found important in the future perspective discussion of curves of the third degree. Whilst I yield to no one in admiration of the surpassing genius with which NEWTON has handled these curves, I cannot withhold the expression of my opinion that every theory of forms in which invariants are ignored must labour under an inherent imperfection, and that NEWTON, from want of acquaintance with the indelible characters which their invariants stamp upon curves, has in the parallel which he has drawn between the generation by shadows of all conics from a common type, and of all cubic curves from a limited number of forms, either himself fallen into error of conception, or at least used language which could scarcely fail to lead others into such error. For no species whatever of cubic curve can be formed for which an infinite number of individuals cannot be found which defy linear or perspective transformation into each other; whereas all conics proper may be propagated as shadows from a single individual. It should be noticed in connexion with this subject, that the *indelible* characters of quartic binary, and cubic ternary forms are two in number, viz. the value of $\frac{s^3}{t^2}$ (where s, t are the two fundamental invariants in either case) and the *sign* of t. The indelibility of the sign of s being implied in the invariability of the value of $\frac{s^3}{t^2}$, does not constitute a distinct character. Of course all symmetrical invariants have an invariable sign; but this is not the case with skew invariants, as *ex. gr.* M. HERMITE'S octodecimal invariant of a binary quintic, which will change its sign with that of the determinant of transformation.

([d]) Whilst upon this subject of invariants, I may allow myself to make a remark bearing upon what will be noticed further on in the text about a case of equality between roots not necessarily being a mark of transition from real to imaginary roots. If a, b, c, d being the roots of a binary quartic we form a secondary cubic, of which the roots are $(a-b)(c-d)$, $(a-c)(d-b)$, $(a-d)(b-c)$, it may be easily shown that two of these quantities become equal, or, in other words, the roots of the original equation mark out a harmonic group of points when t (the cubinvariant) is zero. Notwithstanding which a change of sign in t will not command a change of character in the above three roots of the secondary (nor consequently of the original equation), because it is not an odd but an even power of t, viz. t^2, which enters into the discriminant of the secondary.

It must be remembered that we know, from the form of the criteria-series to the derivatives in respect to either x or y (indifferently), that the equation must have *some* imaginary roots; and the question therefore lies between its having two or four. If the discriminant is negative, the former will be the case, if positive, the latter. I shall show that in each equation the discriminant is positive.

Let s, t represent in general the quartic invariants, then we have to show that s^3-27t^2 is positive.

In case (a), $s=1+4e^2+3e^4$
$$=(1+e^2)(1+3e^2)\qquad t=\begin{vmatrix} 1 & e & e^2 \\ e & e^2 & -e \\ e^2 & -e & 1 \end{vmatrix} \begin{aligned} &=e^2-e^4-e^4-e^2-e^6-e^2 \\ &=-e^2-2e^4-e^6 \\ &=-e^2(1+e^2)^2, \end{aligned}$$
so that
$$s^3-27t^2=(1-e^2)^3\{(1+3e^2)^3-27e^4(1+e^2)\}=(1+e^2)^3(1+9e^2),$$
and is positive.

In case (b),
$$s=(1-4e^2+3e^4)=(1-e^2)(1-3e^2)$$
$$t=\begin{vmatrix} 1 & e & e^2 \\ e & e^2 & e \\ e^2 & e & 1 \end{vmatrix} \begin{aligned} &=e^2+e^4+e^4-e^2-e^6-e^2 \\ &=-e^2+2e^4-e^6=-e^2(1-e^2)^2, \end{aligned}$$
and
$$\begin{aligned} s^3-27t^2&=(1-e^2)^3\big((1-3e^2)^3-27e^4(1-e^2)\big) \\ &=(1-e^2)^3(1-9e^2). \end{aligned}$$

The above can only be negative when e^2 lies between 1 and $\frac{1}{9}$; but in the case supposed $e>1$. Hence the discriminant is positive, and the roots are all imaginary[12]. Thus, then, the theorem is established for $n=4$, as well as for the cases where the criteria are zero (as will have been observed in the course of the demonstration), as for those where they are *plus* or *minus*; and it should be observed that the demonstration proceeds upon our being able to show that the quartic, in the case where it resists reduction to the case of the cubic, viz. where the criteria are negative at the two extremes and positive in the middle, may by real linear transformations be changed into a form where either the middle criterion is zero and the two extremes negative, or the two extremes zero, and the middle one positive.

(12) The reader conversant only with ordinary algebra may easily verify this result. For writing $\frac{x}{y}+\frac{y}{x}=z$, the equation becomes $z^2+4ez+6e^2-2=0$, and this will have its roots impossible unless $4e^2>6e^2-2$, or $2e^2-2$ negative, which it cannot be, since $e^2>1$, and consequently $x:y$ has all its roots impossible. Moreover the same conclusion would (as before shown) hold good unless e^2 lay between 1 and $\frac{1}{9}$; for on making $z=2$, the function above written in z becomes $2+8e+6e^2$, or $2(1+e)(1+3e)$; and making $z=-2$, it becomes $2-8e+6e^2$, or $2(1-e)(1-3e)$, which two quantities evidently have both positive signs unless e lies between 1 and $\frac{1}{3}$, or between -1 and $-\frac{1}{3}$; so that the first and third Sturmian functions are (except on that supposition) respectively positive and negative for $z=2$, and also for $z=-2$, showing that no root of z can lie between 2 and -2, and consequently that all the roots of $x:y$ remain impossible.

Observation.—To make the foregoing demonstration quite exact, it should be noticed that when the criteria L, M, N have been brought to the form $-+0$, and the series of substitutions of $y+\epsilon x$ for y has set in, we have

$$N=0,\quad \delta N=0,\quad \delta M=(cd-be)\epsilon,\quad \delta^2 M=N\epsilon=0,\quad \delta^3 M=0.$$

Consequently if $cd-be$ should become *zero*, we can no longer go on decreasing M. But as soon as $cd-be=0$, since we have also $d^2=ce$, b, c, d, e come to be in geometrical progression, and the transformed equation takes the form

$$ax^4+4\omega x^3y+6\omega^2x^2y^2+4\omega^3xy^3+\omega^4x^4=0,$$

with the condition $\omega^2-a\omega^2$ negative, or $a>1$. Hence we have $q^2x^4+(x+\omega y)^4=0$, which obviously has all its roots impossible([13]).

(6) We may now pass on to equations of the fifth degree, in which the case resisting induction will be that where the criterion-series bears the signs

$$-++-.$$

Let the criteria be called L, M, N, P, so that writing the equation

$$ax^5+5bx^4y+10cx^3y^2+10dx^2y^3+5exy^4+fy^5=0,$$

$$L=b^2-ac,\quad M=c^2-bd,\quad N=d^2-ce,\quad P=e^2-df,$$

and writing for x, $x+\epsilon y$, we have, as before,

$$\delta L=0,\quad \delta M=(bc-ad)\epsilon,\quad \delta^2 M=L\epsilon^2,$$

so that M may be continually diminished.

If M becomes zero before either N or P changes its sign, the criterion-series for the transformed equation becomes $-\ 0+-$, and for its derivative in respect to x, the series is $0+-$, which proves the existence of four imaginary roots in the transformed, and consequently also in the given equation. In like manner, if N becomes zero before M or P have changed their signs, the criterion-series becomes $-+0-$, which obviously leads to the same result. So likewise the same inference may be drawn if L and M, or M and N, or L, M, N become zeros all at the same time, and we have only to consider the case when, L and M retaining their signs, N becomes zero. At this moment the order of the substitutions must be reversed, and for y must be written $y+\epsilon x$; we shall then have

$$P=0,\quad \delta P=0,\quad \delta N=(de-cf)\epsilon\ldots\ldots;$$

([13]) From the first and third criteria it follows that in the form $(a, b, c, d, e \between x, y)^4$, a, c, e have the same sign and may be regarded as all positive; so that writing $a-\frac{b^2}{c}=h^2$, $e-\frac{d^2}{c}=k^2$, the form becomes $h^2x^2+F+k^2y^2$, where

$$F=\frac{b^2}{c}x^4+4bx^3y+bcx^2y^2+4dxy^3+\frac{d^2}{c}y^4,$$

and consequently the given form will have all its roots imaginary when this is true for F, so that we might have proceeded at once to deal with the forms marked (a), (b) at p. 585; but as the method of homographic transformation by infinitesimal substitutions appears to be necessary in passing to the corresponding forms in the case of the fifth degree, and as in treating that case reference is made to what appears above, I have thought that no object would be gained by altering the text.

and reasoning as in the preceding case for $n=4$ (with the sole difference, that if δN vanishes by virtue of $de-cf$ vanishing, we should have $P=0$, $N=0$, and the criterion-series $-\ +\ 0\ 0$, which at once indicates the existence of four imaginary roots), we see that there remains only to consider the case where the criterion-series takes the form $0\ +\ +\ 0$. It is scarcely necessary to observe that all the criteria can never vanish simultaneously; for that would indicate the equality of all the roots in the transformed, and therefore in the given equation, whose own criteria, contrary to hypothesis, would also be all zero. The zero values of the two extreme criteria indicates that the three first and the three last literal parts of the coefficients are in geometrical progression, from which it will immediately be seen that the equation to be considered may be thrown (by substituting in lieu of x and y suitable multiples of x and y, which will not affect the characters of the criteria) into the convenient form

$$x^5+5\epsilon x^4y+10\epsilon^2x^3y^2+10\eta^2x^2y^3+5\eta xy^4+y^5=0,$$

with the two conditions $\epsilon^4-\epsilon\eta^2$ *positive*, $\eta^4-\eta\epsilon^2$ *positive*.

The form of the criterion-series, apocopated from either end, shows that two of the roots must be imaginary; and consequently, in order to establish the existence of two imaginary pairs of roots, it is only necessary to show that the discriminant of the above equation, subject to the above conditions, must remain always positive. That discriminant I proceed to determine; but as a guide to the form under which it is to be expressed, the following observation is important. Let us take the more general form

$$ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0,$$

where

$$a=1,\quad b=\lambda\epsilon,\quad c=\mu\epsilon^2,\quad d=\mu\eta^2,\quad e=\lambda\eta,\quad f=1,$$

λ, μ being any numerical quantities.

The discriminant will evidently be a symmetrical function of e and ϵ.

Let $a^pb^qc^rd^se^t$ be the literal part of any term in the discriminant. By the *law of weight* we must have

$$q+2r+3s+4t=5\times4=20.$$

But in the equation before us, $a^pb^qc^rd^se^t$ (to a numerical factor *près*) is $\epsilon^{q+2r}\eta^{2s+t}$, and

$$(q+2r)-(2s+t)=(q+2r+3s+4t)-5(s+t)$$
$$=5(4-s+t).$$

Hence the difference between the indices of ϵ and η in each term is a multiple of 5, and consequently, since the discriminant is a symmetrical function in ϵ and η, it will be a rational integral function of $\epsilon^5+\eta^5$ and $\epsilon\eta$. Moreover, as no such term as c^4d^4 can figure in the discriminant, which, as we know, must in all cases contain one or the other of the two final and of the two initial coefficients, we see that no term can be of higher than the 14th degree in ϵ, η, nor yet so high, for the only terms that could be of that degree would be bc^3d^3e; but making a and f each zero in the original form, it becomes obvious

that all the terms free from a and f contain b^2e^2 as a factor(14). Hence, in fact, the discriminant will be only of the twelfth degree in ϵ, η, and being therefore of only the second degree in $\epsilon^6+\eta^6$, will admit of comparatively easy treatment.

(7) Before proceeding to the calculation of this discriminant, it will be useful to investigate, as a Lemma ancillary to the subsequent discussion, under what conditions four of the roots of the supposed equation will become imaginary when $\epsilon=\eta$.

In this case writing $\frac{x}{y}+\frac{y}{x}=z$, the equation

$$\frac{1}{x+1}(1,\ \epsilon,\ \epsilon^2,\ \epsilon^2,\ \epsilon,\ 1 \between x,\ y)^5=0$$

becomes

$$z^2-2-z+1+5\epsilon(z-1)+10\epsilon^2=z^2+(5\epsilon-1)z+10\epsilon^2-5\epsilon-1=0,$$

or say $fz=0$.

The determinant of $f(z)$ is thus $(5\epsilon-1)^2-40\epsilon^2+20\epsilon+y$, i. e. $5(1-\epsilon)(1+3\epsilon)$; and all the roots of z, and consequently of (x, y), will be impossible, unless z lies between 1 and $-\frac{1}{3}$.

Now
$$f(2)=1+5\epsilon+10\epsilon^2,$$
$$f'(2)=3+5\epsilon;$$

so that when z has any real roots, *i. e.* when ϵ lies between 1 and $-\frac{1}{3}$, $f(2)$, $f'(2)$ are both positive, and the Sturmian functions are of the signs $+++$.

Again,
$$f(-2)=5-15\epsilon+10\epsilon^2=5(1-\epsilon)(1-2\epsilon),$$
$$f'(-2)=-5+5\epsilon;$$

so that, on the same supposition as before, the Sturmian functions are $\pm-+$, viz.

$$+-+ \text{ when } \tfrac{1}{2}>\epsilon>-\tfrac{1}{3},$$
$$--+ \text{ when } 1>\epsilon>\tfrac{1}{2}.$$

In the former case two real roots, in the latter one real root of z lies between $2, -2$. Hence in the former case no real roots of z lie between the limits ∞, 2, and the limits -2, $-\infty$, and in the latter case one real root lies between those limits. Hence x, y will have four imaginary roots, unless ϵ lies between 1 and $\frac{1}{2}$, and two such roots in every other case.

Thus the discriminant of $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5$, when $\epsilon=\eta$, is negative when ϵ lies between 1 and $\frac{1}{2}$, but for every other value of ϵ is positive, save that it vanishes when

$$\epsilon=1, \text{ or } \epsilon=\tfrac{1}{2}\,(15), \text{ or } \epsilon=-\tfrac{1}{3}.$$

(8) I now proceed to calculate the discriminant of the form

$$x^5+5\epsilon x^4y+10\epsilon^2x^3y^2+10\eta^2x^2y^3+5\eta xy^4+y^5$$

(14) For the discriminant of $xy\phi(x, y)$ = the discriminant of $\phi(x, y)$ multiplied by the square of the product of the resultant of (x, ϕ) and of (y, ϕ).

(15) When $\epsilon=\frac{1}{2}$ the discriminant of $f(z)$ does not vanish, but $z=-2$ satisfies the equation in z, and consequently $\frac{x}{y}$ has two equal roots -1, so that the discriminant of the original equation vanishes.

for general values of ε, η. This will be accomplished most expeditiously by taking the resultant of the two derivatives of the above form, say U and V, where

$$U=x^4+4\varepsilon x^3y+6\varepsilon^2x^2y^2+4\eta^2xy^3+\eta y^4,$$
$$V=\varepsilon x^4+4\varepsilon^2x^3y+6\eta^2x^2y^2+4\eta xy^3+y^4;$$

so that

$$\varepsilon U-V\ =6(\varepsilon^3-\eta^2)x^2y^2+4(\varepsilon\eta^2-\eta)xy^3+(\varepsilon\eta-1)y^4=y^2P,$$
$$-U+\eta V=(\varepsilon\eta-1)x^4+4(\eta\varepsilon^2-\varepsilon)x^3y+6(\eta^3-\varepsilon^2)x^2y^2=x^2Q.$$

Hence

Resultant of (U, V)$=\frac{1}{(\varepsilon\eta-1)^4}\times$ *Resultant* of $(y^2P, x^2Q)=$ *Resultant* of (P, Q);

where

$$P=6(\varepsilon^3-\eta^2)x^2+4(\varepsilon\eta^2-\eta)xy+(\varepsilon\eta-1)y^2,$$
$$Q=(\varepsilon\eta-1)x^2+4(\eta\varepsilon^2-\varepsilon)xy+6(\eta^3-\varepsilon^2)y^2.$$

Hence, calling Δ the discriminant of the original form, we obtain by the well-known formula for the resultant of two binary quadratics, writing for the moment

$$P=(B,\ 4\eta A,\ A)(x,\ y)^2,\quad Q=(A,\ 4\varepsilon A,\ B')(x,\ y)^2,$$

$$\Delta=(4\varepsilon A^2-4\eta AB')(4\eta A^2-4\varepsilon AB)+(A^2-BB')^2$$
$$=(1-16\varepsilon\eta)A^4+16(\varepsilon^2B+\eta^2B')A^3-16\varepsilon\eta BB'A^2-2BB'A^2+B^2B'^2.$$

Hence writing $\varepsilon\eta=q$, $\varepsilon^5+\eta^5=S$,

$$\Delta=(1-16q)(q-1)^4+96(S-2q^2)(q-1)^3-72(8q+1)(q^3+q^2-S)(q-1)^2$$
$$+36^2(q^3+q^2-S)^2.$$

Let $S-q^2-q^3=\sigma$, $q-1=p$, so that

$$S-2q^2=\sigma-q^2+q^3=\sigma+(p+1)^2p.$$

Then

$$\Delta=36^2\sigma^2+72(8p+9)p^2\sigma+96p^3\sigma+96(p+1)^2p^4-(16p+15)p$$
$$=1296\sigma^2+(648p^2+672p^3)\sigma+96p^6+176p^5+81p^4,$$
$$=\tfrac{1}{9}\{108\sigma+27p^2+28p^3)^2+729p^4+1584p^5+864p^6-(27p^2+28p^3)^2\},$$

or

$$9\Delta=(108\sigma+27p^2+28p^3)^2+72p^5+80p^6.$$

(9) Hence we see at once that Δ can be negative only when p lies between 0 and $-\frac{9}{10}$, *i. e.* when $\varepsilon\eta$ (which is $p+1$) lies between 1 and $\frac{1}{10}$. Accordingly when Δ is negative, ε and η must be both positive or both negative. The latter supposition may easily be disproved as follows: treating the equation $\Delta=0$ as a quadratic equation in σ, in order that Δ may be capable of becoming negative, its discriminant in respect to σ must be negative, and its value when $\sigma=-\infty$ is positive. Now

$$S=\varepsilon^5+\eta^5,\quad p+1=\varepsilon\eta,\quad \sigma=S-(p+1)^2-(p+1)^3;$$

so that when ε and η are real we have

$$S>2(p+1)^{\frac{5}{2}}\ (^{16}),\ \text{i. e. } \sigma>-(p+1)^2+2(p+1)^{\frac{5}{2}}-(p+1)^3$$

[16] It is of course understood that $(p+1)^{\frac{5}{2}}$ is to be taken *positive.*

when ϵ, η are both positive, and

$$S<-2(p+1)^{\frac{5}{2}}\text{(16 bis)}, \text{ i. e. } \sigma<(p+1)^2-(p+1)^3-2(p+1)^{\frac{5}{2}}$$

when ϵ, η are both negative.

If now we substitute $(p+1)^2+(p+1)^3-2(p+1)^{\frac{5}{2}}$ for σ in Δ, I say that the resulting value will be positive whatever positive value be given to $(p+1)$; in fact, if we write $p=\nu^2-1$, and make $\sigma=-\nu^4+2\nu^5-\nu^6$, so that Δ becomes a function of the twelfth degree in ν, this function is what the discriminant of the equation in x, y becomes when we have $\epsilon=\eta=\nu$; but in the antecedent Lemma it has been shown that this discriminant is only negative when the two equal quantities ϵ or η, or, which is the same thing, when ν lies between 1 and $\frac{1}{2}$; hence Δ is positive when ν is negative, and consequently when

$$\sigma=(p+1)^2+(p+1)^3-2(p+1)^{\frac{5}{2}}.$$

Thus Δ, a quadratic function in σ, and its discriminant are respectively $+$ and $-$ for this value of σ, as well as for $\sigma=-\infty$. Hence no real root of σ lies between such value of σ and $-\infty$, and consequently Δ must be always positive when ϵ and η are both negative. Hence, if Δ is negative, we must have $1>\epsilon\eta>\frac{1}{10}$; $\epsilon>0$; $\eta>0$. But our *criteria* give

$$\epsilon^4-\epsilon\eta^2>0, \quad \eta^4-\eta\epsilon^2>0,$$

which, when $\epsilon>0$, $\eta>0$, imply $\epsilon^3>\eta^2$, $\eta^3>\epsilon^2$, and consequently $\epsilon\eta>1$, which is in contradiction to the inequality $1>\epsilon\eta$. Hence when these criteria are satisfied the determinant is necessarily *positive*, and all the roots are imaginary, which completes the proof of NEWTON'S rule for equations of the fifth degree.

(10) It follows as a corollary to the Lemma employed in the preceding investigation, that if in Δ we write $\sigma=-(\nu^2-\nu^3)^2$ and $p=\nu^2-1$, and distinguish this particular value by the symbol (Δ), then (Δ) ought to break up into the product of odd powers of $\nu-1$, $\nu-\frac{1}{2}$ of some even power of $(\nu+\frac{1}{3})$, and of a factor incapable of changing its sign, and remaining always positive. This may be easily verified; for dividing (Δ) by $(\nu-1)^4$, we obtain

$$1296\nu^8\big(648(\nu+1)^2+24(\nu^2-1)(\nu+1)^2\big)\nu^4+96(\nu^2-1)^2(\nu+1)^4+176(\nu^2-1)(\nu+1)^4+81(\nu+1)^4;$$

and collecting the terms $1296\nu^8-648\nu^6(\nu+1)^2+81(\nu+1)^4$ whose sum contains the factor $(\nu-1)$, we have

$$\begin{aligned}\frac{(\Delta)}{(\nu-1)^5}= &\ 648(\nu^7+\nu^6+\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\ &-1296(\nu^6+\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\ &-648(\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\ &+81(\nu^3+5\nu^2+11\nu+15)\\ &-24(\nu^7+3\nu^6+3\nu^5+\nu^4)\\ &+96(\nu^7+5\nu^6+9\nu^5+5\nu^4-5\nu^3-9\nu^2-5\nu-1)\\ &+176(\nu^5+5\nu^4+10\nu^3+10\nu^2+5\nu+1)\\ =&\ 720\nu^7-240\nu^6-328\nu^5+40\nu^4+65\nu^3+5\nu^2-5\nu-1.\end{aligned}$$

Hence

$$\begin{aligned}(\Delta)&=(\nu-1)^5(2\nu-1)^3\{90\nu^4+105\nu^3+49\nu^2+11\nu+1\}\\ &=(\nu-1)^5(2\nu-1)^3(3\nu+1)^2\{10\nu^2+5\nu+1\};\end{aligned}$$

(16 bis) It is of course understood that $(p+1)^{\frac{5}{2}}$ is to be taken *positive*.

showing, agreeably with what was seen in the Lemma, that the discriminant of

$$(1, \epsilon, \epsilon^2, \epsilon^3, \epsilon, 1\between x, y)^5$$

vanishes then, and then only, when

$$\epsilon=1, \text{ or } \epsilon=\tfrac{1}{2}, \text{ or } \epsilon=-\tfrac{1}{3},$$

but does not *change its sign*, except as ϵ passes through the limits 1 and $\frac{1}{2}$, and only within those limits can become negative(17).

(11) Although the theory of the possibility of the roots of $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1\between x, y)^5=0$ has now been completely investigated, so far as is necessary for the proof of NEWTON'S theorem applied to equations of the fifth degree, it will be found that the labour will not be ill spent of considering more closely the real nature of the criteria which separate the case of one pair from that of two pairs of impossible roots in the above equation. NEWTON'S *criteria* being constructed so as to cover every possible case for equations of every degree, will always be found to fit loosely, so to speak, upon each case treated *per se*; so that more precise conditions can be assigned in each particular case than those which are furnished by his rule. So, *ex. gr.*, it may be remembered that in the equation $(1, e, e^2, e, 1\between x, y)^4=0$, NEWTON'S rule implies only that when $e>1$, the roots are all impossible; but we have found further that unless $1>e>\frac{1}{3}$ (a much closer condition), the same thing takes place.

It is obvious from what has been demonstrated above, that if we treat p and σ, which are respectively $\epsilon\eta-1$ and $\epsilon^5+\eta^5-\epsilon^2\eta^2-\epsilon^3\eta^3$, as the abscissa and ordinate of a variable point in a plane, the curve $\Delta=0$, i. e. $(108\sigma+27p^2+28p^3)^2+72p^5+80p^6=0$ will be the line of demarcation between those values of ϵ, η which correspond to one pair, and those which correspond to two pairs of imaginary roots.

For all values of ϵ, η corresponding to internal points of the curve Δ there will be two imaginary and three distinct real roots; for all such as correspond to external points there will be four imaginary roots, and for points *on* the curve two imaginary and two equal roots.

The curve Δ is a curve of the 6th degree whose form will presently be discussed. But there is an important remark to be made in the first instance. Not all the points

(17) In *general* the case of equal roots of an equation is the state of transition of two real roots into imaginary, or *vice versâ*. But we see by the above instance that this is not necessarily the case *always*, for Δ vanishes on making $\epsilon=-\frac{1}{3}$, and two roots become equal without any change in the nature of the roots when ϵ passes from being greater to being less than $-\frac{1}{3}$. In such case, however, there is a sort of unstable equilibrium in the form of the equation, by which I mean that the effect of any general infinitesimal change performed upon the coefficients of the equation would be either to cause the real roots in the neighbourhood of $\epsilon=-\frac{1}{3}$ to disappear by the factor $(\epsilon+\frac{1}{3})^2$ becoming superseded by a quadratic function of ϵ with impossible roots, or else a region in the neighbourhood of $\epsilon=-\frac{1}{3}$ would reappear, for which the equation would acquire two real roots, owing to $(\epsilon+\frac{1}{3})^2$ becoming superseded by a quadratic function of ϵ with real roots, in which case there would be two values in the neighbourhood of $\epsilon=-\frac{1}{3}$, for *each* of which there would be a pair of equal roots in the equation. The above is probably the first instance distinctly noticed of this singular obliteration of the usual effect upon real and imaginary roots of a passage through equality, owing to the appearance of a square factor in the discriminant.

within the curve Δ will correspond to *real* values of ϵ, η. In order that these quantities may be real, we must have

$$\epsilon^3+\eta^3>2(\epsilon\eta)^{\frac{3}{2}},$$

$$\text{i. e. } \sigma+q^2+q^3>2q^{\frac{3}{2}}, \text{ where } q=p+1,$$

or

$$\sigma^2+2(q^2+q^3)\sigma+q^4-4q^6+q^6>0.$$

Writing this inequality under the form $R>0$, we see that the curve $R=0$ will represent a second sextic curve intersecting the former. Δ may be called the curve of the discriminant or *discriminatrix*, and will be a close curve, and R the curve of equal parameters or *equatrix*, and will consist of a single infinite branch. All points on the latter correspond to equal values of ϵ, η, those on one side of it to real values of ϵ, η, and those on the other side of it to conjugate values of the form $\lambda+i\mu$, $\lambda-i\mu$ respectively. Thus the area confined within the curve Δ will be divided into two portions by the equatrix, and it is impossible to shut one's eyes to the inquiry as to the meaning of the variable point lying in that portion which gives conjugate values to ϵ, η. It becomes clear by analogy that some kind of distinction must be capable of being drawn between the nature of the roots of the equation $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5=0$ when ϵ, η are conjugate, in some sense similar or parallel to that which we know to exist between them when ϵ, η are real; and obviously this inference cannot be confined to equations of the particular form and degree of that above written; in a word, equations whose coefficients are not real but conjugate, must have roots of two kinds, one analogous to the real, the other to the imaginary roots of equations with real coefficients. This inference will be justified in the sequel; but in the meanwhile it will be desirable to complete the investigation of the special equation under consideration, by a discussion of the forms and relations of the two curves Δ and R. These curves we know *à priori*, from what has been already demonstrated, can only meet in the three points corresponding to

$$\epsilon=\eta=1, \quad \epsilon=\eta=\tfrac{1}{2}, \quad \epsilon=\eta=-\tfrac{1}{3};$$

and since $p=\epsilon\eta-1$, the abscissæ of these three points will be 0, $-\frac{1}{4}$, $-\frac{8}{9}$.

Moreover the 3rd point will be distinguished from the other two by the circumstance that Δ does not change its sign as p passes through the value $-\frac{8}{9}$. Consequently the two curves must touch each other at this point.

Since when $\Delta=0$ p lies between 0 and $-\frac{9}{10}$, the curve Δ is confined to the negative side of the axis of σ. It is also confined to the negative side of the axis of p.

For between the limits $p=0$, $p=-\frac{9}{10}$,

$$648p^2+672p^3, \text{ i. e. } 24(27p^2+28p^3) \text{ is obviously positive,}$$

and

$$96p^6+176p^5+81p^4=\frac{p^4}{6}\{(24p+22)^2+2\} \text{ is always positive.}$$

Hence the two values of σ are both negative throughout the extent of the curve Δ.

Thus $\epsilon^5+\eta^5-\epsilon^2\eta^3-\epsilon^3\eta^2$ being negative, $\epsilon^2-\eta^2$ and $\eta^3-\epsilon^3$ have the same signs when ϵ, η

are *real*, as should be the case; for in order that Δ may be capable of vanishing, $\varepsilon(\varepsilon^2-\eta^2)$ and $\eta(\eta^2-\varepsilon^2)$ must, by NEWTON'S rule, be *both* negative, which could not be the case if either ε or η were negative; so that $\varepsilon^2-\eta^2$ and $\eta^2-\varepsilon^2$ must have the same signs, in fact each must be negative.

The curve Δ under consideration has a multiple point of the 4th order of multiplicity at the origin, where it is touched by the axis of p. Its distance from the axis for the extreme value of p, viz. $p=-\frac{9}{10}$, is $\frac{27}{2000}$.

It has three real maxima and minima, two belonging to its upper portion and one to the lower portion at the points, for which p has the *approximate* values $-\frac{9}{16}$, $-\frac{15}{22}$, and $-\frac{7}{8}$ (18).

The curve R, i. e. $\sigma=\left((p+1)\pm(p+1)^{\frac{3}{2}}\right)^2$, has the values 0 and -4 at the origin, a cusp at its extremity corresponding to $p=-1$, where both of its branches meet and touch the axis of p, and a negative maximum in its upper branch at the point where $p=-\frac{5}{9}$.

At all points within the curve R, ε and η are conjugate, and for the points outside real. Its lower branch will meet and touch the lower portion of Δ at the point where $p=-\frac{8}{9}$, and its upper branch will intersect and pass out of the upper branch of Δ at the point where $p=-\frac{3}{4}$. The only part of the area Δ therefore which corresponds to real values of ε, η, is that which is included between the upper segment of Δ and the upper branch of R, and extends only from $p=0$ to $p=-\frac{3}{4}$, i. e. from $\varepsilon\eta=1$ to $\varepsilon\eta=\frac{3}{4}$. Hence we may easily find an inferior limit to the values of ε and η when the equation (ε, η) has two real roots; for we have in that case ε, η, $\eta^3-\varepsilon^3$, $\varepsilon^2-\eta^2$ all positive. Hence

$$\eta^5>\varepsilon^3\eta^3>q^3, \quad \eta^5<\varepsilon^2\eta^2<q^2.$$

Consequently ε, η must each of them always lie between $q^{\frac{3}{5}}$, $q^{\frac{2}{5}}$; and since the least value of q is $\frac{1}{4}$, ε, η must each be always greater than $(\frac{1}{4})^{\frac{3}{5}}$, i. e. than $\cdot 33499$ (19).

(18) The large numbers which enter into Δ may be usefully reduced, and the equation $\Delta=0$ made more manageable, by aid of the simple substitutions $\sigma=-\frac{27v}{64}$, $p=-\frac{9u}{4}$. The equation $\Delta=0$ then becomes

$$(v-3u^2+7u^3)^2=2u^5-5u^6,$$

whose maxima and minima will be given by the equation

$$(v-3u^2+7u^3)(-6u+21u^2)=5u^4-15u^5;$$

which, making $1-3u=\omega$, becomes

$$270\omega^3-46\omega^2-9\omega+1=0,$$

whose roots are all real, and are one just a little greater than $-\frac{1}{6}$, another a little less than $\frac{1}{4}$, and the third a very little less than $\frac{1}{11}$ respectively; whence $p=\frac{3}{4}(\omega-1)$ will have the approximate values given in the text.

(19) $\varepsilon:\eta$ will have a maximum value, which can be found by writing $\delta\varepsilon:\delta\eta::\varepsilon:\eta$; and consequently, remembering that $q=p+1$, $S=\varepsilon^5+\eta^5$, $\sigma=S-q^2-q^3$,

$$\delta S:\delta q::5S:2q,$$

and therefore

$$\delta\sigma:\delta p::5\sigma+q^2-q^3:2q::5\sigma+p(p+1)^2:2(p+1).$$

Substituting the values of $\delta\sigma:\delta p$ in $\delta\Delta=0$, and combining the result with the equation $\Delta=0$, p and σ may be found by the solution of a numerical equation of the 5th degree, and then ε and η may be found by the solution

There is a third curve not undeserving of notice, of only the 3rd degree, which embodies the joint effect of the two middle criteria (the two extremes being supposed to be each zero) in the two cases where NEWTON'S rule will prove all the roots of the equation under consideration to be impossible. These criteria are $c_1=\varepsilon^4-\varepsilon\eta^3$, $c_2=\eta^4-\eta\varepsilon^3$. But

$$c_1\eta^4+c_2\varepsilon^4=q(2q^3-\mathrm{S})=q(2q^3-q^3-q^2-\sigma)=q(q^3-q^2-\sigma),$$

which for all values of q on the positive side of the line $p=-1$ (i. e. $q=0$) will have the same sign as $q^3-q^2-\sigma$, which we may call K[20]; and K positive will evidently imply that c_1, c_2 are one or both of them positive. The whole plane will be divided by the curve K into an *upper* region (commencing at $\sigma=\infty$), for which K is negative, and a lower region, in which K is positive. For any point of the curve K, $\sigma=q^3-q^2$, which within the limits of q with which we are concerned, viz. those within which Δ lies, is negative; for any point of the curve R, the smaller absolute value of σ is

$$-q^3-q^2+2q^{\frac{5}{2}}=q^3-q^2+2(q^{\frac{5}{2}}-q^3),$$

which $<q^3-q^2$ within the limits in question. So that, remembering that each of these values of σ is negative, we see that the portion of the area Δ corresponding to real values of ε, η will be completely above the curve K, *i. e.* in the negative region of K, and that accordingly Δ for *real values* of ε, η can never vanish when K is positive, as should be the case. This remark does not, however, apply to the conjugate region of Δ; for the curve K will *pass through*[21] the lower or conjugate portion of the area Δ.

(12) I may now say a few words on the signification of that portion of Δ in which ε and η are conjugate imaginary quantities.

---

of a quadratic and the extraction of 5th roots. To find the maxima and minima values of ε and η themselves exactly would lead to the solution of an equation of a degree quite unmanageable.

But we may first find the greatest maximum and least minimum values of S, i. e. $\varepsilon^5+\eta^5$, by making $\delta\sigma=(2q+3q^2)\delta q$ in $\delta\Delta=0$, which leads to an equation (I forget whether) of the 3rd or 5th degree (it is one of the two): calling this maximum and minimum m, μ respectively, and naming ρ (which of course must exceed unity) the greatest quotient of $\frac{\varepsilon}{\eta}$ or $\frac{\eta}{\varepsilon}$, we shall have

$$\sqrt[5]{\frac{\rho^5}{1+\rho^5}m}>\varepsilon\,;\ \eta>\sqrt[5]{\frac{1}{1+\rho^5}\mu}.$$

These limits will be tolerably near to the absolute maximum and minimum values of ε or η. It may be noticed that we know, from what has gone before, that ρ can never exceed $\left(\frac{1}{q}\right)^{\frac{1}{5}}$; and consequently ρ^5 cannot exceed 4, since q is always $>\frac{1}{4}$.

(20) I call K the Indicatrix, as exhibiting the joint effect of the *indicia* or criteria of the Rule.

(21) This may easily be verified; for at the point $p=-\frac{3}{4}$ it will be found that the ordinate in K and the lower ordinate in Δ are equal, and at the point $p=-\frac{9}{10}$ the lower ordinate in Δ is $-\frac{27}{2000}$, and in K is $-\frac{18}{2000}$; which shows that the curve K entering the area Δ when at the lower half of the curve, at a point where $p=-\frac{3}{4}$, must pass through its upper contour in order to cut the line $p=-\frac{9}{10}$ as it does above the point where Δ is touched by that line.

The curve K has its negative maximum at the point $q=\frac{2}{3}$, i. e. $p=-\frac{1}{3}$. It passes through the origin, and begins with sweeping under the curve Δ, which it enters exactly under the point where R quits Δ, and passes

In general, let

$$(a+i\alpha,\ b+i\beta,\ c+i\gamma, \ldots\ldots,\ c-i\gamma,\ b-i\beta,\ a-i\alpha \between x, y)^n=0$$

be an equation in which all the coefficients, reckoning simultaneously from the two ends, are conjugate to one another, and the central coefficient, if there is one, which can only be when n is even, *real.*

Let $\frac{x}{y}=p+iq$ satisfy this equation. Then evidently $\frac{y}{x}=p-iq$ will also satisfy it; or, which is the same thing, $\frac{x}{y}=\frac{p+iq}{p^2+q^2}$ will satisfy it.

Now either this root will be identical with the former one, or a distinct root; in the former case we must have $p^2+q^2=1$, and the root will be of the form $\cos\alpha+i\sin\alpha$; in the second case p^2+q^2 will differ from unity, and there will be a pair of imaginary roots of the form $\varrho(\cos\alpha+i\sin\alpha)$, $\frac{1}{\varrho}(\cos\alpha+i\sin\alpha)$, in which the real parts ϱ, $\frac{1}{\varrho}$ are reciprocal to one another, and the directive parts $e^{-i\alpha}$ identical. Moreover, if we write the given equation under the form $U+iV=0$, and suppose, as can always be done, that U and V have been divested of any algebraical common factor, it may easily be shown that the equation so prepared, and which may be called a Conjugate Equation *proper*, can have no real roots and no *pairs of imaginary roots* in the sense in which that term is employed in the theory of equations with real coefficients; but the distinction between *simple* or solitary and *twin* or associated roots reappears in the theory of conjugate equations, under a different form. It will of course be understood that the class of simple roots for which the modulus is unity is quite as general as that of twin roots, for each of which the modulus may be anything different from unity, just as in the ordinary theory the case of real is quite as general as that of imaginary roots, although the former may be represented by points on a fixed straight line, whilst the points representing the latter may be anywhere in the plane, this liberty of displacement being balanced, so to say, by the constraint of coupling. The general geometrical representation of the roots of a real equation is a system of points in a line, and a system of pairs of points at equal distances on opposite sides of the line. So the general geometrical representation of the roots of a conjugate equation will be system of points in the circumference of a circle to

---

through Δ at a point very close indeed to the horizontal extremity of Δ. It may be noticed that when $p=-\frac{3}{4}$, the smaller ordinates of R and Δ are each $-\frac{1}{64}$, the ordinate of K and the larger ordinate of Δ being each $-\frac{3}{64}$.

I have found the points of contact of K with Δ by actually substituting q^3-q^2, i. e. $p(p+1)^2$ for σ in $\Delta=0$. This gives the equation

$$2064p^4+7352p^3+9823p^2+5832p+1296=0,$$

one factor of which is $4p+3$, dividing out which we have

$$516p^3+1451p^2+1368p+432=0.$$

The Newtonian criterion applied to the three first coefficients of the above gives $-1362\frac{5}{9}$, showing that two of the roots are impossible; the remaining real root I find to be ·8946, &c. It does not appear to be a rational number.

radius unity, and of points situated in pairs in the same radii at reciprocal distances from the centre. In a word, in each case we may say that the roots can be geometrically represented by points on a circle, and pairs of points electrical images of each other in respect to the circle, but the radius of the circle in the one case will be infinity, in the other unity. Conjugate like real equations will have all their invariants of an even degree real, and those of an odd degree will be pure imaginaries, or real quantities affected with the multiplier i. Their morphological derivatives (covariants, contravariants, &c.) will be also conjugate forms. The whole doctrine of equations, as regards the separation of real from imaginary roots, and the determination of the limits within which the former lie, will reproduce itself with suitable modifications in the theory of conjugate equations, in which simple, on the one hand, and coupled or twin roots, on the other, will correspond respectively as analogues to the real and imaginary roots of the ordinary theory. Thus the following theorem may be demonstrated without difficulty, viz., in any conjugate equation the number of coupled roots is congruent to 0 in respect to the modulus 4 when the discriminant is positive, and to 2 in respect to the same modulus when the discriminant is negative(22). We see now how to interpret the

(22) (a) A very simple linear transformation shows the immediate connexion between the solitary and associated roots of conjugate with the real and paired imaginary roots of ordinary equations. For if $f(x, y)=0$ be a conjugate equation, writing

$$y=v+iu, \quad x=v-iu,$$

$f(x, y)$ becomes $F(u, v)$, a real form in u, v.

When u, v are real, we have

$$\frac{y}{x}=\frac{v+iu}{v-iu}=\cos\left(\tan^{-1}\frac{v}{u}\right)+i\sin\left(\tan^{-1}\frac{v}{u}\right);$$

when $\frac{v}{u}=c\pm i\gamma$, the two values correspond to

$$\frac{y}{x}=\frac{c+i\gamma+i}{c+i\gamma-i}, \quad \left(\frac{y}{x}\right)'=\frac{c-i\gamma+i}{c-i\gamma-i}.$$

Thus

$$\frac{y}{x}:\left(\frac{y}{x}\right)'::c^2+(\gamma+i)^2:c^2+(\gamma-i)^2;$$

also

$$\frac{y}{x}\times\left(\frac{y}{x}\right)'=\frac{c^2-1+\gamma^2+2ci}{c^2-1+\gamma^2-2ci},$$

of which the modulus is obviously unity.

(b) Now it is known that if t be the number of real, and τ of imaginary roots in the real form, $(u, v)^n$, its discriminant, bears the sign $(-)^{\frac{t(t-1)}{2}}$. Hence the sign of the discriminant of the conjugate form $(x, y)^n$ (since the determinant of $v+iu$, $v-iu$ is $2i$) will be $(-)^q$, where

$$q=\frac{n(n-1)}{2}+\frac{t(t-1)}{2}=\frac{(t+\tau)(t-1+\tau)+t(t-1)}{2}=t(t-1)+t\tau+\frac{\tau(\tau-1)}{2}.$$

Hence since τ and $t(t-1)$ are both even, $(-)^q=(-)^{\frac{\tau(\tau-1)}{2}}$, and the sign of the discriminant of a conjugate form is + or − according as the number of imaginary roots does or does not contain 4 as a factor.

It must be remembered that the sign of the discriminant is not in general the same as that of the *zeta* or squared product of differences of the roots. The sign of the *zeta* for real equations follows precisely the same law as the sign of the *discriminant* for conjugate ones.

effect of the variable point whose coordinates are $\epsilon^5+\eta^5$ and $\epsilon\eta$ lying within the area Δ, in that portion of it for which ϵ, η became imaginary; viz. it is that in such case the equation (ϵ, η), which then becomes of a conjugate form, will have three simple and two twin roots; and thus the unity of the interpretation is restored if we choose, as we very well may, to extend the use of these terms to the real roots and the paired imaginary roots of ordinary equations. We may neglect the curve of reality R altogether, and affirm that all over the area Δ, ϵ, η will have such values as will give rise to three simple and two coupled roots.

(13) That part of the theorem of NEWTON which had received a demonstration from MACLAURIN and CAMPBELL in the generalized form in which I have enunciated it in this paper, may be easily extended to the case of conjugate equations. It will, as applied to them, read thus: If the $(n-1)$ quadratic derivatives of a conjugate form of the nth degree, all whose roots are simple, be multiplied respectively by the coefficients of any other conjugate form, all whose roots are also *simple*, of the degree $(n-2)$, and the sum of these products be taken as a new quadratic form, the discriminant of this latter must be positive, or, which is the same thing, its determinant must be negative.

(14) So much for the case of $n=5$. If we were to proceed to the consideration of equations of the 6th degree, *two* cases of resistance would present themselves in the demonstration of NEWTON's rule, viz. one in which the signs of the criteria are $-+++-$, the other $-+-+-$. In the latter it would only be necessary to show that the discriminant is necessarily negative, since we know from the derivatives that the equation must have four imaginary roots, and the choice would lie between the alternatives of there being four or six. In the former case the derivatives only indicate the necessary existence of two real roots, and it would become requisite to prove that there must be four or six—an alternative which depends not on the sign of one function of the coefficients, but on the nature of the signs of two such functions given by STURM's or any equivalent theorem. It would thus become requisite to prove that two functions of the coefficients, say L, M, could not *both* be negative; and this might be shown by demonstrating the existence of two quantities, L′, M′, other functions of the coefficients incapable of assuming any but the positive sign such that L′L+M′M would be necessarily positive.

PART II.—ON THE LIMIT TO THE NUMBER OF REAL ROOTS IN EQUATIONS OF THE FORM $\Sigma(ax+b)^n$.

(15) I shall now proceed to the consideration of a theorem relating to a particular class of ordinary equations, which occurred to me in the course of and in connexion with the preceding investigations. The theorem itself, but unaccompanied by proof, has appeared in the 'Comptes Rendus' of the Academy for the month of March 1864.

Both as regards its nature and the processes involved in the proof, it stands in close relation to NEWTON's rule, my study of which in fact led me to its discovery. It will therefore take its place most appropriately in this paper.

Certain preliminary properties of circulation introducing some new notions of polarity must be first established, by way of Lemmas to the proof in question.

By a *type* let us understand a succession of symbols of any subject matter whatever susceptible of receiving the signs $+-$, or any suchlike indications of opposite polarity.

Let $a, b, c, \ldots l, k, l$ be any such type, where the *elements* $a, b, c, \ldots$ may be regarded either as points in a line or rays in a pencil affected respectively with the signs of $+$ and $-$.

Then by a *per-rotatory* circulation of such type, I mean the act of passing from the first element to the second, from the second to the third, &c., from the last but one to the last, and from the last to the first.

By a *trans-rotatory* circulation of the same, I mean the act of passing from the first to the second, the second to the third, &c., from the last but one to the last, and from the last to the first, *with its sign reversed.*

A type considered subject to per-rotatory circulation may be termed a Per-rotatory Type; one subject to the other sort of circulation, a Trans-rotatory Type.

If a, b, c, d, e be a per-rotatory type, its direct *phases* are

$$\begin{array}{ccccc}
a, & b, & c, & d, & e, \\
b, & c, & d, & e, & a, \\
c, & d, & e, & a, & b, \\
d, & e, & a, & b, & c, \\
e, & a, & b, & c, & d,
\end{array}$$

and its retrograde phases

$$\begin{array}{ccccc}
a, & e, & d, & c, & b, \\
e, & d, & c, & b, & a, \\
d, & c, & b, & a, & e, \\
c, & b, & a, & e, & d, \\
b, & a, & e, & d, & c.
\end{array}$$

If, on the other hand, a, b, c, d, e be a trans-rotatory type, its direct *phases* will be

$$\begin{array}{ccccc}
a, & b, & c, & d, & \bar{e}, \\
b, & c, & d, & \bar{e}, & \bar{a}, \\
c, & d, & \bar{e}, & \bar{a}, & b, \\
d, & \bar{e}, & \bar{a}, & \bar{b}, & \bar{c}, \\
\bar{e}, & \bar{a}, & \bar{b}, & \bar{c}, & \bar{d},
\end{array}$$

and its retrograde phases

$$\begin{array}{ccccc}
a, & \bar{e}, & \bar{d}, & \bar{c}, & \bar{b}, \\
\bar{e}, & \bar{d}, & \bar{c}, & \bar{b}, & \bar{a}, \\
\bar{d}, & \bar{c}, & \bar{b}, & \bar{a}, & e, \\
\bar{c}, & \bar{b}, & \bar{a}, & e, & d, \\
\bar{b}, & \bar{a}, & e, & d, & c,
\end{array}$$

where the sign $(-)$ is, for greater convenience of writing, placed over instead of before the elements which it affects; and so on in general a type of n elements, whether per-rotatory or trans-rotatory, will admit of n direct and n retrograde phases.

If we count the number of variations of sign in the circulations of any phase of a per-rotatory type, this number will be the same for all the phases, and will be an even number; this even number may be termed the variation-index of the type.

So, again, if whatever be the original signs of the element in a trans-rotatory type, we count the number of variations in the circulation of any of its phases, this number also will be constant and will be odd, and this odd number may then be termed the variation-index of the type.

(16) Let any phase be taken of a per-rotatory type, and out of such phase let any element be *suppressed*; then we obtain a type one degree lower in the elements, which, if we please, we may consider as a trans-rotatory type, and such trans-rotatory type may be termed a derivative of the original per-rotatory one.

In like manner any phase being taken of a trans-rotatory type, one element may be suppressed, and the reduced type treated as a per-rotatory one, and termed a derivative of the original trans-rotatory one.

We may now enunciate the following important general proposition, viz.

Any trans-rotatory type or any per-rotatory type whose variation-index is different from zero being given, a per-rotatory derivative of the one and a trans-rotatory derivative of the other may be found such that the variation-index of the derived types in either case shall be less by a unit than the variation-index of the types from which they are derived.

Case (1). Let the given type be per-rotatory. Then by hypothesis, since it has some variations, we may find a phase of it beginning with $+$ and ending with $-$, by which I mean beginning with an element that is positive and ending with one that is negative. This gives rise to two sub-cases.

$$\text{T, the phase in question, will be } +\ldots\ldots+-$$
$$\Theta\text{, the phase in question, will be } +\ldots\ldots--.$$

In either sub-case let the last sign be suppressed, and the result treated as a trans-rotatory type; then T, Θ become respectively T′, Θ', where

$$\mathrm{T}' \text{ is } +\ldots\ldots+$$

and

$$\Theta' \text{ is } +\ldots\ldots-$$

and evidently the variation-index of T $-$ variation-index of T′$=$ number of changes of sign in $+-+$ less changes of sign in $+-=2-1=1$; and again variation-index of $\Theta-$ variation-index of $\Theta'=$ number of changes of sign in $--+$ less changes of sign in $--=1-0=1$. Hence the theorem is proved for the case where the given type is per-rotatory.

Case (2). Let the given type be *trans-rotatory*.

Then, again, there must either be a phase of the form P, or one of the form Φ, where

P represents a *continual succession* of signs of the same name as $+ + \ldots +$ or $- - \ldots -$, and Φ represents a succession beginning with one sign as $+$ and ending with one or more signs $-$, or else beginning with $-$ and ending with a succession of signs $+$. Essentially, then, as a change of signs throughout a whole succession does not affect the variation-index, we may suppose

$$P=+\ldots\ldots++,$$
$$\Phi=-\ldots-+\ldots+,$$

the signs intervening between the two expressed signs $-$ in Φ being filled up in any manner whatever, and those between the two signs $+$ with signs exclusively $+$.

Let now that phase of Φ be taken which commences with the first sign of the final succession of $+$. Then Φ becomes

$$(\Phi)=+\ldots++\ldots.+,$$

which is of the form

$$+\ldots..++,$$

so that P is only a particular case of (Φ). If the last sign in (Φ) be suppressed and the result treated as a per-rotatory type be called $(\Phi)'$, so that $(\Phi)'=+\ldots\ldots+$, we have variation-index in $(\Phi)-$ variation-index in $(\Phi)'=$ changes of sign in $-+$ less changes of sign in $++=1-0=1$.

Hence the proposition is established for both cases.

(17) The theorem to which this Lemma-proposition is to be applied concerns equations of the form

$$\varepsilon_1 u_1^m+\varepsilon_2 u_2^m+0\ldots+\varepsilon_n u_n^m=0,$$

where $u_1, u_2, \ldots, u_n$ are any linear functions of x, y; m is any positive integer, and $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ are each respectively and separately, either *plus unity* or *minus unity*.

Such an equation for convenience of reference may be termed a superlinear equation, and the function equated to zero a superlinear function.

Every superlinear function may be conceived as having attached to it a pencil of rays constructed in a manner about to be explained.

1. We may conceive the function to be prepared in such a manner, that supposing $ax+by$ to be any one of the n linear elements u, every b shall be positive. If m is even, this can be effected by writing when required for $ax+by$, $-ax-by$ without further change. If m is odd, we may write when required $-ax-by$ in place of $ax+by$, changing at the same time the factor ε, which appertains to $(ax-by)^m$ from $+1$ to -1, or *vice versâ*, from -1 to $+1$.

Now take in a plane any two axes of coordinates $O\xi$, $O\eta$, and consider a, b as the ξ and η coordinates of a point. All the n points thus obtained, on account of every b being positive, will lie on the same side of the axis $O\eta$, and thus the entire n linear functions will be represented by a pencil of n rays, the two extreme rays of which make an angle less than two right angles with each other; but each term of the superlinear function contains, besides $(ax+by)^m$, a definite multiple $+1$, or -1, and we must accordingly, to

completely express such term, conceive every ray affected with a distinct sign $+$ or $-$. A pencil thus drawn with its rays so polarized will give a complete representation of any given superlinear function, and may be called its type-pencil([23]).

I am now able to state the following proposition:

(18) *The number of real roots in a superlinear equation cannot exceed the variation-index of its type pencil, regarded as a per-rotatory type, if the degree of the equation be even, and as a trans-rotatory type if the degree of the equation be odd.* I prove this inductively as follows.

1. Suppose the theorem to be true when the variation-index of the type-pencil is not greater than the even number ν, and consider an equation of the odd degree $(2i+1)$, for which the type-pencil viewed as trans-rotatory has the variation-index $\nu+1$.

Let a *phase* of this type be taken, say corresponding to the rays $\rho_n, \rho_{n-1} \ldots \rho_2, \rho_1$, such that the per-rotatory type obtained by striking out the term ρ_1 has the variation-index ν (as we know may be done by virtue of the Lemma).

Take for new axes $O\xi'$, $O\eta'$, when $O\xi'$ coincides with ρ_1; then it is clear that the pencil $\rho_n, \rho_{n-1} \ldots \rho_2, \rho_1$ will still serve as a type-pencil to the given function, the only change being that some of the rays, namely those that did lie on one side of ρ_1, have been inverted in direction and changed in sign (corresponding to a change in the coefficient a, b, accompanied with a change in the sign of the corresponding ε), whilst the rays on the other side of ρ_1 have been left unaltered.

The points (a_1, b_1), $(a_2, b_2) \ldots (a_n, b_n)$ corresponding to the rays $\rho_1, \rho_2, \ldots \rho_n$ will, with respect to the new axes, change their values, becoming converted into $(\alpha_1, 0)$, (α_2, β_2), $(\alpha_3, \beta_3), \ldots (\alpha_n, \beta_n)$, where $\beta_2, \beta_3, \ldots \beta_n$ will still all be positive, the angle between ρ_1 and ρ_n being the same as between the two extreme rays in the original figure of the type-pencil, and the superlinear equation may now be written in the form

$$F(u, v)=\varepsilon_1(\alpha_1 u)^{2i+1}+\varepsilon_2(\alpha_2 u+\beta_2 v)^{2i+1}+\varepsilon_3(\alpha_3 u+\beta_3 v)^{2i+1}+\varepsilon_n(\alpha_n u+\beta_n v)^{2i+1}=0,$$

where u, v are real linear functions of x, y.

([23]) Let a circle be imagined pierced by a pencil containing any number of rays protracted in both directions, say in the opposite points a, α; b, β; c, γ; d, δ; and let these points, taken in order of natural succession from left to right, or right to left, be $a, b, c, d, \alpha, \beta, \gamma, \delta$. Then, commencing with any point c, a *complete* circulation will be represented by the succession of transits

$$c \text{ to } d, \quad d \text{ to } \alpha, \quad \alpha \text{ to } \beta, \quad \beta \text{ to } \gamma, \quad \gamma \text{ to } \delta, \quad \delta \text{ to } a, \quad a \text{ to } b, \quad b \text{ to } c.$$

But whether $\alpha, \beta, \gamma, \delta$ bear respectively the same signs or signs contrary to those of a, b, c, d, the transit between any two points β to γ will be of the same nature, as regards continuance or change of sign, as the transit from b to c, and thus we see that the complete cycle or total revolution above indicated is only a reduplication of, and may be fully designated by the hemicyclic succession c to d, d to α, α to β, β to γ, for which the number of variations therefore will be the same as for any similar succession obtained by commencing with any other element in the original system of points instead of c. If the opposite points bear like signs, the above succession of transits may be indicated by the order c, d, a, b, c; if they bear contrary signs by the order $c, d, \bar{a}, \bar{b}, \bar{c}$, and thus it is that the idea arises of the two kinds of so-called circulation, but which are in fact only more or less disguised species of semicirculation.

Let the derivative of this function be taken in regard to v, and we have

$$\frac{1}{2i+1}\mathrm{F}'(u,v)=\beta_2\varepsilon_2(\alpha_2u+\beta_2v)^{2i}+\beta_3\varepsilon_3(\alpha_3u+\beta_3v)^{2i}\ldots+\beta_n\varepsilon_n(\alpha_nu+\beta_nv)^{2i},$$

where $\beta_2\varepsilon_2$, $\beta_3\varepsilon_3\ldots\beta_n\varepsilon_n$ have the same signs as $\varepsilon_2, \varepsilon_3, \ldots \varepsilon_n$ respectively.

Now the pencil-type of $\mathrm{F}'(u, v)$ will be the per-rotatory type $\varrho_n, \varrho_{n-1}, \ldots \varrho_2$, of which by construction the variation-index is ν. Hence by hypothesis $\mathrm{F}'(u, v)$ has not more than ν real roots, i. e. at least $2i-\nu$ imaginary roots. Hence $\mathrm{F}(u, v)$ has at least that number of imaginary roots, i. e. at most $(2i+1)-(2i-\nu)$, i. e. $\nu+1$ real roots. Hence if the theorem is true for ν an even number, it is true for $\nu+1$.

In like manner let us proceed to show that when it is true for ν an odd number, it would remain true for $\nu+1$.

The reasoning will be precisely similar to that followed in the antecedent case. We must find a phase of the *per-rotatory* type $\varrho_n, \varrho_{n-1}, \ldots \varrho_2, \varrho_1$ having the variation-index ν such that the trans-rotatory reduced type $\varrho_n, \varrho_{n-1}, \ldots \varrho_2$ shall have the variation-index $\nu-1$; the new pencil will still continue to be a type-pencil of the given superlinear function, the change of direction in the bunch of rays one on side of ϱ_1 being now unaccompanied with change of sign, such change corresponding to $\varepsilon(ax+by)^{2i}$ becoming changed into $\varepsilon(-ax-by)^{2i}$ without ε undergoing a change of sign.

As before, the axes of coordinates are transformed from ξ, η into ξ', η', and we obtain

$$\mathrm{F}(u, v)=\varepsilon_1(\alpha_1u)^{2i}+\varepsilon_2(\alpha_2u+\beta_2v)^{2i}+\ldots+\varepsilon_n(\alpha_nu+\beta_nv)^{2n+1},$$

$$\frac{1}{2i}\mathrm{F}'(u, v)=\beta_2\varepsilon_2(\alpha_2u+\beta_2v)^{2i-1}+\ldots+\beta_n\varepsilon_n(\alpha_nu+\beta_nv)^{2i}.$$

for which the type-pencil is the trans-rotatory type $\varrho_n, \varrho_{n-1}, \ldots \varrho_2$, of which by construction the variation-index is $\nu-1$, so that its number of imaginary roots is $2i-(\nu-1)$, and consequently the number of real roots of $\mathrm{F}(u, v)$ will be $\nu+1$.

Thus, then, if the theorem be true for ν, whether ν be even or odd, it will be true for $\nu+1$.

But when $\nu=0$, the superlinear function becomes a sum of even powers of linear functions of x, y, all taken with the same sign, of which the number of roots is evidently 0. Hence, being true for this case, the proposition is true universally.

It will be noticed that the algebraical part (as distinguished from the purely polar-tactic part of the above demonstration) depends on the same principle of which such abundant use has been made in the former part of this dissertation, viz. that the number of imaginary roots in any ordinary algebraical equation in x cannot be increased when we operate any homographic substitution upon x, and take the derivative of the equation thus transformed in lieu of the original([24]).

([24]) For greater clearness I present in an inverted order of arrangement a summary of the foregoing argument.

By an ith derivative of $f(x, y)$ is meant any derived form

$$\left(\lambda_1\frac{d}{dx}+\mu_1\frac{d}{dy}\right)\left(\lambda_2\frac{d}{dx}+\mu_2\frac{d}{dy}\right)\ldots\left(\lambda_i\frac{d}{dx}+\mu_i\frac{d}{dy}\right)f(x, y),$$

(19) The proposition above established leads immediately to the theorem and corollary following, viz.

THEOREM. If $c_1, c_2, \dots c_n$ be a series of ascending or descending magnitudes, and m any positive integer, the equation

$$\lambda_1(x+c_1)^m+\lambda_2(x+c_2)^m+\dots+\lambda_n(x+c_n)^m=0$$

cannot have more real roots than there are changes of sign in the sequence $\lambda_1, \lambda_2, \dots \lambda_n, (-)^m\lambda_1$.

For obviously $(1, c_1)$, $(1, c_2), \dots (1, c_n)$ will be points corresponding to rays within a semirevolution, and therefore forming a type-pencil.

Corollary. If the above equation be transformed by any real homographic substitution into the form

$$\mu_1(y+\gamma_1)^m+\mu_2(y+\gamma_2)^m+\dots+\mu_n(y+\gamma_n)^m=0,$$

where $\gamma_1, \gamma_2, \dots \gamma_n$ are taken in ascending or descending order, the number of changes of sign in the series $\mu_1, \mu_2, \dots \mu_n, (-)^m\mu$ is *invariable*(25); for the effect of any such formation will be to leave the type-pencil unaltered except in its *phase*.

(20) If we look to the undeveloped form of the superlinear function

$$S=\varepsilon_1u_1^m+\varepsilon_2u_2^m+\dots+\varepsilon_nu_n^m,$$

and are supposed to possess no knowledge of the coefficients which enter into the linear elements u, we may still draw some general inferences as to the limit of the number of real roots in $S=0$. Thus if the number of positive units ε is j, and of the negative units k, and j is not greater than k, it is obvious that, whatever may be the form of the type-pencil to S, its variation-index cannot be more than $2j$ when m is even, nor more than $2j+1$ when m is odd; for the arrangement the most favourable to the largeness of the number of the real roots is that where every two rays with the signs belong-

the λ, μ quantities being any real quantities whatever. Then I say—

1. If T is the type-pencil (per-rotatory or trans-rotatory) of any superlinear form F, every derivative of T of the contrary name is the type-pencil of some first derivative of F, as shown in art. (18).

2. A derivative of T of contrary name may be found such that its variation-index shall be less by a unit than that of T itself, as shown in art. (16).

3. Hence if i is the variation-index of the type-pencil of F, an ith derivative of F may be found such that its variation-index shall be zero, and consequently having no real roots.

Hence, finally, since the number of real roots of any rational integral homogeneous function in x, y cannot exceed by more than i the number of the real roots in any of its ith derivatives, F cannot have more real roots than there are units in the variation-index of its type-pencil.

The subtle point of the argument, it will be noticed, lies in forming the conception of the variation-index to a trans-rotatory pencil, in which the singular phenomenon occurs of a reversal of *relative polarity* in passing from the last ray to the first, whereas in a per-rotatory pencil any ray indifferently may be regarded as the initial ray, no such reversal in that case taking place.

(25) It may be noticed that, contrariwise, the limit to the number of real roots given by NEWTON's criteria is *not* an invariant; it fluctuates with the homographic transformations operated upon the equation; and a question suggests itself as to the maximum value the number of imaginaries indicated by the rule can attain. I presume this maximum is not in all cases necessarily the actual number of the imaginary roots possessed by the equation.

ing to the j group of ε are separated by one or more of the rays with a contrary sign to themselves. Thus it appears that when only the units $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ are given, we may impose a maximum upon the number of real roots in the superlinear equation; this limit may be called the *absolute maximum*, being the double of the inferior number of like signs in the series $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ when the degree is even, and one more than such double when the degree is odd([26]).

The *specific maximum*, on the other hand, will depend on the form of the type-pencil, and cannot be ascertained until the coefficients of the linear elements are given. It can never exceed, but may be less than the absolute maximum. It may, indeed, be easily proved that *in general* the specific maximum will be less than the absolute maximum. Thus, by way of example, suppose the degree to be even, and the inferior number of like signs to be 2; the absolute maximum number of real roots will be four, but the specific maximum will more generally be only two. For let the number of linear terms in the superlinear function be $2+n$, n being 2 or any greater number; and first, to fix the ideas, suppose $n=2$. The type-pencil, which is to be read per-rotatorily, consists of four rays, say a, b, c, d, following each other in uninterrupted circular order, of which two are to bear positive and two negative signs. If the two negative signs fall on a, c or on b, d, the variation-index will be 4, but in the other four cases of incidence such index will be only 2. Consequently the chance is 2 to 1([27]) that the specific maximum, which may be 4, is not greater than 2; and consequently the chance that there will be four real roots in the equation will be only a chance (too difficult to be calculated, but which is a function of the degree of the equation) *of* the chance $\frac{1}{3}$ that there will be as many as four real roots in the equation $u_1^n+u_2^n-u_3^n-u_4^n=0$, where u_1, u_2, u_3, u_4 are

([26]) (*) If a superlinear form of an odd degree contains an odd number of terms, say $2k+1$, the greatest value of the *inferior* number of like signs is k, and the extreme limit to the number of real roots will be $2k+1$.

If it contain an even number of terms, say $2k$, the greatest value of the inferior index is k; but for this particular case it will readily be seen that a limit may be assigned to the variation-index closer than that given by the rule in the text; in fact the variation-index cannot in that case exceed $2k-1$, which will therefore be the extreme limit to the number of real roots. Now suppose the canonizant of an odd-degreed function of x, y to have all its roots real, then it may be expressed by a superlinear form of which the number of terms will be $2i+1$ or $2i$, according as the degree is $4i+1$ or $4i-1$. In the one case the number of real roots cannot exceed $2i+1$, in the other $2i-1$. Hence the following somewhat curious theorem:

(*) *If the canonizant of an odd-degreed quantic in* x, y, *of the degree* $4i\pm1$, *has* no *imaginary roots, the quantic itself must have at least* i *pairs of imaginary roots.* From the fact that when the roots of the canonizant of a quintic are all real there must be one pair at least of imaginary roots, we can infer that when the discriminant of a quintic is positive and that of its canonizant is negative, the equation has one real and four imaginary roots. This observation has led to a long train of reflections, which will be found embodied in the 3rd part of the memoir.

([27]) This, in fact, is identical in substance with the noted problem of determining the chance that two straight lines drawn on a black board will cross. Mr. Cayley, of whom it may be so truly said, whether the matter he takes in hand be great or small, "nihil tetigit quod non ornavit," suggests the following independent proof of this. Taking unity as the length of the contour, fixing the extremity of one of the lines, and calling s the distance of its other end from it measured on the contour, the chance of the second line crossing this is easily seen to be $2s(1-s)$, which, integrated between $s=0$, $s=1$, gives $\frac{1}{3}$, as before obtained.

unknown linear functions of x: thus we are entitled to say that *in general* the number of real roots in such an equation is *not* the maximum four, but a less number. This remark is of importance, as showing that on this subject it is possible to speak with scientific certainty, and on other than empirical grounds, of what may *in general* be expected to take place. Thus we find NEWTON declaring twice over in the chapter quoted, that *in general* his rule will give not merely the maximum, but the actual number of the imaginary roots in an equation. I am strongly inclined to doubt the truth of this assertion; but it is important to be satisfied by analogy that such an assertion may rest on a scientific and demonstrative basis, and not on the utterly fallacious foundation of arithmetical empiricism([28]).

([28]) A few additional words on this question of probability may not be unacceptable. In order to meet the case of the degree of the superlinear form or equation being odd as well as even, let it be supposed known under the form

$$\Sigma_n^1\lambda_i(x+c_i)^m,$$

the values of the quantities c_i being supposed to be left wholly indeterminate, and only the signs of the quantities λ to be given. Let ω be the inferior number of like signs in the λ series, meaning thereby that the number of signs of one sort is ω, and of the other sort ω, or more than ω.

Let the probability of the specific maximum of real roots being $2k$ when m is even, be represented by p_{2k}, and of its being $2k+1$ when m is odd by π_{2k+1}; also let s_{2k}, σ_{2k+1} represent the number of cases when ω and n are given which correspond to the specific maximum being $2k$, $2k+1$ respectively. Suppose $\omega=1$, then obviously, when m is even, we have $s_2=n, p_2=1$. But when n is odd $\sigma_1=2$ (for when either extreme element alone is negative the trans-rotatory cycle has the variation-index unity), and $\sigma_3=n-2$, so that

$$\pi_1=\frac{2}{n},\quad \pi_3=\frac{n-2}{n}.$$

Again, suppose $\omega=2$, m being even; then obviously s_2 is the number of contiguous duads in a cycle of n elements, and s_4 is the remaining number of duads; hence

$$s_2=n,\ s_4=n\frac{n-1}{2}-n=n\frac{n-3}{2};$$

so that

$$p_2=\frac{2}{n-1},\quad p_4=\frac{n-3}{n-1}.$$

2nd. Suppose $\omega=2$, m being odd, so that σ_1, σ_3, σ_5 will have to be separately estimated. To fix the ideas, let the λ series be termed a, b, c, d, e, f, g, in which two of the elements are supposed of one sign, say negative, and the rest of the opposite sign, say positive; then the only dispositions of sign which correspond to the specific maximum being 1 are those in which a, b or else f, g are both negative. Hence $\sigma_1=2$. Again, the dispositions of sign which make the specific maximum equal to 3 are those in which a, g are both negative, those in which a and c, d, e, or f are negative, those in which g and e, d, c, or b are negative, and, finally, those in which any two contiguous elements except the a and g are negative. Hence $\sigma_3=1+2(n-3)+(n-3)=3n-8$; and it should be observed that this result cannot be prejudiced in its generality by the supposition of any of the components of σ_3 becoming negative, since $\omega=2$ implies that n is at least 4. Hence, finally,

$$\sigma_5=\frac{n^2-n}{2}-(3n-8)-2=\frac{n^2-7n+12}{2}=\frac{(n-3)(n-4)}{2};$$

so that

$$\pi_1=\frac{4}{n^2-n},\quad \pi_3=\frac{6n-20}{n^2-n},\quad \pi_5=\frac{n^2-7n+16}{n^2-n}.$$

This example serves to show how much more difficult is the computation of the respective probabilities when m is odd than when m is even, owing to the break of continuity in the cycle of readings on passing from the last to the first term.

And the total number of arrangements, which is the number of ways in which μ things can be distributed over $(\mu+\nu)$ places, is $\frac{\pi(\mu+\nu)}{\pi\mu\pi\nu}$. Hence we obtain

$$[\mu, \nu, g]=\frac{\pi\mu\pi\nu}{\pi(\mu+\nu-1)}\left\{\frac{(\mu-1)(\mu-2)..(\mu-g+1)\times(\nu-1)(\nu-2)..(\nu-g+1)}{1.2.(g-1)(1.2..g)}\right\}$$

$$=\frac{\pi\mu\pi(\mu-1)\pi\nu\pi(\nu-1)}{\pi g\pi(g-1)\pi(\mu-g)\pi(\nu-g)\pi(\mu+\nu-1)}.$$

[Throughout these investigations $\pi(x)$ is used in the same sense as Πx, to signify the factorial $1.2.3\ldots x$.]

If there should appear any obscurity in the statement of the method by which has been obtained the number of distinct distributions of the μ, ν elements into g groups of each, the reader is referred to the equation in differences obtained further on in this Note, by which all doubt of the correctness of the result will be removed.

(22) For a *trans-rotatory* pencil of rays, to ascertain the probability of the variation-index being $2g+1$.

Imagine a circular arrangement of μ positive elements and ν negative elements containing 2γ variations.

Let this circle be supposed opened out at any point and the variations of the open pencil so formed to be reckoned according to the trans-rotatory law, which is that in passing from one extremity to the other a change is to be seen as a variation, and a variation as a change. If the break is made between two negative or between two positive elements, the number of variations obviously becomes *increased* by one unit; but if between a positive and a negative element, that number becomes decreased by one unit. The number of these latter intervals is 2γ, and of the former $\mu+\nu-2\gamma$.

Hence the probability of the index becoming $2\gamma+1$ is $\frac{\mu+\nu-2\gamma}{\mu+\nu}$, and of its becoming $2\gamma-1$ is $\frac{2\gamma}{\mu+\nu}$.

If, then, we denote the probability to be calculated by $[\mu, \nu, g+\frac{1}{2}]$, it is obvious that we shall have

$$[\mu, \nu, g+\tfrac{1}{2}]=\frac{\mu+\nu-2g}{\mu+\nu}[\mu, \nu, g]+\frac{2(g+1)}{\mu+\nu}[\mu, \nu, g+1].$$

But by the formula previously obtained it will easily be seen that

$$[\mu, \nu, g+1]=\frac{(\mu-g)(\nu-g)}{g(g+1)}[\mu, \nu, g].$$

Hence

$$[\mu, \nu, g+\tfrac{1}{2}]=\frac{[\mu, \nu, g]}{\mu+\nu}\left\{(\mu+\nu-2g)+\frac{2(\mu-g)(\nu-g)}{g}\right\}$$

$$=\left(\frac{2\mu\nu}{g(\mu+\nu)}-1\right)[\mu, \nu, g] \quad \ldots\ldots\ldots\ldots \quad (*)$$

$$=2\frac{(\pi\mu)^2(\pi\nu)^2}{\pi(g-1)\pi(g+1)\pi(\mu+\nu)\pi(\mu-g)\pi(\nu-g)}-\frac{\pi\mu\pi(\mu-1)\pi\nu\pi(\nu-1)}{\pi(g-1)\pi g\pi(\mu+\nu-1)\pi(\mu-g)\pi(\nu-g)}.$$

The integral must satisfy the further condition that $[\mu, 1, g]$ shall be unity when g is 1, and zero for all values of g greater than 1.

Assume the value of $[\mu, 1, g]$ obtained by the method given in art. (21). This obviously satisfies the initial conditions corresponding to $g=1$. Moreover we may easily deduce from it the equalities

$$[\mu, \nu-1, g-1]=\frac{(g-1)g}{(\mu-g+1)(\nu-g)}[\mu, \nu-1, g], \text{ and } [\mu, \nu, g]=\frac{(\nu-1)\nu}{(\mu+\nu-1)(\nu-g)}[\mu, \nu-1, g].$$

Hence the equation in differences will be satisfied if it be true that

$$\frac{(\nu-1)\nu}{\nu-g}=(\nu-1+g)+\frac{(g-1)g}{\nu-g},$$

which is obviously the case, since $\nu^2-\nu-g^2-g=(\nu-g)(\nu+g-1)$.

Since, then, the assumed value of $[\mu, \nu, g]$ is correctly determined when $\nu=1$, it is obvious, from the form of the equation, that it holds good for all other values of ν, as was to be shown.

(26) From the equation

$$\frac{[\mu, \nu, g+1]}{[\mu, \nu, g]}=\frac{(\mu-g)(\nu-g)}{g(g+1)}$$

making $(\mu-g)(\nu-g)=g(g+1)$ or $g=\frac{\mu\nu}{\mu+\nu+1}$, we may readily infer that the value of g for which the probability $[\mu, \nu, g]$ is greatest is the integer part of $\frac{\mu\nu}{\mu+\nu+1}$, if that quantity is non-integer, or the quantity itself and the number next below it (indifferently) if it is an integer.

(27) If we apply a similar method to $[\mu, \nu, g+\frac{1}{2}]$, we obtain by aid of the formula above given,

$$\frac{[\mu, \nu, g+\frac{1}{2}]}{[\mu, \nu, g-\frac{1}{2}]}=\frac{2\mu\nu-(\mu+\nu)\gamma}{2\mu\nu+\mu+\nu-(\mu+\nu)\gamma}\cdot\frac{(\mu+1)-\nu(\nu+1-\gamma)}{\gamma^2};$$

and equating this ratio to unity, we obtain

$$\frac{2\mu\nu-(\mu+\nu)\gamma}{2\mu\nu+\mu+\nu-(\mu+\nu)\gamma}=\frac{\gamma^2}{(\mu+1)(\nu+1)-(\mu+\nu+2)\gamma};$$

or writing $\mu+\nu=p$, $\mu\nu=q$,

$$(p^2+p)\gamma^2-(3pq+4q+p^2+p)\gamma+2q(q+p+1)=0.$$

The roots of this equation will be both of them real, for its *determinant* is

$$p^2q^2+16pq^2+16q^2+(p^2+p^3)(\mu^2+\nu^2),$$

which is necessarily positive. Hence it follows that there are two positive roots of the equation. Whether there will exist values of g which give actual maxima or minima values, or one and the other to $[\mu, \nu, g+\frac{1}{2}]$, depends on the further condition being satisfied that the values of g in the above equation shall come out, one or both of them, not greater than either of the two numbers μ, ν. The inquiry connected with the satisfaction of this condition may be conducted by means of repeated applications of the

(30) Moreover, we thus see that the average number of variations in an open line with μ positive and ν negative signs, which is

$$\Sigma(2g-1)[\mu, \nu, g-\tfrac{1}{2}]+\Sigma 2g[\mu, \nu, g],$$

or

$$\Sigma 2g\big([\mu, \nu, g-\tfrac{1}{2}]+[\mu, \nu, g]\big)-\Sigma[\mu, \nu. g-\tfrac{1}{2}]\big)$$

will be equal to

$$\Sigma 2g[\mu, \nu, g]-\Sigma\frac{2g}{\mu+\nu}[\mu, \nu, g]=\frac{\mu+\nu-1}{\mu+\nu}\Sigma 2g[\mu, \nu, g]=\frac{\mu+\nu-1}{\mu+\nu}\cdot\frac{2\mu\nu}{\mu+\nu-1}=\frac{2\mu\nu}{\mu+\nu}.$$

The total number of variations and continuations together is $\mu+\nu-1$. Hence the difference between the two is $\frac{4\mu\nu}{\mu+\nu}-(\mu+\nu-1)$, or $\frac{(\mu+\nu)-(\mu-\nu)^2}{\mu+\nu}$; so that the average number of variations is greater than, equal to, or less than that of the continuations, according as the difference between the numbers of the two sets is less than, equal to, or greater than the square root of the entire number of signs. Obviously the average should be the same for the variations as for the continuations if the number of signs, say $n+1$, is given, and each is supposed equally likely to be positive or negative. This is easily verified; for multiplying the probable value of each distribution of signs by the probable value of the number of variations corresponding thereto, we obtain the series

$$\frac{1}{(n+1)2^n}\left\{1.n.(n+1)+2(n-1)(n+1)\frac{n}{2}+3(n-2)\frac{(n+1)n.(n-1)}{1.2.3}+\dots\right\}=\frac{n(n+1)2^{n-1}}{(n+1)2^n}=\frac{n}{2}.$$

This is the final average of the number of variations of sign, and will be equal to that of the continuations, since the entire number of the two together is n.

Received October 27, 1864.

PART III.—ON THE NATURE OF THE ROOTS OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

(31) In a foot-note, Part II. of this memoir, I have shown that when the discriminant of the canonizant (constituting an invariant of the twelfth order) of an equation of the fifth degree bears a particular sign, the character of the roots becomes completely determined by the sign of the discriminant of that equation.

This has naturally led me to investigate *de novo* the whole question of the character of the roots of an equation of that degree; and I have succeeded in obtaining under a form of striking and unexpected simplicity the invariantive criteria which serve to ascertain in all cases the nature of the equation as regards the number of real and imaginary roots which it contains; then passing to the expression for these criteria in terms of the roots themselves, I obtain expressions which exhibit the intimate connexion between this subject and a former theory of my own relative to the construction of the conditions for the existence of a given number and grouping of equal roots, which can hardly fail to lead eventually to the extension of the results herein obtained to equations of any odd degree whatever. It is the more needful that these results in a question of so high moment to the advancement of algebraical science should be made public, inasmuch as they do not seem to accord with those obtained by my eminent friend M. HERMITE, who has preceded me in this inquiry in a classic memoir, published in the year 1854 in

Hence since T to a constant factor *près* is identical with C, the coefficients of η^3 and ξ^3 in the above determinant must vanish in order that $\xi\eta$ may be contained in T.

Hence the two determinants

$$\begin{matrix} \alpha & \beta & \gamma \\ \beta & \gamma & \delta \\ \gamma & \delta & \varepsilon \end{matrix} \quad \text{and} \quad \begin{matrix} \beta & \gamma & \delta \\ \gamma & \delta & \varepsilon \\ \delta & \varepsilon & \iota \end{matrix}$$

both vanish.

Hence either α, β, γ, or otherwise γ, δ, ε, or else the first minors of

$$\begin{vmatrix} \beta & \gamma \\ \gamma & \delta \\ \delta & \varepsilon \end{vmatrix}$$

are each zero.

The first two suppositions must be excluded, since either of them would lead to the conclusion of T, and therefore C, being a perfect cube, contrary to hypothesis. The last supposition implies either that β, γ, δ, or otherwise that γ, δ, ε, or else that $\beta\delta-\gamma^2$ and $\gamma\varepsilon-\delta^2$ are each zero.

If β, γ, δ are each zero, T becomes a multiple of $\eta^2\xi$; if γ, δ, ε are each zero, T becomes a multiple of $\eta\xi^2$; that is to say, T, and consequently C, contains a square factor; and obviously the converse is true, so that when C contains a square factor F is reducible to the form $au^5+5euv^4+fv^5$. When this is not the case $\delta=\frac{\gamma^2}{\beta}$, $\varepsilon=\frac{\delta^2}{\gamma}=\frac{\gamma^3}{\beta^2}$. Hence

$$F=\left(\alpha-\frac{\beta^2}{\gamma}\right)\xi^5+\frac{\beta}{\gamma}\left(\xi+\frac{\gamma}{\beta}\eta\right)^5+\left(\iota-\frac{\varepsilon^2}{\delta}\right)\eta^5,$$

which is of the form $\omega^5+\phi^5+\psi^5$, ω, ϕ, ψ being linear functions of x, y.

(33) We have supposed C not to be a perfect cube. When it is a perfect cube, say ξ^3, we may assume η any second linear function of x, y; and expressing F in the same manner as before in terms of ξ, η, it is clear that all the first minors of

$$\begin{matrix} \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \varepsilon \\ \gamma & \delta & \varepsilon & \iota, \end{matrix}$$

except the one obtained by cancelling the last column in the above matrix, must vanish, consequently δ, ε, i must all vanish, so that Φ, and consequently F, must contain a cube factor identical with the canonizant itself.

Lastly, if the canonizant vanish entirely, every first minor in the above matrix, when we write again a, b, c, d, e, i in lieu of α, β, γ, δ, ε, ι, will be zero. Hence either a, b, c, d, or b, c, d, e, or c, d, e, i must each vanish, or else that must be the case with the first minors of

$$\begin{matrix} a & b & c & d \\ b & c & d & e, \end{matrix}$$

Thus, then, if $a=0$ and $i=0$, all the coefficients, or else all except one, viz. b or e, are zero;

if $a=0$ and $d=0$, all the coefficients, or else only not e and i or only not b or only not i are zero;

so if $i=0$ and $c=0$, all must be zero except b and a or e or a;

if $c=0$ and $d=0$, only e and i or else a and b or else a and i will differ from zero.

Hence, then, in any case there will be at least four equal roots, or else F is of the form ax^5+iy^5.

Thus, then, for the first time has been here rigorously demonstrated, free from all doubt and subject to no exceptions, the following important proposition:

Every binary quantic function *not containing three or more equal roots* is reducible to one or the other of the two following forms,

$$u^5+v^5+w^5, \text{ or } au^5+5euv^4+fv^5.$$

The former is the case when the discriminant of the canonizant is different from zero, the latter when it is equal to zero; for it will be observed that, whether the canonizant has equal roots or totally disappears, its discriminant in both cases alike is zero.

(34) It has been seen that when the quintic has three equal roots the canonizant becomes a perfect cube; and it may not be out of place here to point out what the conditions (necessary and sufficient) are to ensure the quintic having four equal roots. These are all comprised in that of the quadratic covariant vanishing. To prove this, let η be a factor of $F(x, y)$, so that

$$F(x, y)=\Phi(x, \eta)=(\alpha, \beta, \gamma, \delta, \varepsilon, 0\between x, \eta)^5.$$

Then, since the similar covariant *quoad* x, y must also vanish, we have

$$\alpha\varepsilon-4\beta\delta+\gamma^2=0, \quad -3\beta\varepsilon+2\gamma\delta=0, \quad -4\gamma\varepsilon+3\delta^2=0.$$

If $\varepsilon=0$, then $\delta=0$, $\gamma=0$ by virtue of the two extreme equations, and Φ, and therefore F, contains four equal factors. If ε is not zero,

$$\gamma=\frac{3\delta^2}{4\varepsilon}, \; \beta=\frac{\delta^3}{2\varepsilon^2}, \; \alpha=\frac{5\delta^4}{16\varepsilon^3}, \text{ and } \Phi \text{ becomes } \frac{5\varepsilon}{16}x\left(\frac{\delta}{\varepsilon}x+2\eta\right)^4;$$

so that, as before, there are four equal factors. Conversely, it is obvious that if there are four equal factors u, so that $\Phi=au^5+5bu^4v$, the quadratic covariant of Φ disappears.

(35) The quadratic covariant also it was which led me to perceive the transformation applied in the antecedent article. For when the first minors of

$$\begin{vmatrix} a & b & c & d \\ c & d & e & f \end{vmatrix}$$

So when the canonizant has two equal roots and is of the form $C(x+py)(x+qy)^2$; in which case the reduced form is $au^5+5euv^4+fv^5$. The canonizant in respect to u, v becomes

$$\begin{array}{cccc} a & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e & f \\ v^3 & -v^2u & vu^2 & -u^3, \end{array}$$

i. e. ae^2uv^2. Hence, writing

$$u=x+py, \quad v=x+qy, \quad F=au^5+5euv^4+fv^5,$$

a, e, f may be obtained, as before, by means of three linear equations, and the terms au^5, $5euv^4$, fv^5 form a single and unique system.

Finally, when the canonizant vanishes entirely, so that the form becomes au^5+fv^5, the quadratic covariant will take the form $C(x+ey)(x+fy)$; and making $u=x+py$, $v=x+qy$, a, f become determined by means of two linear equations, so that au^5, fv^5 form a single and unique system, as in the preceding cases.

(37) When the canonizant has three distinct roots, they may be all real, or one real and the other two imaginary. In the former case, in the expression $ru^5+sv^5+tw^5$, u, v, w may be considered as all real functions of x, y, and r, s, t will then also all of them be real. In the latter case w may be taken as a real function of x, y, u, v as conjugate imaginary functions; and consequently it is easy to see that, except when r, s are equal to each other, they will constitute a pair of conjugate imaginary quantities: in this case we may take for our canonizant form

$$r\left(\frac{-u+iv}{2}\right)^5+s\left(\frac{-u-iv}{2}\right)^5+tu^5;$$

or, if we please,

$$ru_{\prime}^5+sv_{\prime}^5+tw^5$$

understanding by $u_{\prime}$, $v_{\prime}$ $\frac{-u+iv}{2}$, $\frac{-u-iv}{2}$ respectively. And it should be noticed that the determinant of $u_{\prime}$, $v_{\prime}$ in respect to u, v will be

$$\begin{vmatrix} -\frac{1}{2} & \frac{i}{2} \\ -\frac{1}{2} & \frac{-i}{2} \end{vmatrix}$$

which is i.

(38) Let us proceed briefly to express the invariants of $ru^5+sv^5+tw^5$, which call Φ, with respect to u, v; the corresponding ones of $ru_{\prime}^5+sv_{\prime}^5+tw^5$, which call $\Phi_{\prime}$, in respect to the same variables u, v will be found by attaching to these suitable powers of i.

$$\Phi=(r-t,\ -t,\ -t,\ -t,\ -t,\ s-t)(u,\ v)^5.$$

Hence its quadratic covariant is the quadratic invariant of

$$\left((r-t)u-tv,\ -tu-tv,\ -tu-tv,\ -tu-tv,\ -tu+(s-t)v \between u', v'\right)^4,$$

which is obviously

$$-rtu^2-stv^2+(rs-rt-st)uv.$$

Of this the quadratic invariant is

$$rt\,.\,st-\tfrac{1}{4}(rs-rt-st)^2;$$

or writing $\varrho=st$, $\sigma=tr$, $\tau=rs$, and calling this invariant (I),

$$(\mathrm{I})=-\tfrac{1}{4}(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\sigma\tau-2\tau\varrho).$$

Again, the cubic covariant or canonizant has been already shown to be $rst(u^2v+uv^2)$. Calling the discriminant of this (L), we have

$$(\mathrm{L})=-\tfrac{1}{27}r^4s^4t^4\,(^{30})=-\tfrac{1}{27}\varrho^2\sigma^2\tau^2.$$

Again, to find the discriminant (D) in respect to u, v.

When $ru^5+sv^5+tw^5=0$ has two equal roots, and $u+v+w=0$, it is easy to see that we have $ru^4+\lambda=0$, $sv^4+\lambda=0$, $tw^4+\lambda=0$.

Hence to a constant factor *près* (D) will be the *Norm* of

$$(st)^{\frac{1}{4}}+(tr)^{\frac{1}{4}}+(rs)^{\frac{1}{4}},\ \text{i. e. of } \varrho^{\frac{1}{4}}+\sigma^{\frac{1}{4}}+\tau^{\frac{1}{4}}\,(^{31}).$$

To find the value of this norm, suppose $\varrho^{\frac{1}{4}}+\sigma^{\frac{1}{4}}+\tau^{\frac{1}{4}}=0$, then

$$\varrho+\sigma+\tau=2(\varrho^{\frac{1}{2}}\sigma^{\frac{1}{2}}+\sigma^{\frac{1}{2}}\tau\ +\tau^{\frac{1}{2}}\varrho^{\frac{1}{2}}),$$

and

$$\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau=8\varrho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\tau^{\frac{1}{2}}(\varrho^{\frac{1}{2}}+\sigma^{\frac{1}{2}}+\tau^{\frac{1}{2}}).$$

Hence

$$(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau)^2=64\varrho\sigma\tau\{(\varrho+\sigma+\tau)+2(\varrho^{\frac{1}{2}}\sigma^{\frac{1}{2}}+\sigma^{\frac{1}{2}}\tau^{\frac{1}{2}}+\tau^{\frac{1}{2}}\varrho^{\frac{1}{2}})\}=128\varrho\sigma\tau(\varrho+\sigma+\tau).$$

Hence (D) must contain $(\mathrm{J})^2-128\varrho\sigma\tau(\varrho+\sigma+\tau)$ as a factor; and since when $t=0$, $\varrho=0$, $\sigma=0$, and $(\mathrm{D})=\tau^4=(\mathrm{J})^2$, it is clear that $(\mathrm{D})=(\mathrm{J})^2-128(\mathrm{K})$, where

$$(\mathrm{K})=\varrho\sigma\tau(\varrho+\sigma+\tau).$$

(39) Although in the investigation in view (K) will only figure as an abbreviation of $\frac{(\mathrm{D})-(\mathrm{J})^2}{128}$, it may not be amiss to indicate a direct process for finding it. Let us for this purpose act upon the Hessian of Φ, treated as a function of u, v twice with the canonizant of Φ converted into an operator by substituting $\frac{d}{dv}$, $-\frac{d}{du}$ in place of u and v.

(30) For this is $\left(0, \frac{rst}{3}, \frac{rst}{3}, 0 \between u, v\right)^3$, and the discriminant of $\left(a, b, c, d \between u, v\right)^3$ is

$$a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd.$$

(31) It is worthy of observation that (J) is also a Norm, viz. of $\varrho^{\frac{1}{3}}+\sigma^{\frac{1}{3}}+\tau^{\frac{1}{3}}$, so that (J) is the discriminant of $ru^3+sv^3+tw^3$. I have not been able to perceive the morphological significancy of this relation.

The Hessian of Φ may be obtained without difficulty under the form

$$rsu^3v^3+stv^3w^3+trw^3u^3 \text{ or } \tau u^3v^3+\varrho v^3w^3+\sigma w^3u^3\ (^{32}).$$

Operating upon this with

$$r^2s^2t^2\left(\frac{d}{dv}\cdot\frac{d}{du}\ \left(\frac{d}{du}-\frac{d}{dv}\right)\right)^2,$$

we obtain $\varrho\sigma\tau(A\tau+B\varrho+C\sigma)$, where

$$A=-2\left(\frac{d}{du}\right)^3\left(\frac{d}{dv}\right)^3u^3v^3=-72;$$

and as we know that this quantity must be of the form $\lambda(K)+\mu(J)^2$, we have $\mu=0$, $\lambda=-72$; so that, denoting the *operator* corresponding to the canonizant by T, and the Hessian by H, we have $(K)=-\frac{1}{72}T^2H\Phi$ (33). This gives a ready practical method for finding the discriminant of a general quintic F by means of the identity $D=J^2+\frac{16}{9}T^2H$, where D is the discriminant, H the Hessian, T the canonizantive operator, and J the quadratic invariant of F in respect to its own variables.

(40) If now we suppose the determinant of u, v in respect to x, y to be μ, where μ is by hypothesis a real quantity, and if we call the

Quadratic invariant in respect to x, y	. .	$-\frac{1}{4}J$,
Discriminant of primitive „ „	. .	D,
Discriminant of the canonizant „	. .	$-\frac{1}{27}L$,

we have obviously

$$\left.\begin{array}{l} J=\mu^{10}(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau), \\ K=\mu^{20}\varrho\sigma\tau(\varrho+\sigma+\tau), \quad D=J^2-128K, \\ L=\mu^{30}\varrho^2\sigma^2\tau^2, \end{array}\right\}\text{invariants of } \Phi.$$

This applies to the case where the reduced form is Φ, *i. e.* where the roots of the canonizant are all real, and consequently where $-L$ is negative, *i. e.* L positive.

When L is negative and the reduced form is $\Phi_{\prime}$, then, since the determinant of $u_{\prime}$, v, in respect to $u_{\prime}$ v is ι, we have

$$\left.\begin{array}{l} J=-\mu^{10}(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau), \\ K=\ \ \mu^{20}\varrho\sigma\tau(\varrho+\sigma+\tau), \quad D=J^2-128K, \\ L=-\mu^{30}\varrho^2\sigma^2\tau^2, \end{array}\right\}\text{invariants of } \Phi_{\prime}$$

By means of the ratios $\frac{L}{J^3}$, $\frac{K}{J^2}$, it is obvious that in either case alike the ratios of ϱ, σ, τ

(32) It will be the quadratic invariant of $ru^3\xi^2+sv^3\eta^2+tw^3\zeta^2$ with respect to ξ, η, $\zeta+\eta+\zeta$ being zero; just as the quadratic covariant of Φ is the quadratic invariant of $ru\xi^4+sv\eta^4+tw\zeta^4$ with regard to the same variables. This latter is in fact $rsuv+stvw+trwu$.

(33) The intervening covariantic form of degree 3 in the variables and 5 in the coefficients, viz. $TH\Phi$, will easily be seen to be

$$rst^2(u^2v-uv^2)+str^2(v^2w-vw^2)+trs^2(w^2u-wu^2).$$

become determinable by means of the same cubic equations, viz.

$$\theta^3-K\theta^2+\frac{K^2-JL}{4}\theta-L^2=0\,;$$

ρ, σ, τ will be to each other as the roots of this equation([34]).

(41) Since $ru^5+sv^5+tw^5$ represents a function in x, y with real coefficients, it follows that when L is positive, u, v as well as w being real, $\alpha:\beta:\gamma$ are ratios of real quantities, and the roots of the preceding cubic will be real; when L is negative, u, v becoming conjugate imaginary functions of x, y, whilst w remains real, r, s, unless they are equal, must become conjugate imaginary constants. When r, s, t are all real, ρ, σ, τ will be so too; and when r, s are imaginary and t real, ρ, σ will be imaginary and τ real. Thus according as L is positive or negative the roots of θ are or are not all real. Hence understanding by Δ the discriminant of the preceding equation with respect to θ and 1, $\frac{\Delta}{L}$ must be always either zero or negative. We see *à priori* that $\frac{\Delta}{L}$ must be integer, because when $L=0$ the cubic has two equal roots, $\frac{L}{2}$. To compute its value more conveniently, write $K=6k$, $J=12j$. Then the equation becomes

$$(1,\ 2k,\ 3k^2-jL,\ L^2\between\theta,\ -1)^3,$$

of which the discriminant is

$$L^4+4(3k^2-jL)^3+32k^3L^2-12k^2(3k^2-jL)^2-12kL^2(3k^2-jL).$$

Hence

$$\begin{aligned}\frac{\Delta}{L}&=L^3-108k^4j+36k^2j^2L-4j^3L^2+32k^3L\\&\qquad+\ 72k^4j-12k^2j^2L\qquad\quad-36k^3L+12jkL^2\\&=L^3-\ 36k^4j+24k^2j^2L-4j^3L^2-\ 4k^3L+12jkL^2.\end{aligned}$$

Accordingly, multiplying the above equation by $-3\cdot12^2$ in order to avoid fractions, replacing k, j by their values in terms of K, J, and naming G the quantity $-432\,\frac{\Delta}{L}$,

([34]) For since the absolute values of ρ, σ, τ are not in question, we may consider ρ, σ, τ as the roots of $\theta^3-K\theta^2+q\theta-r$, so that $\rho+\sigma+\tau=K$. We have then

$$\frac{\rho^4\sigma^4\tau^4}{(\rho\sigma\tau)^3(\rho+\sigma+\tau)^3}=\frac{L^2}{K^3},\quad\text{or}\quad\frac{r}{K^3}=\frac{L^2}{K^3},$$

which gives $r=L^2$. Again,

$$\frac{\rho\sigma\tau K^2}{(K^2-4q)^2}=\frac{K^2}{J^2},\quad\text{or}\quad\frac{(K^2-4q)^2}{r}=J^2,\quad\text{or}\quad(K^2-4q)^2=L^2J^2,\quad\text{or}\quad q=\frac{K^2\mp IL}{4}.$$

As regards the sign to be given to JL in q, since

$$\frac{J^3}{L}=\frac{(K^2-4q)^3}{r^2}=\frac{(K^2-4q)^3}{L^4},$$

we have $(K^2-4q)^3=J^3L^3$. Hence

$$q=\frac{K^2-1^{\frac{1}{3}}JL}{4}.$$

Consequently

$$q=\frac{K^2-JL}{4},\ \text{and } \textit{not}\ \frac{K^2+JL}{4}.$$

positive, or to speak more strictly non-negative, we have

$$G=JK^4+8LK^3-2J^2LK^2-72JL^2K-432L^3+J^3L^2\,(^{25}).$$

It is evident that G must be identical to a positive numerical factor *près* with the function which M. HERMITE denotes by I^2 ([26]).

([25]) It will be observed that when $J=0$ and $L=0$, G vanishes. This is easily verifiable *à priori*; for when $J=0$ and $L=0$, the reduced form has been seen to be ax^5+5exy^4, of which the canonizant is

$$\begin{vmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e & 0 \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix}$$

which equals $aexy^2$.

Hence the form and its canonizant have a common factor x, and consequently their resultant vanishes; hence $I=0$ and $G=I^2=0$. G also vanishes when $K=0$ and $L=0$, which is also easily verifiable; for then the reduced form becomes u^5+v^5, of which the canonizant vanishes, and consequently the resultant of the form and its canonizant becomes intensely zero; which accounts for the high power of K in (JK^4), the sole term of G in which L does not appear.

([26]) (*) Compare expression for $16I^2$, Cambridge and Dublin Journal, p. 203. This will be found to contain nine terms, and to rise as high as the fifth power in Δ (which to a constant factor *près* is identical with my J); whereas in $\frac{-\Delta}{L}$ there are only six terms, and no power of J beyond the third. This seems to indicate that the K and L are more fortunately chosen than M. HERMITE's J_2, J_3, which are invariants of the like degrees 8 and 12. It is of course evident that the following relations exist between M. HERMITE's Δ_1, J_2, J_3 and the J, K, L of this paper,

$$\Delta=lJ,$$
$$J_2=mJ^2+nK,$$
$$J_3=pJ^3+qJK+rL,$$

where l, m, n, p, q, r are certain numerical quantities. Until these are ascertained, it is impossible to confront M. HERMITE's results with my own, to ascertain whether or not they are identical in substance, and, if not, wherein the difference consists. I therefore subjoin the necessary calculations for effecting this important object.

Let us first take the form $x^5+5exy^4+y^5$. The quadratic covariant of this is $x(ex+y)$.

Accordingly, to obtain M. HERMITE's A, B, C, C′, B′, A′ (Cambridge and Dublin Journal, vol. ix. p. 179), we must make

$$x=X; \quad ex+y=Y,$$

which gives (vide C. and D. J. p. 180)

$$F=X^5+5eX(Y-eX)^4+(Y-eX)^5$$
$$=(A, B, C, C', B', A')(X, Y)^5,$$

where

$$A=1+4e^5,\ B=-3e^4,\ C=2e^3,\ C'=-e^2,\ B'=0,\ A'=1.$$

Accordingly (vide C. and D. J. p. 184),

$$AA'-3BB'+\ 2CC'=1+4e^5-\ 4e^5=1 \qquad =\sqrt{\Delta},$$
$$AA'+\ BB'-\ 2CC'=1+4e^5+\ 4e^5=1+\ 8e^5=\frac{I_1}{2\sqrt{\Delta^3}},$$
$$AA'+5BB'+10CC'=1+4e^5-20e^5=1-16e^5=\frac{I_2}{2\sqrt{\Delta^5}}.$$

Hence

$$\Delta=1,\ \ I_1=2+16e^5,\ \ I_2=2-32e^5.$$

Again (vide C. and D. J. p. 186. § vii.),

$$8J_1=I_1-\Delta^2=1+16e^5,\ \ 24J_2=I_2-2I_1\Delta+\Delta^2=-1-64e^5;$$

functions,

$$ru^5+sv^5-t(u+v)^5; \quad rstuv(u+v),$$

---

and conversely,

$$\begin{aligned} J&=\Delta, \\ K&=-\frac{J}{2}+\frac{1}{8}\Delta^2 \\ L&=-\frac{J_3}{18}-\frac{2}{27}\Delta J_2+\frac{1}{8}\Delta^3. \end{aligned}$$

Unhappily a further step is wanting to bring M. HERMITE's results to the final test of comparison; for the value of AA′ (p. 192) does not agree with that given for AA′ (p. 186) by simply changing J_1, J_2 into J_2, J_3 respectively; a further change of Δ into 2Δ becomes necessary to make the ratios of AA′, BB′, CC′ (p. 192) accord with the ratios of the same quantities at p. 186. Finally, even after making this change the expression for $16I^2$ (p. 203) does not accord (even to a constant coefficient *près*) with that with which it is meant to be identical, viz. $16I_3^2$ (p. 187); so that after great labour I am still baffled in my attempt to ascertain the agreement or discrepancy of my conclusions with those of my precursor in the inquiry. As will appear hereafter, the two sets of conclusions are undoubtedly discrepant in form; but whether they are so in substance or not, or rather whether they are or not in contradiction to each other, requires a close examination to discover, the more especially because, as will hereafter be shown, there is a certain necessary element of indeterminateness in the scheme of invariantive conditions which serve to fix the character of the roots. It is greatly to be lamented that so valuable a paper as M. HERMITE's should be to some extent marred, in respect of the important end it would serve as a term of comparison, by the existence of these numerical and notational inaccuracies. I have spent hours upon hours in endeavouring to reconcile these several texts of the same memoir, and, after all my labour, the work is left unperformed without which the truth as between the two methods cannot be elicited. I feel, however, as confident of the correctness of my own conclusions as of the truth of any proposition in Euclid.

([b]) It is worthy of notice that there is a failing case in M. HERMITE's process for finding I^2 in terms of Δ, J_2, J_3, just as there is one in mine for finding G in terms of J, K, L,—the failure of the process, however, in neither case entailing any corresponding defect in the results obtained. The process employed in this memoir fails when $L=0$: for then the general form $ru^5+sv^5+tw^5$ is superseded by the supplementary one, $au^5+5euv^4+fv^5$. M. HERMITE's fails when J (the J of *this* memoir) $=0$; for then the quadratic invariant becomes a perfect square, and the substitution of its factors in place of the original variables becomes inadmissible, since the two former coincide.

([c]) It may be as well here to notice the form which M. HERMITE's two linear covariants assume when referred to the canonical form above written. The quadratic covariant being $rsuv+stvw+truw$, if we operate with the correlative of this obtained by writing in it $\frac{d}{dv}$, $-\frac{d}{du}$, $\frac{d}{du}-\frac{d}{dv}$ in lieu of u, v, w, viz.

$$-rs\,\frac{d}{du}\frac{d}{dv}-st\frac{d}{du}\left(\frac{d}{du}-\frac{d}{dv}\right)+tr\frac{d}{dv}\left(\frac{d}{du}-\frac{d}{dv}\right)$$

upon the primitive, we obtain to a factor *près* the canonizant $rstuvw$, which has been already obtained; repeating the process, it is easy to see that the first linear covariant of the fifth degree in the coefficient assumes the simple form $rst(stu+trv+rsw)$, or $rst(\rho u+\sigma v+\tau w)$. Taking again the correlative of this, viz.

$$rst\left(\rho\frac{d}{dv}-\sigma\frac{d}{du}+\tau\left(\frac{d}{du}-\frac{d}{dv}\right)\right),$$

and operating with it upon $rsuv+stvw+truw$, it will be found without difficulty that the second linear covariant of the seventh degree in the coefficients becomes

$$rst\{(\sigma-\tau)(\sigma+\tau-\rho)u+(\tau-\rho)(\tau+\rho-\sigma)v+(\rho-\sigma)(\rho+\sigma-\tau)w\},$$

which is distinguishable in species from the former one by its symmetry being only of the hemihedral kind.

([d]) It may not be out of place to notice here that the Hessian of the canonical form will be found to be

$$\rho v^2w^2+\sigma w^2u^2+\tau u^2v^2.$$

their resultant in respect to u, v is obviously

$$(rst)^5(r-s)(s-t)(t-r)\text{([37])},$$

([*]) Again, if we write

$$\begin{aligned} &rst(\rho u+\sigma v+\tau w)=\xi \\ &rst(w-\tau)(\sigma+\tau-\rho)u+(\tau-\rho)(\tau+\rho-\sigma)v+(\rho-\sigma)(\rho+\sigma-\tau)w)=\eta, \\ &\qquad u+v+w \qquad\qquad\qquad\qquad\qquad\qquad =0, \end{aligned}$$

and from these equations deduce the values of u, v, w, and substitute them in $ru^5+sv^5+tw^5$, we shall obtain M. Hermite's "forme-type" expressed in terms of the parameters of the reduced form, and every coefficient therein will be invariantive.

The resultant of the equations above written (on making $\xi=0$, $\zeta=0$) will appear in the denominator of each such coefficient. Hence it appears, from M. Hermite's expressions (Camb. and Dubl. Math. Journal, vol. ix. p. 193), where J_3 will be seen to enter into the denominator of A, B, C, C′, B′, A′, that this resultant to a factor *près* is his J_3. Its value may easily be calculated, and will be found to be

$$\rho\sigma\tau(\rho+\sigma+\tau)^3-4(\rho+\sigma+\tau)(\rho\sigma+\rho\tau+\sigma\tau)+9\rho\sigma\tau=JK+9L.$$

Accordingly as L (to use Dr. Salmon's convenient elliptical expression) is the condition of the failure of my *general* reduced form, so is $9L+JK$ the condition of the failure of M. Hermite's "forme-type." As particular cases of this last failure, we may suppose $J=0$, $L=0$, or $K=0$, $L=0$. In the former case the reduced form is ax^5+5ex^4y, of which the simplest quadratic and cubic covariants are respectively aex^2; ae^2y^2x. Thus to find L, the first linear covariant, we have to operate upon ae^2y^2x with $ae\left(\frac{d}{dy}\right)^2$, which gives a^2e^3x; and to find L_2, we have to operate on $(aex^2)^2$ with $ae^2\left(\frac{d}{dx}\right)^2\frac{d}{dy}$, or, if we please (according to M. Hermite's method), with $\left(a^2e^3\frac{d}{dy}\right)$ on aex^2, showing that L_2 vanishes, but L_1 continues to subsist. When, secondly, $K=0$, $L=0$, the reduced form is ax^5+ey^5, and the canonizant disappears entirely, so that the first, and consequently also the second, linear covariants, each of them becomes a *null*.

([37]) By aid of the reduced forms of the invariants J, K, L, I given in the text, it is easy to prove that every other invariant, say Ω of a quintic, is a rational integral function of these four. In what follows, let a parenthesis enclosing the symbol of any invariant signify its value when any two of the quantities u, v, w in the reduced form $ru^5+sv^5+tw^5$; $[u+v+w=0]$ are taken as the independent variables. We have then

$$(J)=\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau,\quad (K)=\rho\sigma\tau(\rho+\sigma+\tau),\quad (L)=\rho^2\sigma^2\tau^2,\quad (I)=\rho^2\sigma^2\tau^2(\rho-\sigma)(\sigma-\tau)(\tau-\rho),$$

ρ, σ, τ meaning st, tr, st.

The degree of Ω must be of the degree $4m$ or $4m+2$. 1. Let it be of the form $4m$. Then, since the interchange of any two of the variables u, v, w must leave (Ω) unaltered, (Ω) will be unaltered by the interchange of any two of the letters r, s, t, and is consequently a symmetric function of ρ, σ, τ, the roots of the equation

$$\theta^3-\frac{(K)}{(L)^{\frac{1}{2}}}\theta^2+\frac{(K)^2-(J)(L)}{(L)}-(L^{\frac{1}{2}})=0.$$

Hence

$$(\Omega)=\frac{F((J),(K),(L))}{(L)^{2m}},$$

F denoting a rational integral function-form of the quantities it affects. Consequently

$$\Omega=\frac{F(J,K,L)}{L^{2m}}.$$

Hence since Ω cannot become infinite when $L=0$, which merely implies that the general form reduces to

$$(a, 0, 0, 0, e, i\between x, y)^5,$$

$\Omega=\Phi(J, K, L)$, a rational integral function of J, K, L.

2. If the degree Ω is of the form $4m+2$, (Ω) will be a function of r, s, t, which changes its sign when u and v

and consequently, if we call I the resultant in respect to x, y, we have

$$\pm \mathrm{I} = \mu^{45}\varrho^2\sigma^2\tau^2(\sigma-\varrho)(\tau-\sigma)(\varrho-\tau^2),$$

and

$$\mathrm{I}^2 = \mu^{90}\varrho^4\sigma^4\tau^4(\sigma-\varrho)^2(\tau-\sigma)^2(\varrho-\tau)^2$$
$$= \mu^{30}(\sigma-\varrho)^2(\tau-\sigma)^2(\varrho-\tau)^2\mathrm{L}^2.$$

(43) Thus we see that the two quantities G, I^2, which are both rational integral functions of the degree 36 in the coefficients of $\mathrm{F}(x, y)$, cannot one vanish without the other, at all events when L is not equal to zero. This is sufficient to show that they are identical to a numerical factor *près*, whatever L may be, zero or not zero([38]), and consequently that the quantity called G, proved to be positive upon the supposition of L not being zero, must also remain positive when L is zero, because it is in fact the square of a rational function of the coefficients. But we may also prove this independently by virtue of the supplementary reduced form $au^5+5euv^4+fv^5$ applicable to the case of L zero.

For when $\mathrm{L}=0$, G becomes JK^4; so that the condition "G not negative" implies simply that J is positive unless K vanishes.

Now the canonizant, when it does not vanish, i. e. when e is not zero, contains v^2u as a factor, and, its coefficients being real, u, v are both of them necessarily real functions of x, y. Consequently J, which by definition is $-4\times$ discriminant of quadratic covariant, becomes $-4\mu^{10}\times$ discriminant of $au(eu+fv)$ in respect to u, v, which $=\mu^{10}a^2f^2$, μ being real. Consequently J is positive, since the reality of u, v implies that of a, e, f, when e is not zero. When e *is* zero u, v may be either real or imaginary; for u^5+v^5 may be real whether u, v be real or conjugate imaginary functions of x, y; but in that case K, which is found by operating twice upon the Hessian with a canonizant turned into an operator, vanishes, since then all the coefficients of the canonizant vanish([39]). Hence the rule that G cannot be negative is seen to be true, whatever L may be.

---

or any two of its quantities u, v, w, are interchanged, such interchange having the effect of introducing as a multiplier the $5(2m+1)$th power of the determinant of substitution (-1). Hence (Ω) is of the form

$$(\varrho-\sigma)(\sigma-\tau)(\tau-\varrho)\mathrm{F}(\varrho, \sigma, \tau), \text{ i. e. } \frac{(\mathrm{I}).\mathrm{F}(\varrho, \sigma, \tau)}{(\mathrm{L})^{\frac{1}{2}}},$$

which again is of the form

$$\frac{(\mathrm{I}).\mathrm{F}\big((\mathrm{J}), (\mathrm{K}), (\mathrm{L})\big)}{(\mathrm{L})^{2m-8}},$$

so that Ω is of the form

$$\frac{\mathrm{I}.\mathrm{F}(\mathrm{J}, \mathrm{K}, \mathrm{L})}{\mathrm{L}^{2m-8}}.$$

Hence since, as before, Ω cannot become infinite when $\mathrm{L}=0$, and since, furthermore, I does not vanish (for if so then G, which is I^2, would vanish) when $\mathrm{L}=0$, Ω must be of the form $\mathrm{I}\Phi(\mathrm{J}, \mathrm{K}, \mathrm{L})$. Q. E. D.

([38]) For if $\mathrm{Q}^2=\mathrm{KI}^2$ for an indefinite number of systems of values of a, b, c, d, e, f, of which Q, I are rational integral functions, Q^2 and KI^2 must be *absolutely* identical; this of course is the case when Q^2 and KI^2, as proved in the text, are known to be identical for all values of a, b, c, d, e, f which do not make L zero.

([39]) ([a]) In the more general form $au^5+5euv^4+fv^5$, taking $\mu=1$. The canonizant is ae^2uv^2; this squared and

It may be said that the case of three or more equal roots existing in F (x, y) has been

---

turned into an operator becomes $a^2e^4\left(\frac{d}{dv}\right)^2\left(\frac{d}{du}\right)^4$, which, applied to the Hessian, viz. $3aeu^4v^2+afu^3v^3-e^2v^6$, after multiplying by $-\frac{1}{72}$, gives $K=-2a^3e^5$, so that $D=J^2-128K=a^4f^4+256a^3e^5$, which is capable of easy verification. In fact D becomes the resultant of au^4+ev^4 and $v^3(4eu+fv)$; v^3 introduces the factor a^3 into D; and further, making $u:v::-f:4e$ and substituting in au^4+ev^4, we obtain the other factor af^4+256e^5.

If we adopt $u^5+5euv^4+v^4$ as the reduced form for the failing case (a form analogous to the well-known one, $u^4+6cu^2v^2+v^4$, for the general quartic), to find e we have $J=\mu^{10}$, $K=-2\mu^{20}e^5$. Hence $e^5=-\frac{K}{2J^2}$; thus when $K=0$, $e=0$.

(^{b}) By a linear transformation we may always take away any two (except the two first or last) coefficients of a given quintic, but the vanishing of more than two coefficients always corresponds to some invariantive condition. Thus, *ex. gr.*, in the form

$ax^5+5exy^4+fy^5$	$L=0$		
ax^5+fy^5	$L=0$	$K=0$	
ax^5+5exy^4	$L=0$	$J=0$	
$ax^5+10dx^2y^3$	$J=0$	$K=0$	
$ax^5+5bx^4y+10cx^3y^2$	$L=0$	$J=0$	$K=0$.

(^{c}) The condition for the existence of four equal roots in a quintic is the vanishing of the quadratic covariant; that is to say, we must have

$$ae-4bd+3c^2=0,\quad af-3be+2cd=0,\quad bf-4ce+3d^2=0.$$

The three quantities equated to zero are not separately invariants, but constitute in their *ensemble* an invariantive plexus.

(^{d}) [It may here be noticed incidentally that the conditions for equal roots in the biquadratic form are as follows. For two equal roots, of course, the discriminant is zero, for three equal roots the two lowest invariants are each zero, and for two pairs of equal roots the Hessian $(A, B, C, D, E)(x, y)^4$ becomes to a factor *près* identical with the primitive $(a, b, c, d, e)(x, y)^4$, so that all the first minors of the matrix

$$\left|\begin{matrix} a, & b, & c, & d, & e, & f \\ A, & B, & C, & D, & E, & F \end{matrix}\right|$$

vanish. *Quære*, whether the character of the five-rayed pencil (centre at origin), in which a, A; b, B; c, C; d, D; e, E mark points, may not serve to distinguish between the case of four real and four imaginary roots.]

(^{e}) When $J=0$ and $K=0$, but *not* $L=0$, it is obvious that $\rho:\sigma:\tau::1:\iota:\iota^2$, ι being any imaginary cube root of unity, and the reduced form is $u^5+\iota v^5+\iota^2w^5$, with the relation $u+v+w=0$.

J and K being zero, D will be so too, and accordingly the equation $u^5+\iota v^5+\iota^2w^5=0$ will have two equal roots. It will easily be found that these equal roots correspond to the system of ratios $u=1$, $v=\iota^2$, $w=\iota$. In fact, if we write $u=1+\rho$, $v=\iota^2+\iota\rho$, $w=\iota+\iota^2\rho$, the equation becomes $u^5+\iota v^5+\iota^2w^5=\rho^2(30\rho+3\rho^3)=0$.

Hence, understanding by ε either of the two prime sixth roots of unity, the complete system of ratios of u, v, w may be expressed as follows:—

$u=1$	$v=\iota^2$	$w=\iota$
$u=1$	$v=\iota^2$	$w=\iota$
$u=1-\sqrt[3]{10}$	$v=\iota^2-\sqrt[3]{10}$	$w=\iota-\iota^2\sqrt[3]{10}$
$u=1+\sqrt[3]{10}\varepsilon$	$v=\varepsilon^4-\sqrt[3]{10}$	$w=\varepsilon^2+\sqrt[3]{10}\varepsilon^5$
$w=1+\sqrt[3]{10}\varepsilon^5$	$v=\varepsilon^4+\sqrt[3]{10}\varepsilon$	$w=\varepsilon^2-\sqrt[3]{10}$.

Thus, when $J=0$ and $K=0$, u, v, w (with the relation $u+v+w=0$) may first be found, in terms of x, y, by

lost sight of; but we know, and it is capable of immediate verification by taking as the

---

solving the cubic equation, obtained by equating to zero the canonizant of $(a, b, c, d, e, f \between x, y)$, and then x, y will be known from the above system of values for any two of the quantities u, v, w.

([f]) It is obvious that the form $ax^5+dx^2y^3$ gives $J=0$ and $K=0$; but it seems desirable to prove the converse, viz. that when $J=0$ and $K=0$, but not $L=0$, the form is always reducible to $au^5+10\delta u^2v^3$, which may be done as follows. Since $J=0$ and $K=0$ the discriminant is zero, and we may assume

$$F=ax^5+5bx^4y+10cx^3y^2+10dx^2y^3,$$

and we have $J=$ discriminant of

$$(-4bd+3c^2)\xi^2+2cd\xi\eta+3d^2\eta^2.$$

Hence

$$3d^2(3c^2-4bd)-c^2d^2=0;$$

d cannot be zero, for then we should have $J=0$, $K=0$, $L=0$, contrary to hypothesis. Hence $8c^2-12bd=0$.

If $b=0$ and $c=0$, F is already reduced to the desired form; but if not, $d=\frac{2c^2}{3b}$, and F becomes

$$ax^5+\frac{5b}{6}x^2\left(6x^2y+\frac{12c}{b}xy^2+\frac{8c^2}{b^2}y^3\right);$$

or, making

$$a-\frac{5b}{6}=\alpha, \quad \frac{b}{6}=2\delta, \quad x+\frac{2cy}{b}=v,$$

$F=\alpha x^5+10\delta x^2v^3$, as was to be shown.

The corresponding converses for the case of $J=0$, $L=0$, and of $K=0$, $L=0$ have been already established.

([g]) It will be observed that under a certain point of view L for binary quintics is the analogue of Δ the *discriminant* for binary quartics, the condition of failure in the *general* reduced form in the two cases being $L=0$ and $\Delta=0$ respectively. The mere vanishing of the discriminant in the case of the quintic function, unattended by any other condition, does not affect the nature of the reduced form.

([h]) It has been shown previously in the text that when $L=0$ the primitive is reducible to the form

$$(a, 0, 0, 0, e, f \between x, y)^5.$$

Hence if I_{12} is any duodecimal invariant which vanishes when $b=0$, $c=0$, $d=0$, I_{12} must vanish whenever L vanishes, and consequently, since L is of as high a degree as I_{12}, I_{12} must be a numerical multiple of L. In Mr. CAYLEY's Third Memoir on Quintics, "No. 29" represents a duodecimal invariant calculated by M. FAÀ DE BRUNO, and characterized morphologically by Mr. CAYLEY as being that duodecimal invariant in which "the leading coefficient a does not rise above the fourth degree." On examining No. 29 it will be found to contain no term in which b, c, d are all simultaneously absent. Hence it is, by virtue of the above observation, a multiple of my L: to determine the numerical factor, let all the coefficients in the primitive except a, d be supposed zero; then the canonizant becomes

$$\begin{vmatrix} a & 0 & 0 & d \\ 0 & 0 & d & 0 \\ 0 & d & 0 & 0 \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix} = d^3y^3+ad^2x^3.$$

Hence L becomes $-27a^2d^{10}$, but "No. 29" becomes $27a^2d^{10}$. Hence we have the important relation "No. 29" $=-$L, so that No. 29 is a discriminant, an *intrinsic* property of the calculated invariant, which, I believe, was not suspected.

([i]) It will at once be recognized that "No. 19" given in Mr. CAYLEY's Second Memoir upon Quantics is identical with the J of this memoir, whence it follows from Mr. CAYLEY's equation (No. 26)=(No. 19)^2−1152 No. 26, that K=9 (No. 25). Thus abstraction made of a mere numerical factor, Mr. CAYLEY and myself agree upon perfectly distinct grounds in recognizing K and L as the true simplest invariants of their respective degrees, an accordance as satisfactory as it was unexpected, and which must be considered as setting at rest the question of what should be deemed the, so to say, *staple* invariants of the Binary Quintic.

round the axis of D the facultative and non-facultative portions may be made to exchange places.

(46) The axis of D itself lies on the surface of G, and like every other portion of this surface is facultative, for there is no reason for disallowing G to become zero. Conversely, if, instead of a real equation, we take one of the conjugate class (described in the second section), the whole of the facultative portion of space (except the separating surface G) becomes non-facultative, and the non-facultative part becomes facultative, but G itself remains facultative. When the invariants, or any of them, become imaginary, we are put out of space altogether, and the system can belong neither to a real nor to a conjugate family, but to one with coefficients at the same time imaginary and non-conjugate. $G=0$ ([43]), it may be remarked, will in all cases be the condition of an equation capable of linear transformation into one of recurrent ([44]) form; for the reduced form then in general becomes $ru^5+rv^5-t(u+v)^5$. The case when G becomes zero by virtue of $J=0$ and $L=0$, that is to say when the function is reducible by real or imaginary linear substitutions (see footnote ([39]) (^{f})) to the form $u(u^4\pm v^4)$, is the one which might for a moment be supposed to offer an exception to the rule; but only the exception is only apparent, since $u(u^4-v^4)$, on writing $u=p+q$, $v=p-q$, becomes $16(p+q)pq(p^2+q^2)$.

(47) To every point in space, it has been remarked, will correspond one particular family of equations all of the same character as regards the number they contain of real or imaginary roots, because capable of being derived from one another by real linear substitutions, such family consisting of an infinite number of ordinary or conjugate equations according as the point is facultative or non-facultative; but it may be well to notice that, conversely, every point does not correspond to a distinct family. In fact every point in the curves $D=pJ^2$, $L=qJ^3$ (p, q being constants) will denote a curve divided into two branches by the origin of coordinates, one of which will be facultative and the other non-facultative; but in each separate branch every point will represent the very same family. Any such separate branch may be termed an isomorphic line; and we see that the whole of space may be conceived as permeated by and made up of such lines radiating out from the origin in all directions.

(48) The origin at which $J=0$, $D=0$, $L=0$, as already noticed, corresponds to the case of three equal roots. The theorem that, when more than half as many roots are equal to each other as there are units in the degree of any binary form, all the invariants vanish, was remarked by myself originally in the very infancy of the subject, before Mr. CAYLEY's paper, alluded to by M. HERMITE, appeared in Crelle. The method of proof which then occurred to me is the simplest that can be given. For instance, in

([43]) I shall hereafter allude to the surface denoted by $G=0$ under the name of the Amphigenous Surface, as being the locus of the points which give birth to real and conjugate forms indifferently.

([44]) The roots of recurring equations, geometrically represented, in general go in quadruplets, A, A′; B, B′, where A and B, as also A′, B′, are mutual optical images of each other in respect to a fixed line, and A, A′, as also B, B′, are electrical images of each other in respect to a circle of which the fixed line is a diameter—with liberty, of course, for the images taken in either mode of combination to coalesce so as to reduce the quadruplet to a simple pair.

the case before us, if the quintic have three equal roots, we may reduce it to the form

$$ax^5+5bx^4y+10cx^3y^2.$$

Suppose now, if possible, an invariant of the degree m; the *weight* of each term therein, say $a^rb^sc^t$, in respect to x or y would be the same $\left(\text{viz. } \frac{5m}{2}\right)$, so that we should have

$$5r+4s+3t=\frac{5m}{2}=s+2t, \text{ or } 5s+3s+t=0,$$

and therefore $r=0$, $s=0$, $t=0$, $m=0$. So for a sextic with three equal roots reduced to the form $(a, b, c, 0, 0, 0 \between x, y)^6$. Supposing any term in one of its invariants to be $a^rb^sc^t$, we should have

$$6r+5s+4t=\frac{6m}{2}=s+2t, \text{ or } 6r+4s+2t=0,$$

which is absurd, unless $r=0$, $s=0$, $t=0$, $m=0$, and so in general for a binary form of any degree. If in the above example for the degree m only three roots were equal *inter se* (the form assumed being $(a, b, c, d, 0, 0, 0 \between x, y)^6$, any term in a supposed invariant being $a^rb^sc^td^u$, where $r+s+t+u=m$, we should have

$$6r+5s+4t+3u=3m=s+2t+3u,$$

and, as before,

$$6r+4s+2t=0, \ r=0, \ s=0, \ t=0;$$

no longer, however, $m=0$, but $m=u$, which is left undetermined.

(49) Before proceeding further it will be proper to consider under what circumstances a variation (in the coefficients of any equation) arbitrary, except that the coefficients are to remain real, can affect the character of the roots.

Let $F(x)=0$ be any algebraical equation with real coefficients, and let $\delta(Fx)$ be the variation of F due to the variation of the coefficients, $dF(x)$ the variation due to the change of x into $x+dx$. If, now, r be a root of $Fx=0$, and $r+dr$ the corresponding root of $F(x)+\delta F(x)=0$, we have

$$Fr=0, \quad F(r+dr)+\delta F(r)=0, \quad \text{or } \delta F(r)+\frac{d}{dr}F(r)dr+\frac{1}{1.2}\left(\frac{d}{dr}\right)^2 Fr(dr)^2+\&c.=0.$$

Hence, unless $\frac{dF}{dr}=0$, i. e. unless there are two equal roots r, we shall have $dr=-\frac{\delta F(r)}{\frac{d}{dr}F(r)}=$ a real quantity; so that the character of the root $r+dr$ will be the same as that of r.

But if

$$\frac{dF}{dr}=0, \ \frac{d^2F}{dr^2}=0, \ \dots \ \left(\frac{d}{dr}\right)^{i-1}F=0,$$

so that there are i roots r, i being any integer greater than zero, then to find dr we have the equation

$$(dr)^i+\frac{\Pi(i)\delta Fr}{\left(\frac{d}{dr}\right)^i F(r)}=0.$$

Thus dr will have i distinct values; of these, if i is odd, all but one will be imaginary, but if i is even they will be all imaginary, or only all but two imaginary and the remain-

ing two real, according as the sign of $\delta F(r)$ is the same as or the contrary to that of $\left(\frac{d}{dr}\right)^i F(r)$. Accordingly, if r is real (45) and i even, the nature of the *ensemble* of the i roots $r+dr$ will not be the same when $\delta F(r)$ is positive as when $\delta F(r)$ is negative.

(50) So, further, if $Fx=0$ have $2m$ equal roots r, $2n$ equal roots s, and so on, the deduced corresponding groups of roots in $F(x)+\delta F(x)=0$ will, or may at least each of them, undergo a change of character to the extent of one pair of the r group changing their nature with the sign of $\delta F(r)$, one pair of the s group changing their nature with the sign of $\delta F(s)$, and so on; but in no case, except $F(x)$ possess some equal roots (*i. e.* unless its discriminant be zero), can an infinitesimal variation in the constants affect the character of the roots(46).

(51) To every facultative point corresponds a certain set of values of J, D, L; and when these are given, it has been shown that the equation $(a, b, c, d, e, f\between x, y)^5$ is reducible to the form $ru^5+sv^5+tw^5$, where $u+v+w=0$, or to the form $ru_{\prime}^5+sv_{\prime}^5+tw^5$, where

$$u_{\prime}+v_{\prime}+w=0, \text{ and } u_{\prime}=\frac{-w+iv}{2}, \; v_{\prime}=\frac{-w-iv}{2},$$

or to the form $au^5+5euv^4+fv^5$, u, v, w being always real linear functions of x, y, with the sole exception that when $J=0$, $K=0$, $L=0$, the reduced form is

$$au^5+5bu^4v+10cu^3v^2.$$

When these three invariants are not all zero, the coefficients in the reduced form r, s, t or a, e, f are known functions of J, D, L, and the character of the roots is perfectly determinate; so that to every facultative point corresponds an infinite family of equations with real linear coefficients all deducible from each other by real linear substitutions. Thus then, with the sole exception of the origin, every facultative point corresponds to a determinate character of equation, viz. to an equation with four, or two, or no imaginary roots; so that by a bold figure of speech we may be permitted to speak of every point but one in facultative space having a determinate quality, as masculine, feminine, or neuter. The origin alone is exempt from this law, and may be considered to be of epicene gender, since the factor $au^2+5buv+10v^2$ may have its roots real or imaginary. As we travel continuously from point to point in the facultative portion of space we pass from family to family, or, if we please, from an individual of one family to an individual of another family, differing from the former individual by an infinitesimal variation of the constants.

(45) r, although supposed to be one of a group of equal roots, is not *necessarily* real, for it may belong to a factor $(x^2+2e\cos\theta+e^2)^2$.

(46) Compare this statement with the corresponding one given by M. HERMITE, Camb. and Dub. Journal, vol. ix. p. 204, where only one parameter is supposed to undergo a change. I think that greater breadth and at the same time greater precision and clearness are gained by the mode of exposition employed in the text above. It will be observed that for a change of character to be possible when the function passes through a phase of equal roots, it is not enough that there shall exist a group of equal roots r, but there must be an even number of such roots in the group, and, furthermore, the equal roots must be *real*; when this last supposition is not satisfied, no change in the character of dr will affect the character of $r+dr$: an instructive exemplification of this remark will occur in the sequel.

(52) If, then, we insulate any portion of facultative space, and in the block so insulated it is possible to pass from one point to any other—that is to say, if we can draw a *continuous* curve of any sort from one point to another without passing out of the block, and without cutting or touching the plane $D=0$, then by virtue of the principle just laid down, we see that all the points in such block have the same character, and the nature of the roots will be the same in the infinite number of families, each containing an infinite number of individuals which the points in that block severally represent. Now imagine a block taken so extensive as to admit of no further augmentation, except accompanied with a violation of the condition of the capability of free communication between point and point without cutting or touching the surface D; such a block may be termed a *region*, and the whole of facultative space will be capable of subdivision into a certain number of these regions. This being supposed effected, the character of each region will be known when we know the character of a single point in it; that is to say, every region will have a determinate character of positive, negative, or neuter. It will presently be shown that the number of such regions is only three([47]) (the least number it could be to meet the three cases of four, two, or no imaginary roots), one masculine, one feminine, one neuter; and consequently there will be but three cases to consider when the invariantive coordinates J, D, L are given; according as J, D, L belong to one or the other of these three regions, the equation to which they belong will have all its roots real, or only one real, or three real and two imaginary. The origin, it need hardly be added, constitutes a region *per se*, in which, so to say, the characters of masculine and feminine are blended.

(53) Let it be observed that we can see *à priori* that, were it not for the distinction between facultative and non-facultative portions of space, it would be impossible for each point corresponding to a given system of invariants to possess an unequivocal character; for in such case there would necessarily be free continuous communication possible between all the points on each side of D *inter se*, and consequently we should be landed in the absurdity of conceiving the general equation of the fifth degree not to admit of division into cases of four, two, or no imaginary roots; D being negative, we know, would imply two roots, and not more than two, being imaginary; and accordingly D positive would imply either that four roots are imaginary or none—not sometimes one and sometimes the other, but in all cases alike four imaginary, to the exclusion of the supposition of the roots being all real, or else of all the roots being real and never four imaginary. Thus we see that the mere fact of a given system of invariants communicating a definite character to the roots, implies the necessity of the invariants exercising a restraining action over each other's limits, and that where this restraint does not exist it is impossible that the character of the roots can be determined by the values of the invariants.

([47]) It is clear from the definition, that a *region* can only be bounded by G the amphigenous surface, and D the plane of the discriminant: and granted (as will be shown hereafter) that G and D *touch* each other in only one continuous line, it becomes obvious *à priori* that there can be but two regions on one side of D and a single region on the other.

(54) This is precisely what happens in biquadratic equations. In such we know the fundamental invariants t, s, or, if we please, t, Δ (where $\Delta=s^3+27t^2$), are perfectly independent and subject to no equation of condition; so that if we consider t, Δ as the coordinates of points in a plane, the whole of the plane will be made up of facultative points. When Δ is negative, *i. e.* for representative points lying on one side of the line Δ, it is true we know that there is just one pair of imaginary roots constituting what may be termed the neuter case; but when the representative points lie on the other side of this plane, they cannot be said to be either masculine or feminine, but will every one of them possess that epicene character which is peculiar to the origin alone in the case of quintic forms. A single example will make this clear.

Take the two reduced forms

$$u^4+6(1+\epsilon)u^2v^2+v^4,$$
$$\omega^4+6(1-\epsilon)\omega^2\theta^2+\theta^4,$$

where u, v are real linear functions of x, y, and ω, θ conjugate imaginary ones of the same; and suppose s, the quadrinvariant in respect to x, y, to be the same for both forms. For greater convenience of computation consider ϵ to be infinitesimal.

Then in the one case the t is of the same sign as

$$(1+\epsilon)\big(1-(1+\epsilon)^2\big), \text{ i. e. } -2\epsilon,$$

and in the other the t is of the contrary sign to

$$(1-\epsilon)\big(1-(1-\epsilon)^2\big), \text{ i. e. } \quad 2\epsilon,$$

so that t is of the same *sign* (viz. negative) in each case.

Again, in the two cases respectively

$$\frac{t^2}{s^3}=\frac{4\epsilon^2}{1+3(1\pm\epsilon)^2}=4\epsilon^2.$$

Hence t as well as s, and consequently t and Δ are alike for both forms.

But in the one first written the roots are of the same nature as those of $u^4+6u^2v^2+v^4$, i. e. are all impossible, and in the other of the same nature as in

$$\left(\frac{u+iv}{2}\right)^4+6\left(\frac{u+iv}{2}\right)^2\left(\frac{u-iv}{2}\right)^2+\left(\frac{u-iv}{2}\right)^4=0,$$

where u, v are real linear functions of x, y and $i=\sqrt{-1}$, in which case the roots are all possible. Thus we see that the very same values of t, Δ may correspond either to the case of four real or four imaginary roots, showing that the point t, Δ is what we have termed *epicene*. If we choose to take s, t as the coordinates, the same remarks would apply, except that Δ instead of a straight line would become a semicubical parabola. All the points on one side of this curve would have a definite neuter character, but those on the opposite side would be neither masculine nor feminine, but epicene.

(55) With a view to its subsequent distribution into regions, I now proceed to ascertain the form of that moiety of space which I have termed facultative.

Let $J^2=qK$, $J^3=\nu L$. Then

$$\frac{G}{J^9}=\frac{1}{q^4}+\frac{8}{\nu q^3}-\frac{2}{\nu q^2}-\frac{72}{\nu^2 q}-\frac{432}{\nu^3}+\frac{1}{\nu^3}, \text{ and } \frac{D}{J^2}=1-\frac{128}{q}.$$

We may for the moment make abstraction of the section of G made by the plane of L; that being done, and J, K, L being referred to the form $ru^5+sv^5+tw^5$ or $ru_{\prime}^5+sv_{\prime}^5+tw^5$, calling μ^{10}, M, and, as before, using ϱ, σ, τ to denote st, tr, rs, we have

$$\begin{aligned}\pm J &= M(\varrho^2+\sigma^2+\tau^2-2\varrho\sigma-2\varrho\tau-2\sigma\tau),\\ K &= M^2\varrho\sigma\tau(\varrho+\sigma+\tau),\\ \pm L &= M^3\varrho^2\sigma^2\tau^2.\end{aligned}$$

Now when $G=0$, we may suppose $\varrho=\sigma$, $\frac{\tau}{\varrho}=\frac{\tau}{\sigma}=\theta+4$, θ being a new auxiliary variable. We have then

$$\begin{aligned}\pm J &= M(\tau^2-4\varrho\tau) &&= M\varrho\tau\theta,\\ K &= M^2\varrho^2\tau(2\varrho+\tau) &&= M^2\varrho^2\tau^2\left(1+\frac{2}{\theta+4}\right),\\ \pm L &= M^3\varrho^4\tau^2 &&= M^3\varrho^3\tau^3\frac{1}{\theta+4},\end{aligned}$$

and consequently

$$\nu=\frac{J^3}{L}=\theta^4+4\theta^3,$$

$$q=\frac{J^2}{K}=\frac{\theta^2(\theta+4)}{\theta+6}.$$

(56) In general we have $\theta^4+4\theta^3-\nu=0$.

By a well-known corollary to DESCARTES's rule this equation can never have more than two real roots; when ν is positive there will always be two real roots of opposite signs; but when ν is negative and inferior to a certain negative limit, *all the roots become imaginary*. When ν lies between zero and that limit, two roots of θ will be real and both negative. To find that limit we may make $4\theta^3+12\theta^2=0$, or $\theta=-3$, which gives $\nu=81-108=-27$.

(57) When $D=0$, $q=\frac{J^2}{K}=128$, i. e. $\theta^3+4\theta^2-128\theta-768=0$, or $(\theta+8)^2(\theta-12)=0$; so that the roots of θ, when $D=0$, are -8, -8, 12, and the corresponding values of ν are 2^{11}, 2^{11}, $2^{10}27$.

If now we make $\theta^4+4\theta^3=2^{11}$, one of the real values of θ we know is -8, and the other will be the real root of the cubic equation $\theta^3-4\theta^2+32\theta-256=0$.

When $\theta=5$, the left-hand side of the equation $=125+160-100-256=-71$.

When $\theta=6$, the left-hand side of the equation $=216+192-144-256=\ \ 8$.

Hence the real root lies between 5 and 6, and q lies between $\frac{225}{11}$ and $\frac{360}{12}$. Thus $q<30$ and $\frac{D}{J^2}=1-\frac{128}{q}$ is negative.

Again, if we take $\theta^4+4\theta^3=27\cdot2^{10}$, and take out the root $\theta=12$, the resulting cubic becomes

$$\theta^3+16\theta^2+192\theta+2304=0,$$

where it will easily be seen the real root lies between -12 and -16.

When $\theta=-12$,

$$q=\theta^2\frac{\theta+4}{\theta+6}=144\times\frac{8}{6}=192\,;$$

and when $\theta=-16$,

$$q=256\times\frac{12}{10}=307\tfrac{1}{5}.$$

Moreover, when q is a maximum or minimum, it will readily be found that $\theta^2+11\theta+24=0$; so that $\theta=-3$, or $\theta=-8$. Hence for the value of θ found from the above cubic $q<192$ and $\frac{\mathrm{D}}{\mathrm{J}^3}=1-\frac{128}{q}$ is *positive.*

(58) When $\mathrm{J}=0$, $\nu=0$; and when $\mathrm{L}=0$, $\nu=\infty$.

For these two cases it will be more simple to dispense with the auxiliary variable θ, and to revert to the original equation between J, K, L.

Accordingly, when $\mathrm{J}=0$, we find $8\mathrm{LK}^3-432\mathrm{L}^3=0$. Hence

$$\mathrm{L}=0,\ \text{or}\ \mathrm{K}^3=54\mathrm{L}^2,\ \text{i. e.}\ \left(\frac{-\mathrm{D}}{128}\right)^3=54\mathrm{L}^2;$$

so that the complete section of G made by the coordinate plane J becomes a straight line, viz. the axis of D, and a semicubical parabola whose axis is the negative part of D. When J is very nearly zero, ν becomes a positive or negative infinitesimal in the equation $\theta^4+4\theta^3=\nu$.

One real root of this equation is $\theta=\left(\frac{\nu}{4}\right)^{\frac{1}{3}}$.

The other is $-4+\delta$, where $\left(4(-4)^3+12(-4)^2\right)\delta=\nu$,

or
$$\delta=-\frac{\nu}{64}.$$

Now
$$\frac{\mathrm{K}^3}{\mathrm{L}^2}=\left(\frac{\theta+6}{\theta+4}\right)^3(\theta+4)^2=\frac{(\theta+6)^3}{(\theta+4)}.$$

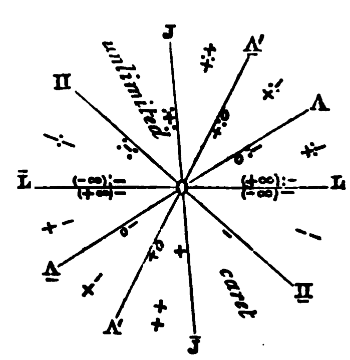
The first value of θ gives $\mathrm{K}^3=54\mathrm{L}^2$ to an infinitesimal *près*; the other value gives

$$\mathrm{K}^3=-\frac{512}{\nu}\mathrm{L}^2,$$

or, to an infinitesimal *près*,

$$\left(\frac{\mathrm{D}}{128}\right)^3=\frac{512}{\nu}\mathrm{L}^2;$$

so that D passes from $+\infty$ to $-\infty$, *i. e.* $\frac{\mathrm{J}^3}{\mathrm{L}}$ passes through zero.



(59) In the annexed figure([48]), the plane of the paper represents the plane of D, *i. e.* the plane for which $\mathrm{D}=0$; $\mathrm{JO\bar{J}}$ is the axis of J, OJ being the positive and $\mathrm{O\bar{J}}$ the negative direction; $\mathrm{LO\bar{L}}$ is the axis of L, OL being the positive and $\mathrm{O\bar{L}}$ the negative direction. In order to avoid any appearance of an attempt at a practicably impossible accuracy of drawing, I use straight lines to

([48]) I shall refer, when I have occasion to do so, to this figure, which contains a synopsis of the whole theory, under the name of the Dial figure.

denote cubical parabolas, and pay no attention whatever to relative magnitudes, but only to the order or progression of magnitudes, using the lines which are drawn in the figure not as *copies* but as *symbols* of the actual curves which are to be mentally imagined.

Thus the line $\mathrm{JO\bar{J}}$ is used to represent the straight line $L=0$; $\Lambda'\mathrm{O}\underline{\Lambda}'$ the cubical parabola $J^3=27\cdot2^{10}L$; $\Lambda\mathrm{O}\underline{\Lambda}$ the cubical parabola $J^3=2^{11}L$; $\Pi\mathrm{O}\underline{\Pi}$ the cubical parabola $J^3=-27L$([49]).

It will be observed that certain combinations of *plus*, *zero*, *minus*, positive and negative *infinity* are placed along the lines and inside the sectorial spaces. The meaning of these will be sufficiently obvious from what has preceded. They refer to the signs of the two values of D in the surface G for each point in the line or sector along or within which they are placed. At every point along the line $\mathrm{O\bar{J}}$, $\frac{D}{J^2}$ has only one value, and that positive; along $\Lambda'\mathrm{O}\underline{\Lambda}'$, $\frac{D}{J^2}$ has two values, one positive and the other zero. Along $\Lambda\mathrm{O}\Lambda$, $\frac{D}{J^2}$ has two values, one positive the other negative. Immediately below $\mathrm{\bar{L}OL}$ two values, one $+\infty$, the other finite and negative. Immediately above $\mathrm{\bar{L}OL}$ two values, one $-\infty$, the other finite and negative. Along $\Pi\mathrm{O}\underline{\Pi}$ one value, finite and negative.

Moreover D has been shown to be never zero, except along $\Lambda'\mathrm{O}\underline{\Lambda}'$, $\Lambda\mathrm{O}\underline{\Lambda}$. Hence it is obvious that *inside* $\Lambda'\mathrm{O\bar{J}}$ and the opposite sector D has two values, both *plus*; inside the next pairs of opposite sectors two values, one *plus*, the other *minus*; inside the next pair of sectors also two values, one plus, the other minus; inside the next pair of sectors two values both *minus*, and in the pair of sectors left vacant, for which $\nu<-27$, it has been shown that D becomes impossible.

([49]) It has been shown in the preceding articles that corresponding to the line $\mathrm{\bar{J}OJ}$ and to the line $\Pi\mathrm{O}\underline{\Pi}$, the vertical ordinate D of the amphigenous surface ($G=0$) has only one value positive for the former, negative for the latter; along the line $\Lambda'\mathrm{O}\underline{\Lambda}'$ two values, one positive the other negative; for the space between $\Lambda\mathrm{O}\Lambda'$, $\mathrm{\bar{L}OL}$ indefinitely near to the latter two values, one positively infinite, the other negative; and for the space indefinitely near to the same on the opposite of it, two values, one negatively infinite, the other negative. These results are collected and represented symbolically in the Table annexed.

$\bar{J}$	Λ'	Λ	$\bar{L}$		Π
	$+$	0	$(+\infty)$	$-$	
$+$					$-$
	0	$-$	$-$	$(-\infty)$	

Thus, corresponding to the upper sheet of G, we have the succession

	$+$	$+$	0	$(+\infty)$	$-$	$-$
and to the lower sheet						
	$+$	0	$-$	$-$	$(-\infty)$	$-$

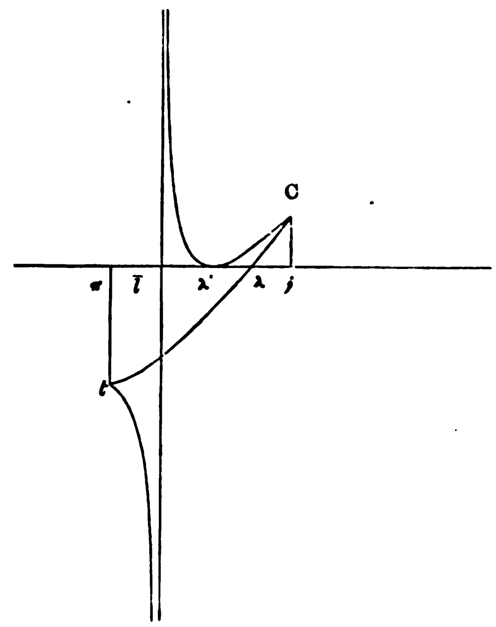
the two sheets coming together at a cuspidal edge above $\mathrm{\bar{J}OJ}$ and below $\Pi\mathrm{O\bar{\Pi}}$.

Moreover these are the only positions of the line revolving in the plane of D corresponding to which a change in the nature of D can take place, and thus we can without further examination fill up the Table, giving the nature of D for the intervening spaces, and may thus obtain the Table embodied in the *dial-figure* above, viz.,

$\bar{J}$		Λ'		Λ		$\bar{L}$			Π
	$+$	$+$	$+$	0	$+$	$(+\infty)$	$-$	$-$	
$+$									$-$
	$+$	0	$-$	$-$	$-$	$-$	$(-\infty)$	$-$	

(60) Thus it will be seen that the surface G consists of two opposite portions precisely similar and symmetrical in respect to the axis of D.

Let us trace that one of these whose ground-plan is comprised within the sector $\Pi O\bar{J}$. It will consist of two sheets coming to a cuspidal edge (a common parabola) in the superior part of the plane of L. The upper sheet will touch the plane of D in $O\Lambda$ ([60]), and, remaining above the plane of D, approach continually to the plane of J as an asymptotic plane. The lower sheet will cut the plane of D in $O\Lambda'$, pass under the plane of D, cut the plane of J, progress to a maximum distance from it, and then approach indefinitely to J as its asymptotic plane. This will become apparent by taking a vertical section of this portion, cutting the lines $O\bar{L}$, $O\bar{J}$; for the nature of the flow of the two branches of the section will evidently be as figured below, where j, λ, λ', l, π represent the points in which the lines $O\bar{J}$, $O\Lambda'$, $O\Lambda$, $O\bar{L}$, $O\Pi$ are cut by the secant plane. [It should be particularly noticed that this figure is only intended to exhibit, under its most general aspect, the nature of the flow of the two branches of the curve; it is drawn in other respects almost at random, and makes no pretension whatever to giving a representation of the actual form of the curve.]



No part of the surface G lies under or above the sector ΠOJ, except the axis of D. The cusp C, where the two branches meet, is the intersection of the cutting plane with the parabola $J=D^2$ lying in the plane of L, and there will be another cusp at t, the point of maximum recession from the plane of J.

(61) I now proceed to discriminate, by aid of this surface, the facultative from the non-facultative portion of space.

If in the expression for G as a function of J, K, L we substitute for K its value $-\frac{D}{128}+\frac{J^2}{128}$, we obtain $G=\frac{J}{(128)^4}D^4+$ terms involving only lower powers of D; so that, calling D_1, D_2 the two real values of D in the upper and lower sheets of G respectively corresponding to any point J, L,

$$G=J(D-D_1)(D-D_2)Q,$$

Q being a quantity essentially positive.

Hence when J is negative the *facultative* points in any line parallel to D will be those for which D lies between D_1, D_2, but when J is positive, the facultative points must be exterior to the segment D_1D_2; I denote this difference in the figure by placing a colon between the signs in each sector for which J is positive, indicating thereby that the facultative points lie between $+\infty$ and D_1, and between D_2 and $-\infty$; but where no

([60]) For the value of D for this sheet is zero all along $O\Lambda$, and positive on either side of it.

colon is interposed, then it is to be understood that the facultative points lie between D_1 and D_2. Thus, if we turn back for a moment to the section of G last drawn, the whole of the space included between the two branches and the asymptote is facultative, because up to the asymptote J is negative, and beyond the asymptote the whole of the space not included between the asymptote and the lower branch is facultative, because beyond the asymptote J becomes positive. Thus, then, we see that the whole of that portion of the plane which lies on the left-hand side of the entire curve is facultative, and the portion on the right-hand side of the same non-facultative; the curve separating facultative from non-facultative space as a coast-line, indefinitely extended, separates land from water; so that there is, as of course we might have anticipated, no break of continuity in passing through the plane J.

If we take a corresponding section of the opposite portion of space corresponding to the ground-plan $JL\underline{\Pi}$, it is obvious that precisely the contrary takes place, because the sign of J is opposite in the opposite sectors; so that what was facultative becomes non-facultative, and *vice versâ*.

(62) It is now clear that the whole of the facultative part of space is divided into three, and only three of the *regions* previously defined. One region will consist of that portion of it which is entirely under the plane of D: the second region will be so much of the upper portion as stands upon the acute sector $\bar{J}O\Lambda$; and the third of so much of the remainder of this portion as stands on the sector $\Lambda OJJO\underline{\Pi}$ [51]. Again, as regards the second region, the line $O\Lambda'$ is quite inoperative against its unity, because we have vertical ordinates above $O\Lambda'$ through which free communication can take place between the blocks over $JO\Lambda'$ and $\Lambda'O\Lambda$; but when we come to $O\Lambda$, where G touches the plane of D, there we have an effective line of demarcation between the adjoining blocks *above* the plane of D; for it is impossible to pass from one into the other without going under D and coming up again through that plane, or else descending to the line $O\Lambda$ and so meeting the plane of D [52].

[51] It will be borne in mind that the whole of the infinite prism, both above and below, standing on ΠOJ belongs to *facultative* space: the prism standing on the opposite section $\bar{J}O\bar{\Pi}$, or, to speak more strictly, on the *inside* of this last-named sector, is wholly unfacultative. The facultative line D which passes through O is completely isolated from the facultative portion which stands over $\Lambda O\bar{J}$, except at the point O (which we are forbidden to pass through if we would remain in the same region), and is of course a rectilinear edge to the facultative prism above referred to.

[52] Two superior regions we know *à priori* must exist to correspond respectively to the two cases of five and of one real root. Moreover we know *à priori* that two regions can only meet on the plane of D, and an inspection of the *dial-figure* shows that only $O\Lambda$ can be such line. Thus without completely making out the geometry of the question as regards the remarkable line ($J=0$, $L=0$) (the axis of D) which lies on the surface G, we may feel assured that the upper part of this line (which is easily found to belong to the 1-real-root region) cannot have any point except the origin in common with the 5-real-roots region, since otherwise these two regions would communicate along this line and merge into one. When it is considered that G is a surface of the ninth order in J, D, L, it will not appear surprising that some difficulty arises in forming a mental conception of certain of its local properties; on the contrary, the subject of wonder rather is that enough can be ascertained about it in a very brief compass to shed all the needful light upon the analytical problem which it illustrates.

(63) It remains only to fix the characters of the several regions; but this requires no calculation to effect, for we know that when D is negative there is one and only one pair of imaginary roots. This disposes of the first of the regions above enumerated. Again, we know that when L is positive so that the reduced form is the superlinear equation $ru^5+sv^5+tw^5=0$, u, v, w being *real* functions, D being also positive, there must be four imaginary roots, as follows from the theory of the second section. Hence the third region has for its character two pairs of imaginary roots; and consequently the only remaining region, the second described, must correspond to the case of no imaginary roots, since otherwise we should be absurdly assuming the impossibility in any case of a quintic equation having all its roots real.

(64) It may, however, be an additional satisfaction to see how the change of character comes to pass at the critical line OΛ from one to five real roots.

Along the line OΛ we have found that, calling the reduced form $ru_i^5+sv_i^5+tw^5$,

$$r=s \quad \frac{\tau}{\varrho}=\frac{rs}{st}=\frac{r}{t}=\theta+4=-4.$$

Hence the equation becomes

$$4u_i^5+4v_i^5+(u_i+v_i)^5=0,$$

u_i, v_i being of the form $\frac{-u+iv}{2}$, $\frac{-u-iv}{2}$, because L is negative.

Hence $u_i+v_i=0$, or

$$4(u_i^4-u_i^3v_i+u_i^2v_i^2-u_iv_i^3+v_i^4)+(u_i+v_i)^4=0,$$

$$\text{i. e. } 5u_i^4+10u_i^2v_i^2+v_i^4=0,$$

$$\text{i. e. } (u_i^2+v_i^2)=0;$$

so that there are two pairs of equal roots of $\frac{u_i}{v_i}$, viz. $\pm\iota$; to these values of $\frac{u_i}{v_i}$ correspond

$$\frac{u-iv}{u+iv}=\iota, \quad \frac{u-\iota v}{u+\iota v}=-\iota.$$

Hence

$$(1-\iota)u=(\iota-1)v, \text{ or } (1+\iota)u=(\iota+1)v;$$

so that the two pairs of equal roots of $\frac{u}{v}$ are ± 1, the outstanding root corresponding to $u_i+v_i=0$ being $\frac{u}{v}=0$.

Now, *still keeping upon the surface* G, which we know is facultative, let θ become $-8+4\epsilon$, where ϵ is an infinitesimal, then

$$\delta\left(\frac{J^3}{L}\right)=\delta\nu=(4\theta^3+12\theta^2)\delta\theta=-5120\epsilon;$$

also the supposed equation becomes

$$(4-4\epsilon)(u_i^5+v_i^5)+(u_i+v_i)^5=0,$$

or

$$(\iota v-u)^5-(\iota v+u)^5+8(1+\epsilon)u^5=0;$$

or, calling $\frac{v}{u}=\varrho$,

$$(\iota\varrho-1)^5-(\iota\varrho+1)^5+8(1+\epsilon)=0.$$

Let $\varrho=\pm 1+\sigma$, where σ is an infinitesimal. Hence

$$\left(-10(\pm\iota-1)^4+10(\pm\iota+1)^5\right)\sigma^2-8\epsilon=0,$$

or

$$20(1-10+5)\sigma^2-8\epsilon=0,$$

or

$$\sigma^2=\frac{-\epsilon}{10}=+\frac{1}{51200}\delta\left(\frac{J^3}{L}\right).$$

Hence calling σ_1, σ_2 the two values of σ, the four roots that at $O\Lambda$ were 1, 1, -1, -1 become $1+\sigma_1$, $1+\sigma_2$, $-1+\sigma_1$, $-1+\sigma_2$, when $\frac{J^3}{L}$ becomes varied by $\delta\left(\frac{J^3}{L}\right)$, and consequently become all real if $\frac{J^3}{L}$ is increased, and all imaginary if $\frac{J^3}{L}$ is decreased, *i. e.* become real or imaginary according as the line $O\Lambda$ sways towards or away from $O\bar{J}$, conformably with what has been shown on other grounds.

It will be noticed that in the line $O\Lambda$ produced in the opposite direction, *i. e.* along the line $O\underline{\Lambda}$, L being positive, the reduced form is

$$4(u^5+v^5)+(u+v)^5=0,$$

and the roots of $\frac{u}{v}$ become $\frac{u}{v}=-1$, $\frac{u}{v}=\pm\iota$, $\frac{u}{v}=\pm\iota$; so that, according to the canon laid down at the commencement of this discussion (see foot-note ([46])), no change in the character of the roots can possibly take place along $O\Lambda$, and accordingly we have seen that this curved line does not correspond to any demarcation of regions.

(65) It is easy to express the conditions to be satisfied by the coordinates of a point according as it lies in one or another of the three regions which have now been mapped out, and it is clear that we have the following rule:

When D is negative the equation has two imaginary roots.

When D is positive the equation has *no* imaginary roots, provided the two criteria J and $2^{11}L-J^3$ are both negative([53]); but if either of these is zero or positive, there are two pairs of imaginary roots([54]).

The duodecimal criterion-invariant, $2^{11}L-J^3$, and the invariants of the like order, $27\cdot 2^{16}L-J^3$, $-27L-J^3$, I shall henceforth call Λ, Λ', Π respectively. It has been just above shown that the three invariants J, D, Λ of the 4th, 8th, and 12th orders respectively are sufficient for ascertaining the character of the roots of the quintic to which they appertain.

([53]) Observe that this implies L also being negative; so that $2^{11}-\frac{J^3}{L}$ is positive and $\frac{J^3}{L}<2^{11}$.

([54]) ([a]) Observe that in general when $2^{11}L-J$ is zero there are no facultative points above the plane of D, but when J and $2^{11}L-J$, and consequently L and J are both simultaneously zero, a facultative right line springs into existence, viz. the axis of D extending both above and below the plane of D. The reduced form of equation (as previously demonstrated) corresponding to this singular line is $u^5\pm uv^4=0$.

([b]) It may further be noticed that on each side of the line $O\underline{\Lambda}$ the limits of D are between positive infinity and a positive quantity, and between negative infinity and a negative quantity; so that as we pass from $O\underline{\Lambda}$ to either side of it no facultative point can be found lying in the plane of D, showing that we cannot pass by a real infinitesimal variation of coefficients from an equation with two pairs of equal imaginary roots to an equation with a single pair of equal roots, as is apparent also on purely analytical grounds.

(66) The assertion that the *whole* of facultative space is divisible into three regions, in strictness requires a slight modification. It is obvious that the plane of D itself cannot be said to belong to any of the regions; and in order to make our theory quite complete, so as to furnish criteria applicable to equations having equal roots, and to enable us to distinguish between the case of the unequal roots being all three real, or two imaginary and one real, we must examine what takes place in this plane, and under what circumstances a passage from one point of it to another will or may be accompanied with a change of character in the roots.

If the roots of $f(x)=0$ are supposed to be a, a, c, d, e, where c, d, e are unequal, on varying the constants of fx in such a manner that the variation of the discriminant D is zero, the two equal roots a, a will remain equal. Now *in general* we have $\delta f(a)+f''(a)\frac{(da)^2}{2}=0$; if this, under the particular supposition made, continued to obtain, da would have two distinct values, and the two equal roots would cease to continue to be equal, contrary to hypothesis. Hence we see that $D=0$, $\delta D=0$ necessarily implies $\delta f(a)=0$ [55], and consequently $\delta f(a+da)$ is no longer δfa, but $\delta f'ada$; so that we obtain $da=0$, or $da=-\frac{2\delta f'a}{f''a}$, and no change of character in the five roots results. If, however, the original roots are a, a, c, c, e, then, as shown in the general case, δc will have two distinct values, which will be both real or both imaginary. Accordingly we see that in

[55] [a] This is a somewhat curious theorem (whether new or otherwise I know not) thus incidentally established in the text, viz. that if $D(f)$ represent the discriminant of f, and if $D(f)=0$ and $\delta D(f)=0$, then when $f=0$ we must have $\delta(f)=0$. The very simplest example that can be chosen will serve to illustrate this proposition. Let

$$f=ax^2+2bxy+cy^2.$$

Suppose

$$D(f)=ac-b^2=0,$$

and also

$$\delta D(f)=a\delta c+c\delta a-2b\delta b=0,$$

we have

$$\delta(f)=x^2\delta a+2xy\delta b+y^2\delta c.$$

Now if $f=0$ we may write $x=b$, $y=-a$, and δf becomes

$$\begin{aligned}&b^2\delta a-2ab\delta b+a^2\delta c\\&=b^2\delta a-2ab\delta b+2ab\delta b-ac\delta a\\&=(b^2-ac)\delta a=0,\end{aligned}$$

according to the theorem.

If we make $f=(x,1)^n$, D we know becomes a syzygetic function of f and f' $\left(\text{meaning by the latter } \frac{df}{dx}\right)$. Hence since δD vanishes when $fx=0$, $D=0$, and $\delta f(x)=0$, we learn that $\delta(D)$ is a syzygetic function of $(f, f', \delta f)$.

The theorem thus stated easily admits of extension to the higher variations of D, and so extended takes I believe the following form:

$$\delta^i(D)= \text{a syzygetic function of } (f, f', f'', \ldots . f^i, \delta f).$$

[b] Professor CAYLEY has since informed me that the theorem in [55] [a], about whose originality I was in doubt, will be found in SCHLÄFLI's 'De Eliminatione.' This is not the first unconscious plagiarism I have been guilty of towards this eminent man, whose friendship I am proud to claim. A much more glaring case occurs in a note by me in the 'Comptes Rendus,' on the twenty-seven straight lines of cubic surfaces, where I believe I have followed (like one walking in his sleep), down to the very nomenclature and notation, the substance of a portion of a paper inserted by SCHLÄFLI in the 'Mathematical Journal,' which bears my name as one of the editors upon its face!

the plane of D no change can possibly take place except in crossing the line which corresponds to a family of *two pairs* of equal roots.

(67) It has already been pointed out, in a foot-note, that we cannot pass facultatively from $O\underline{\Lambda}$ to either side of this curve line. Hence the separation of the plane of D into subregions can only take place along the line $O\Lambda$, and it remains but to ascertain the character of the points on either side of this line, which we know, therefore, *à priori*, must possess opposite characters, since otherwise we should be admitting the absurd proposition of its being impossible to construct an equation of the fifth degree having two equal roots without the remaining three being always of *one character*, either all real or all not real. Let us, then, ascertain the character of the points in OJ for which $D=0$, $L=0$, and J is positive(56).

Since $L=0$, the reduced form is $u^5+5euv^4+v^5$.

This equation, by DESCARTES's rule, must contain imaginary roots. Hence in the sector $\Lambda O\bar{J}$ the roots are all real, and in the remainder of the facultative portion of the plane (from which it may be noticed the sector $\underline{\Lambda}O\bar{J}$ is excluded) two of the roots are imaginary.

Along $O\Lambda$ itself there are, as already observed, two pairs of real equal roots, and along $O\underline{\Lambda}$ two pairs of imaginary equal roots. Thus, finally, we have the *complete rule.*

If D is negative, 2 roots imaginary.

If D is positive.

When J, Λ are both negative, 0 roots imaginary.

„ J, Λ are *not* both negative, 4 roots imaginary.

If D is zero.

$$\left.\begin{array}{ll}\text{When J, }\Lambda\text{ are both negative,} & \text{0 roots imaginary}\\ \text{„ J, }\Lambda\text{ are } \textit{not} \text{ both negative,} & \text{2 roots imaginary}\end{array}\right\}\text{1 pair of equal roots.}$$

$$\left.\begin{array}{ll}\text{„ J is negative, }\Lambda\text{ zero,} & \text{0 roots imaginary}\\ \text{„ J is positive, }\Lambda\text{ zero,} & \text{4 roots imaginary}\end{array}\right\}\text{2 pairs of equal roots.}$$

„ J is zero, Λ zero, 3 equal roots(56 bis).

Thus we see that our space referred to an arbitrary origin, and with the invariants J, D, Λ for the coordinates, has been first divided into facultative and non-facultative space. The former has then been resolved prismatically into two regions above and one below the plane of D. The plane of D itself, or the facultative part of it, into two

(56) We could not take J negative, for the facultative points of D in $\bar{J}$ are two positive quantities. See dial figure.

(56 bis) When $D=0$, $\Lambda=0$, there are two pairs of equal roots. If J is negative these pairs are both real. If J is positive they are both imaginary. When J is zero there are no longer two pairs, but a single triad of equal roots. This perfectly explains what at first sight has the air of a paradox, viz. that the discrimination between the two kinds of double equality of an apparently equal order of generality that may subsist between the roots of an equation, depends on the fulfilment or failure of an algebraical equality. The fact is, as shown above, that there are not, as commonly supposed, two, but three kinds of double equality, according as there are two pairs of real, two pairs of imaginary, or one triad of equal roots; and the last is a sort of transition case between the other two.

planar regions on opposite sides of the line $\Lambda O \underline{\Lambda}$; and again this line into two linear regions on either side of the origin O, which last corresponds to the case of three equal roots, and constitutes a region or microcosm in itself.

(68) It may as well be noticed here that the ambiguity of character in the points representing the different families of biquadratic forms when t and D are taken as the coordinates (and the same would be true if s and D were employed), which prevails when these points lie above the line $D=0$, equally obtains along this line itself. For the reduced form, when $D=0$, is $ax^4+4bx^3y+6cx^2y^2$. In that case, calling the determinant of transformation μ, we have $s=3\mu^{12}c^2$, $D=-\mu^{24}c^3$; and thus, whatever s and D may be, the character of the unequal roots is left undecided.

It may also be noticed that the blending of characters at the *origin* for the quintic form is not precisely of the same nature as that for the points above the line D in the biquadratic form; for at these points it is the cases of 4 and 0 imaginaries which become undistinguishable invariantively; whereas at the origin for quintics the reduced form becomes $ax^5+5bx^4y+10x^3y^2$, and the characters left undistinguished are those of 4 and of 2 imaginary roots—unless, indeed, we consider equal real roots as belonging indifferently to the class of real and imaginary; on which supposition all the three genders (so to say), masculine, feminine, and neuter, become blended together at that point. But if we consider equal real roots as exclusively of the real class, then the *origin* for quartics ceases to be epicene; for when there are three equal roots all of them must be real. Thus the origin in quintics is the only epicene point, and in quartics the only non-epicene point—understanding by epicene the blending of the masculine (4 *imaginary roots*) and feminine (*no imaginary roots*) characters.

(69) We may draw some further important inferences from an inspection of the "dial figure," or the section of facultative space which follows it.

Within the prism $JO\Lambda'$ [57] it will be observed D is always positive [58]. Hence, when J is negative and Λ' is negative, all the roots *must* be real, and the necessity for using the criterion D is done away with.

Again, when J and L are both negative, D is always negative, so that just two of the roots must be imaginary; and in this case also it becomes unnecessary to apply the criterion D.

Again, since there is no facultative prism corresponding to ΠOJ, the combination of L and D, both negative, can never occur unless $\underline{\Pi}$ is negative.

When L is negative, but J not negative, there may be two or four imaginary roots, according to the sign of D; but all the roots cannot be real.

(70) M. HERMITE's rule is as follows. For remarks on the relation between his Δ, J_2, J_3 and the J, K, L of this paper, see foot-note [36]. D is still the discriminant.

If D is negative (of course) two roots are imaginary.

If D is positive.

[57] By which I mean within the facultative prism of which $JO\Lambda'$ is the section made by the plane of D.

[58] The vertical section of facultative space in this supposition (see figure) is the area $\lambda C\lambda'$, which lies wholly *above* the plane of D.

When Δ is negative, $25\Delta^2-3.2^{10}J_3$ negative and J_3 positive, no roots are imaginary.

Δ is negative, $25\Delta^2-3.2^{10}J_3$ positive, $25\Delta^2-2^{11}J_3$ negative, no roots are imaginary.

Δ is positive, ... four roots are imaginary.

Δ is negative, $25\Delta^2-3.2^{10}J_3$ positive, $25\Delta^2-2^{11}J_3$ positive, four roots are imaginary(59).

(71) What is the effect of the condition "Λ *positive* or *negative*," as the case may be? or rather, how does this condition arise? The ground of it is simply this, that $\Lambda=0$ represents a cylindrical surface passing through the curve $O\Lambda$ (see dial figure), which curve is the *edge* of separation between two regions of opposite characters above the plane of D; the cylinder in question cuts the facultative position of space below the plane of D, but above this plane (except along the vertical line $J=0$, $L=0$, *i. e.* the axis of D) it passes exclusively through non-facultative space, never again cutting or meeting the surface G (the facultative boundary). Now it is clear that any surface whatever which passes through $O\Lambda$ and never meets the surface G above the plane $D=0$, except along the axis of D (*i. e.* the line $J=0$, $L=0$), may be substituted for Λ(60) and will serve equally well with Λ to distinguish between the masculine and feminine regions of space. $\Lambda-\rho JD$ will fulfil the condition of passing through the line $O\Lambda$,

(59) (a) The last four conditions ought to tally (and be in effect coextensive) with the two given by me for the case of D positive. The third of them, viz. the case of D positive Δ positive, I have already noticed, as inferences from the dial figure; for M. HERMITE's Δ, if not identical with my J, is at all events a positive multiple of it. I do not see how the case of Δ negative, $25\Delta^2-3.2^{10}J_3$ negative with D positive, is met by this system of criteria, since J_3, as well as Δ, may be negative consistently with the second condition. I have not been able to ascertain whether in the memoir such a combination is shown to be impossible. M. HERMITE admits, and indeed has been always aware of, the existence of a *lacuna* in the conditions above stated, which, I understand from him, it is his intention at some future time to fill up, and thus to complete his original solution. In the meanwhile he has been led to study the question from a different point of view, and has succeeded in obtaining a new set of criteria adequate to a complete solution of the question without calling in the aid of the principle of continuity. In this new system my Λ criterion is replaced by an invariant of the twenty-fourth degree, which is of course an objection as far as it goes, but in no wise diminishes the extraordinary interest that attaches to this altered mode of approaching the question, which bears to his original method and my own the same relation as the proof of STURM's theorem by the law of inertia for quadratic forms bears to that given by STURM himself.

(b) It is apparent from the fact that when $D=0$, G (M. HERMITE's I^2) becomes $(25\Delta^2-3.2^{10}J_3)(25\Delta^2-2^{11}J_3)^2$ (Camb. and Dub. Journal, vol. ix. p. 206), that the factors of this product are respectively of the form $a\Lambda'+bJD$, $c\Lambda+eJD$, a, b, c, e being certain numerical quantities. This gives rise to a singular reflection, *to wit*, that my own criteria for the case of D positive may be varied by the addition of a term λDJ to Λ (λ being a numerical coefficient), provided λ lies within certain limits, the form of the criteria in all other respects remaining unchanged. This proposition, fraught with the most important consequences, and not unlikely to lead to an entire revolution in the mode of attacking the general problem of criteria, I proceed to establish in the text.

(60) The surface to be employed will be $\Lambda-\rho JD$, which call M. Λ and M (or at least their upper portions above the plane of D) may then be regarded as the two sides of a sack, of infinite dimensions, open at the top, and seamed together at the bottom, along the curved line $D=0$, $\Lambda=0$, and in the vertical direction along the straight line $J=0$, $L=0$. The surface Λ serving as a screen of separation between the two upper regions, it is clear that M will serve equally well as such screen, provided no superior facultative points lie in the interior of the sack.

whose equation is $\Lambda=0$, $D=0$, and obviously is the only invariant not exceeding the twelfth order capable of so doing; it only remains to ascertain within what limits the numerical coefficient ϱ must be taken so as to fulfil the condition that the combined equations $\Lambda-\varrho JD=0$, $G=0$ shall be incapable of being satisfied by any positive value of D.

(72) Substituting for Λ and D their values, the equation to be combined with $G=0$ becomes

$$J^3-2^{11}L+\varrho J(J^2-128K)=0.$$

Returning to the notation of art. (55), and dividing by JK, this equation, when $G=0$, becomes

$$q-2^{11}\frac{q}{\nu}+\varrho(q-128)=0,$$

or

$$(1+\varrho)q\nu-2^{11}q=128\varrho\nu,$$

which, substituting for q, ν in terms of θ, gives

$$\frac{(1+\varrho)\theta^5(\theta+4)^2}{\theta+6}=2^{11}\frac{\theta^3+4\theta^2}{\theta+6}-128\varrho\theta^3(\theta+4),$$

or

$$(\theta+4)\theta^2(\theta+8)\big((\theta^3-4\theta^2+32\theta-256)+(\theta^3-4\theta^2-96\theta)\varrho\big)=0.$$

When $\theta+8=0$, $D=0$, see art. (57); neglecting, then, this factor, the condition to be satisfied is that when from the equation

$$(\theta+4)\theta^2\big((\theta^3-4\theta^2+32\theta-256)\varrho+(\theta^3-4\theta^2-96\theta)\big)=0$$

a value of θ has been deduced, the values of D corresponding thereto shall not be a positive finite quantity.

(73) Now

$$\frac{D}{J^2}=1-\frac{128(\theta+6)}{\theta^2(\theta+4)}=\frac{\theta^3+4\theta^2-128(\theta+6)}{\theta^2(\theta+4)}=\frac{(\theta+8)^2(\theta-12)}{\theta^2(\theta+4)}.$$

If $\theta=0$, or $\theta+4=0$, since D cannot be infinite, we have $J=0$, so that $\Lambda-\varrho JD$ becomes identical with the original criterion Λ. Hence the factor $(\theta+4)\theta^2$ in the quantity just above equated to zero may be neglected, and the condition to be fulfilled by ϱ is that if θ be any root of the equation

$$\frac{-\theta^3+4\theta^2-32\theta+256}{\theta^3-4\theta^2-96\theta}=\varrho,$$

θ shall be between -4 and 12; this equation on making $\theta=-4\varphi$, so that $1>\varphi>-3$, becomes

$$-\varrho=\frac{\varphi^3+\varphi^2+2\varphi+4}{\varphi^3+\varphi^2-6\varphi},$$

or, writing $\sigma=\frac{-1-\varrho}{4}$,

$$\sigma=\frac{2\varphi+1}{\varphi^3+\varphi^2-6\varphi}=\frac{2\varphi+1}{(\varphi-2)\varphi(\varphi+3)}.$$

(74) We wish to ascertain what values of σ will be incompatible with the violation of the limits just assigned to φ, and accordingly we must inquire what is the range of values assumed by σ when $\varphi>1$ or $\varphi<-3$; any values of σ *not* included within this range will be admissible for the purpose in view.

When $\phi < -3$, σ is always positive, and proceeds continuously from ∞ to 0 as ϕ passes from $-3-\iota$ (ι being infinitesimal) to $-\infty$. Consequently σ must not be allowed to have any positive value. When $\phi=\infty$, $\sigma=0$, and when $\phi=1$, $\sigma=-\frac{3}{4}$.

Hence, if no minimum value of σ (i. e. no maximum value of $-\sigma$) occurs between $\phi=1$, $\phi=\infty$, σ may have any value between 0 and $-\frac{3}{4}$; but if such a minimum value, $-M$, where $M>\frac{3}{4}$, should exist, the admissible values of σ would become more enlarged, and might be taken between 0 and $-M$.

Making then $\delta\sigma=0$, we have

$$\frac{2}{2\phi+1}=\frac{3\phi^2+2\phi-6}{\phi^3+\phi^2-6\phi},$$

or

$$4\phi^3+5\phi^2+2\phi-6=0;$$

which, substituting $1+\psi$ for ϕ, becomes

$$4\psi^3+17\psi^2+24\psi+5=0;$$

so that there can be no real root of the equation in ϕ greater than unity.

Hence the admissible values of σ are defined by the inequalities $0>\sigma>-\frac{3}{4}$,

$$\text{i. e. } \quad 0>-\frac{1+\varrho}{4}>-\frac{3}{4}, \qquad \text{or } 0>-(1+\varrho)>-3, \qquad \text{or } 2>\varrho>-1.$$

(75) We have thus obtained the complete solution of the problem of assigning invariantive criteria, such that their signs (positive, negative, or zero) shall serve to fix the nature of the roots. These criteria we now see are

$$J,\ D,\ \Lambda+\mu JD,$$

where μ (the negative, it must be noticed, of ϱ) is any numerical quantity intermediate between 1 and -2 (61).

(76) This important modification of the original criteria J, D, Λ I proceed to apply to the problem of obtaining the *simplest* and *most symmetrical* expression for the criteria in terms of the roots of the equation. Let a, b, c, d, e be the roots, and write

$$Z=\Sigma\{(a-b)^2(a-c)^2(b-c)^2(a-d)^4(a-e)^4(b-d)^4(b-e)^4(c-d)^4(c-e)^4\},$$

or say

$$Z=\Sigma\left\{\zeta(a,b,c)\begin{pmatrix} a & b & c \\ & de & \end{pmatrix}\right\} (62).$$

(61) Strictly it has only been proved that the surface $\Lambda+\mu JD$, which passes through the line Λ, D, contains no superior facultative points except those comprised in the line $L=0$, $J=0$. It is, I think, not difficult to see from this, that, if in the "*sack*" formed between Λ and $\Lambda+\mu JD$ any such points were contained, $L=0$, $J=0$, *i. e.* the axis of D would be a double or multiple line on the surface G, which is easily disproved by examining the algebraical form of G in art. 41, where K represents $\frac{-D+J^2}{128}$; any obscurity, however, which may be supposed to cling to this view is immaterial, as a demonstration capable of being followed *in plano* and leaving nothing to be desired in point of perspicuity, will be found in the Note appended to this Part.

(62) Agreeable to the meaning assigned to ζ and to a couple of rows of letters in my memoir on Syzygetic Relations, in the Philosophical Transactions.

Then, since each letter occurs the same number of times (12) in each term, Z will be an invariant.

(77) Again, suppose any two roots to become equal, say that e becomes d, then Z reduces to the single term $\zeta\,(a,\,b,\,c)\begin{pmatrix}a & b & c\\ d & d & \end{pmatrix}$; for any such factor as $\zeta(a,\,b,\,d)$ will be accompanied with the factor $\begin{pmatrix}a & b & d\\ c & d & \end{pmatrix}$ which vanishes.

If, further, we suppose any two of the letters a, b, c to become equal, then Z disappears entirely, since on that supposition $\zeta(a,\,b,\,c)$ vanishes. Hence Z is an invariant of the twelfth order, possessing the property of vanishing when the equation to which it belongs has two pairs of equal roots. Hence Z is of the form $p\Lambda+q\mathrm{JD}$, and it becomes of importance to ascertain the value of the ratio $\frac{q}{p}$.

To do this let us suppose $e=0$, $a=-b$, $c=-d$.

The ten terms in Z correspond to the following ten partitions:—

(1)	(2)	(3)	(4)
abc	abd	acd	bcd
de	ce	be	ae
	(5)	(6)	
	abe	cde	
	cd	ab	
(7)	(8)	(9)	(10)
ace	bde	ade	bce
bd	ac	bc	ad

(78) The corresponding values of the terms will be

$$4a^2(a^2-c^2)^2.16(a^2c^2)8^2(a^2-c^2)^4;\;\; 4a^2(a^2-c^2)^2 16a^2c^2(a^2-c^2)^4;\;\; 4c^2(a^2-c^2)^2.16a^2c^2(a^2-c^2)^4;$$
$$4c^2(a^2-c^2)^2 16a^2c^2(a^2-c^2)^4;\;\; 4a^6c^6(a^2-c^2)^2;\;\; 4c^6a^6(a^2-c^2)^2;\;\; (a-c)^2 256a^4c^4(a+c)^8;$$
$$a^2c^2(a-c)^2 256a^4c^4.a^4c^4(a+c)^8;\;\; (a+c)^2 256a^4c^4(a-c)^8;\;\; (a+c)^2 256a^4c^4(a-c)^8.$$

Collecting and simplifying these terms, and observing that

$$(a-c)^2(a+c)^8+(a+c)^2(a-c)^8=(a^2-c^2)\big((a+c)^6+(a-c)^6\big)=4(a^4-c^4)(a^4+14a^2c^2+c^4),$$

we find

$$\begin{aligned}\mathrm{Z}=128(a^2+c^2)a^8c^8(a^2-c^2)^6+4(a^2+c^2)a^6c^6(a^2-c^2)^8\\ +1024(a^2+c^2)(a^4+14a^2c^2+c^4)(a^2-c^2)^2a^{10}c^{10}.\end{aligned}$$

Let $(a^2-c^2)^2=p$, $a^2c^2=q$, and let $\mathrm{Z}_{\prime}=\dfrac{\mathrm{Z}}{(a^2+c^2)q^3}\cdot$ Then

$$\begin{aligned}\mathrm{Z}_{\prime}&=16384pq^3+1024p^2q^2+128p^3q+4p^4\\ &=2^{14}pq^3+2^{10}p^2q^2+2^7p^3q+2^2p^4.\end{aligned}$$

(79) We must now calculate J, D, L:

$$D=\frac{1}{5^5}\zeta(a,\ -a,\ c,\ -c,\ 0)$$

$$=\frac{1}{5^5}4a^6c^6(a^2-c^2)^4;$$

or writing

$$D=\frac{D}{q^3},$$

$$D_{\prime}=\frac{4}{5^5}p^2.$$

Again, for J. The form to which it belongs is

$$x^5-(a^2+c^2)x^3y^2+a^2c^2xy^4,$$

or

$$\left(1,\ 0,-\frac{a^2+c^2}{10},\ 0,\ \frac{a^2c^2}{5},\ 0\right)\!\!\left(x,\ y\right)^5;$$

so that the coefficients of the biquadratic Emanant are

$$x;\quad -\frac{a^2+c^2}{10}y;\quad -\frac{a^2+c^2}{10}x;\quad \frac{a^2c^2}{5}y;\quad \frac{a^2c^2}{5}x.$$

Hence the quadratic covariant becomes

$$\frac{a^2c^2}{5}x^2+\frac{2}{25}(a^2+c^2)a^2c^2y^2+\frac{3}{100}(a^2+c^2)^2x^2$$

$$=\frac{20a^2c^2+3(a^2+c^2)^2}{100}x^2+\frac{2}{25}(a^2+c^2)(a^2c^2)y^2.$$

Hence, by definition, J (which $=-4\times$ Discriminant of the Quadratic Covariant)

$$=-\frac{4}{1250}(a^2c^2)(a^2+c^2)\left(3(a^2-c^2)^2+32a^2c^2\right);$$

and making

$$J_{\prime}=\frac{J}{(a^2+c^2)q},$$

$$J_{\prime}=-\frac{6}{625}p-\frac{64}{625}q=-\frac{6}{5^4}p-\frac{2^8}{5^5}q.$$

Finally, to calculate L. The canonizant of the form

$$\begin{array}{cccc} 1 & 0 & A & 0 \\ 0 & A & 0 & B \\ A & 0 & B & 0 \\ y^3; & -xy^2; & x^2y; & -x^3 \end{array}$$

is

$$(A^3-AB)x^3+(B^2-A^2B)xy^2,$$

of which the discriminant is

$$-4\frac{AB^3}{27}(A^2-B)^4,$$

where

$$A = -\frac{a^2+c^2}{10}, \qquad B = \frac{a^2c^2}{5}.$$

Hence, by definition,

$$L = AB^2(A^2-B)^4 = -\frac{1}{125.10^9}(a^2+b^2)(a^6b^6)\left((a^2-b^2)^2-16a^2b^2\right);$$

and making

$$L_{,} = -\frac{L}{(a^2+c^2)q^3},$$

$$L_{,} = \frac{1}{125.10^9}(p-16q)^4 = -\frac{1}{5^{12}.2^7}(p^2-16q)^4.$$

(80) Now let us write

$$\frac{1}{5^{12}}Z = \eta L + eJD(^{63}) + \varepsilon J^3.$$

This gives

$$\frac{1}{5^{12}}Z_{,} = eqJ_{,}D_{,} + \varepsilon(p+4q)J_{,}^3 + \eta L,$$

or

$$4p^4+128q^2p^3+1024q^2p^2+16384pq^3$$
$$=125(256p^2q^2+24p^2q)e+(p+4q)(6p+64q)^3\varepsilon+\frac{1}{2^7}(p-16q)^4\eta),$$

by means of which identity we can obtain linear equations for finding the values of e, ε, η.

Thus, equating the coefficients of p^4, q^4, p^3q respectively, we obtain

$$4 = 216\varepsilon + \frac{1}{2^7}\eta,$$

$$4.64^3\varepsilon + \frac{16^4}{2^7}\eta = 0,$$

which gives $\eta = -2^{11}\varepsilon$ (as it ought to do),

$$128 = (24\times125)e + (4\times216+108\times64)\varepsilon + 64\ 2^{11}\varepsilon$$
$$= 3000e + 8800\varepsilon.$$

Hence

$$200\varepsilon = 4, \quad \varepsilon = \frac{1}{50}, \quad \eta = -\frac{2^{10}}{25},$$

$$3000e = 128 - 176 = -48, \quad e = -\frac{2}{125} \text{ and } \frac{e}{\varepsilon} = -\frac{4}{5}.$$

In order to verify the value of e, let $p=-4$, $q=1$; then, assuming the correctness of the above determinations, we ought to find

$$4^5 - 128.4^3 + 1024.16 + 16384 = 125(256.16 - 24.64).\frac{-2}{125} + \frac{1}{128}.160000.-2^{11}.\frac{1}{50},$$

or

$$2^{10}(1-8+16-64) = (-32.256+48.64) - \frac{8}{25}\times160000,$$

or

$$2^{10}(-55) = -5120 - 25.2048 = 2^{10}(-5-50),$$

which is right.

(63) Since Z has been proved to be of the form $p\Lambda + qJD$, we know *à priori* the value of $\frac{\varepsilon}{\eta}$; but I have thought it safer to determine ε, η independently, as an additional check upon the accuracy of the computations.

(81) Thus

$$-Z=\frac{5^{10}}{2}\left(2^{11}L-J^3+\frac{4}{5}JD\right)$$

$$=\frac{5^{10}}{2}\left(\Lambda+\frac{4}{5}JD\right);$$

and accordingly we have proved that $-Z$ is of the form $(\Lambda+\frac{4}{5}JD)$; and consequently, since $\frac{4}{5}$ lies within the allowed limits 1 and -2, $-Z$ may be used to replace Λ in the system of criteria.

(82) On examining the composition of Z, it will be found to have a remarkable relation to the lower criterion J.

J we know is, to a numerical factor *près*, of the form

$$\Sigma\{(d-e)^4\zeta(a, b, c)\},$$

ζ denoting, as usual, the squared product of the differences of the quantities which it affects; and Z, it will readily be seen, is of the form

$$\left(\zeta(a, b, c, d, e)\right)^2\Sigma\frac{1}{\zeta(a, b, c)(d-e)^4};$$

and the squared factor is always positive whatever the roots may be, for ζ is always real.

Hence the essential part of our rule thus transformed comes to this, that if

$$\Sigma\left(\zeta(a, b, c)\times(d-e)^4\right) \text{ and } \Sigma\left\{\left(\zeta(a, b, c)\right)^{-1}(d-e)^{-4}\right\}$$

are both of them positive, then when the discriminant is positive, so that the case of two of the five quantities a, b, c, d, e being conjugate and the other three real is excluded, and the choice lies between supposing all or only one of them real, we are able to affirm that they will all be real. Nothing could be easier than to multiply tests expressed by simple symmetric functions of differences of the roots, any infringement of which would contradict the hypothesis of all the five letters denoting real quantities; the difficulty consists in discovering a system of the least number that will suffice of decisive tests, such that not only their infringement shall contradict the hypothesis of imaginary roots, but whose fulfilment shall ensure the roots being all real. This is what has been proved to be effected by means of the invariants J, D, $\Lambda+\frac{4}{5}JD$.

In the case before us it is clear that when the roots are all real, each of the sums above written must be positive and greater than zero. That their being both positive and greater than zero is inconsistent with four of the letters a, b, c, d, e being imaginary would probably not admit of an easy direct demonstration.

Z we have seen is only a particular value of the general invariant $\Lambda+\mu JD$, which may be called M, where μ is an arbitrary constant limited to lie between 1 and -2.

(83) It may be well to notice the effect of using *as a criterion*, in conjunction with J and D, the value of M corresponding to either extreme value of μ. In such case, supposing M to become zero, it might for a moment appear doubtful to which region

that point representing the family of forms is to be referred. But since the doubt can only arise when J is negative and D positive, and since by hypothesis we have $\Lambda=-\mu JD$, we see that Λ takes the sign of μ; and consequently the sign of M, when it becomes zero, is to be understood as following the sign of μ, i. e. as positive when μ is 1 and negative when μ is -2.

(84) The method above given for ascertaining the nature of the roots of a quintic involves the use of only three criteria. It may be inquired how many would become needful in applying STURM's method. In the case of a cubic equation only the last of the two Sturmian criteria comes into use; and it seems therefore desirable to ascertain whether all four of the Sturmian criteria applicable to that case are required, or whether a smaller number are sufficient. I speak of four criteria, inasmuch as the leading terms fx and $f'x$ cannot be considered as such, their signs being fixed; so that we are at liberty to consider them both positive. Suppose all six Sturmian functions to be written down, including fx (a function of x of the fifth degree) and $f'x$, and let us characterize by the index (r, s) any succession of signs of the leading coefficients which contain r continuations and s variations, and which therefore will correspond to the case of $(r-s)$ roots.

The total number of cases to be considered are the sixteen following:

(5, 0)	+	+	+	+	+	+
(4, 1)	+	+	+	+	+	−
	+	+	+	+	−	−
	+	+	+	−	−	−
	+	+	−	−	−	−
(3, 2)	+	+	+	+	−	+
	+	+	+	−	+	+
	+	+	+	−	−	+
	+	+	−	+	+	+
	+	+	−	−	+	+
	+	+	−	−	−	+
(2, 3)	+	+	+	−	+	−
	+	+	−	+	+	−
	+	+	−	+	−	−
	+	+	−	−	+	−
(1, 4)	+	+	−	+	−	+

the successions corresponding to the indices (2, 3), (1, 4) will become impossible, as corresponding to a *negative* number of real roots. An inspection of the eleven cases corresponding to the indices (5, 0), (4, 1), (3, 2) will show that no *ternary* combination of signs in the third, fourth, and sixth columns belongs to any of the three characters (5, 0), (4, 1), (3, 2) exclusively, and consequently all four signs must be used; and therefore, if the method of STURM is employed, four criteria are indispensable for determining

effectually the character of the roots in an equation of the fifth degree[64]; whereas in the symmetrical and invariantive method which I have employed three have been seen to suffice.

In an equation of the seventh degree the case of 0 or 4 will be distinguishable from that of 2 or 6 imaginary roots by the sign of the discriminant, and then again the case of 0 from that of 4, and of 2 from that of 6, by other invariantive criterion-systems. So for an equation of the ninth degree, the first separation will be that of the 0, 4, or 8 case from that of 2 or 6; then it may be conjectured the 2 case will be invariantively separated from the 6, and the 0 or 8 from that of 4, and, finally, 0 and 8 from each other—the reduction of cases apparently depending upon the relation of the index of the equation to the powers of the number 2. This much we know (from art. 49) as matter of certainty, that no single criterion other than the discriminant can ever serve to distinguish one form of roots from another so that all other criteria must accompany each other in groups; and accordingly the scheme of criteria established in the foregoing investigation is in kind the very simplest *à priori* conceivable.

(64) (a) For an equation of the nth degree there are $n-1$ variable criteria, each capable of being $+$ or $-$, and thus giving rise to 2^{n-1} conceivable diversities of combination. The actual number possible, however, is considerably less than this; and I find by an easy method that this number, when n is odd, is $2^{n-2}+\dfrac{\Pi(n-1)}{2\left(\Pi\frac{n-1}{2}\right)^2}$, and when n is even, is $2^{n-2}+\dfrac{\Pi(n-1)}{\Pi\frac{n}{2}\,\Pi\left(\frac{n}{2}-1\right)}$.

(b) Not quite foreign to this subject is the inquiry as to the comparative probability of each different succession or each different family of successions possessing equivalent characters; and, as connected therewith, the comparative probability of a certain specified number of the roots of an equation of a given degree being real and the remainder imaginary. In the simplest case of a quadratic equation of which the coefficients are real but otherwise arbitrary, I find that upon the particular hypothesis of the squares of the three coefficients being limited by one and the same quantity, the probability of the roots being imaginary is $\frac{31}{72}-\frac{\log 2}{12}$, or $\cdot 3727932$, a little less than $\frac{3}{8}$, this being the value of the integral $\int_0^{\frac{1}{2}} db\int_{b^2}^{1} da\left(1-\frac{b^2}{a}\right)$; but we are not at liberty to infer from this the value of the probability in question when the coefficients are left absolutely unlimited. A case in point, as illustrating the effect of imposing a limit in questions of this kind, occurs in the problem (which I raised in my lectures on Partitions) of finding the probability that four points placed at hazard in a plane will form the angles of a reentrant quadrilateral, which Professor CAYLEY has shown is exactly $\frac{1}{4}$ in the absence of any limit. For if ABCD be the four points, and ABC the greatest of the four triangles of which they may be regarded as the angular points, and if through A, B, C be drawn lines parallel to BC, CA, AB respectively, the triangle $\alpha\beta\gamma$ so formed will be four times as great as ABC, and the point D must be somewhere within $\alpha\beta\gamma$, otherwise ABC would not be less than each of the three other triangles ABD, BCD, CAD; and consequently, since D must lie within ABC when the quadrilateral is reentrant, the probability in question is $\frac{ABC}{\alpha\beta\gamma}$, or $\frac{1}{4}$. Now it is easy to see, by using the very same construction, that if any contour whatever be imposed as a limit upon the positions of the four points, the probability referred to will exceed $\frac{1}{4}$ by a finite quantity—a result somewhat paradoxical, since *à priori* one would have supposed that the value of it for the case of *no limit* would be the *mean* of the values corresponding to the respective suppositions of every possible form of limit.

Received December 8, 1864.

Note on the arbitrary constant which appears in one of the criteria for distinguishing the case of four from that of no imaginary roots, and on the curve whose coordinates express the limiting relations of all the octodecimal invariants of a binary quintic, &c.

(85) The appearance of an arbitrary constant in a criterion is a circumstance so unexampled and remarkable that I have thought it desirable to give a more complete, or at least a more palpable proof of the validity of the substitution of $\Lambda+\mu JD$ for Λ than that furnished in the foregoing text, where some indistinctness arises from the difficulty of raising up in the mind a clear conception of the form of the amphigenous surface, and the two portions of space which it separates. That difficulty is entirely obviated, and the theory rendered palpable to the senses by the following investigation, where the problem is so handled as to involve the contemplation of two dimensions only of space. We have in general

$$D=J^2-128K,\quad \Lambda=2048L-J^3,$$

and at the amphigenous surface (see art. 57)

$$\frac{K}{J^2}=\frac{\theta+6}{(\theta+4)\theta^2},\quad \frac{L}{J^3}=\frac{1}{(\theta+4)\theta^3}.$$

Let

$$\theta=4\phi,\quad y=\frac{D}{J^2},\quad x=\frac{\Lambda}{J^3}.$$

Then

$$y=1-128\frac{\theta+6}{(\theta+4)\theta^2}=1-\frac{8\phi+12}{\phi^2(\phi+1)}=\frac{(\phi+2)^2(\phi-3)}{\phi^2(\phi+1)},$$

$$x=-1+\frac{2048}{(\theta+4)\theta^3}=-1+\frac{8}{\phi^4+\phi^3}=\frac{-(\phi+2)(\phi^3-\phi^2+2\phi-4)}{\phi^3(\phi+1)};$$

and consequently

$$\delta y=\frac{4(\phi+2)(4\phi+3)}{\phi^3(\phi+1)^2}\delta\phi,\quad \delta x=-\frac{8(4\phi+3)}{\phi^4(\phi+1)^2}\delta\phi,\quad \frac{\delta y}{\delta x}=-\frac{\phi^2+2\phi}{2}:$$

x, y may be considered as the coordinates (inclined to each other at any angle) of a curve of the fourth order, whose form, so far as is essential to the object in view, I proceed to determine. It is obvious, furthermore, that this curve will be a section of the amphigenous surface made by the plane $J=1$.

(86) This curve will be seen to consist of four branches, coming together in pairs or two cusps, so as to form two distinct horns[65]. For when $\phi=\infty$, or $\phi=-\frac{3}{4}$, δy, δx will

[65] [a] Since $$\phi^4+\phi^3-\frac{J^3}{256L}=0,$$
we see at once, from DESCARTES's rule, that ϕ can never have more than two real values to one of $\frac{L}{J^3}$, or consequently of x, and consequently there can only be two values of y to each of x.

[b] When $J=0$, the cusp of the left-hand horn and the two points of intersection of the dexter horn with the axis of L coincide at the origin; the upper branch of the latter and the linear of the former become the lower and upper parts of the axis of D, whilst the lower and upper branches of the same respectively become the left and right-hand branches of the semicubical parabola $27.2^{22}L^2=-D^3$.

each of them be zero. Hence there is a cusp at the point where $x=-1$, $y=1$[66], and again at the point where

$$x=-1+\frac{8\times 256}{81-108}=-76\tfrac{23}{27}, \quad y=\frac{(\frac{5}{4})^2(-\frac{9}{4})}{(\frac{3}{4})^2\frac{1}{4}}=-25.$$

(87) When $\varphi=0$, and also when $\varphi=-1$, x and y each become infinite; when $\varphi=\pm\infty$, x and y each become unity.

As φ passes from $+\infty$ to 0, δy is always negative, and x always positive; so that there will be one branch of the curve (CMP in Plate XXV.) extending from $x=-1$ to $x=+\infty$, for which y commences at $y=1$, which cuts the axis of x when $\varphi=3$, i. e. $x=-\frac{25}{27}$[67], and which, for the remaining part of its course, lies completely under the axis of x, becoming infinite when x becomes indefinitely great.

Again, as φ passes from $-\infty$ to -1, δx remains always positive, but δy is negative so long as $\varphi<-2$ vanishes when $\varphi=2$, and ever afterwards continues positive. Thus there is a second branch, COQ, which starts from the cusp C, touches the axis of x at the origin, ever afterwards remaining positive, and increasing up to positive infinity.

Since when $\varphi=\infty$, $\frac{\delta y}{\delta x}=\infty$, the tangent at C is parallel to the axis of y, and consequently the two branches which start from C lie on the same side of the tangent, so that the cusp at this point is of the second or ramphoidal kind; in Professor CAYLEY's nomenclature a cusp-node, and equivalent to the union of a double point and a cusp of the first kind.

There remains to account for the values of φ in the interval between 0 and -1. Throughout this interval y and x remain both of them negative, and $\frac{\delta y}{\delta x}=-\frac{\varphi(\varphi+2)}{2}$[68, 69] is always positive.

There will thus be two branches, in each of which x and y increase simultaneously in the negative direction, coming to a cusp necessarily of the first kind at the point $x=-76\frac{23}{27}$, $y=-25$, one branch corresponding to the values of φ from $-\frac{3}{4}$ to 0, the other to the values of φ from $-\frac{3}{4}$ to -1, both of them lying completely under the axis of x, and becoming respectively infinite at the extreme values of φ (0 and -1).

(66) Where this branch cuts the axis of y we have $\varphi^3-\varphi^2+2\varphi-4=0$, of which the real root will be a trifle less than $\frac{3}{2}$.

(67) From this it is easily seen that, whatever may be supposed to be the inclination of the axes x, y, the curve in question is rectifiable by means of elliptic functions; for $\frac{ds}{d\varphi}$ will be expressible as a rational function of φ and the square root of a quartic function of φ. The same conclusion will hold for the curve obtained by making J constant when J, together with any invariant of the eighth and any of the twelfth order, are taken as the coordinates of the amphigenous surface.

(68) To ascertain which range of φ gives the superior and which the inferior outline of the sinister horn, let $\varphi=\epsilon$, an infinitesimal; then $\varphi^4+\varphi^3=\epsilon^3$, and the other value of φ is $-1-\eta$, where $\eta=\epsilon^3$. Hence the two values of y corresponding to φ nearly zero and φ nearly -1 respectively will be

$$y_1=-\frac{12\epsilon}{\epsilon^3}=-\frac{12}{\epsilon^2} \quad \text{and} \quad y_2=\frac{-4(-1-\eta)}{\epsilon^3}=\frac{4}{\epsilon^3}.$$

Thus y_1 is negative for ϵ positive or negative, but y_2 is positive in the one case and negative in the other, as

Again,

$$2y-x+5=\frac{\phi+2}{\phi^4+\phi^3}\left((2\phi^3-2\phi^2+2\phi)+(\phi^3-\phi^2+2\phi-4)\right)+5$$

$$=\frac{\phi+2}{\phi^3}(3\phi^2-6\phi-4)+5=\frac{8\phi^3-16\phi-8}{\phi^3}.$$

Hence when $\phi=-1$, for which value of ϕ x and y both become infinite, $2y-x+5=0$; hence the straight line $2y-x+5=0$, represented by AN in the diagram, will be an asymptote to the curve (70).

If now we draw the straight line $2y-x=0$, represented by OB in the figure and join OC, the curvilinear triangle OCM will be completely under OC, and the curvilinear infinite sector XOP completely under OB.

(88) What we have to prove is, that so long as μ lies between 2 and 1, so long may $\Lambda+\mu$JD be substituted as a criterion in lieu of Λ, it being remembered that Λ only plays the part of a criterion when D is positive and J is not positive. Hence, since when J$=0$ $\Lambda+\mu$JD and Λ coincide, we have only to show that, so long as D is positive and J is negative, $\Lambda+\mu$JD and Λ will bear the same sign for all such values of J, D, L as constitute a facultative system, *i. e.* coordinates to a facultative point in space.

Now at any facultative point G (the function of the amphigenous surface), or say rather G(J, K, L)>0, or $\frac{1}{J^3}\,G\left(1,\ \frac{D}{J^2},\ \frac{L}{J^3}\right)>0$, and consequently considering $\frac{D}{J^2}$, $\frac{L}{J^3}$ as the coordinates of a plane curve, the line $G\left(1,\ \frac{D}{J^2},\ \frac{L}{J^3}\right)=0$ (the sign of J being fixed) will separate those points for which J, K, L constitute a facultative system from those

---

already seen for the dexter horn. We see also that y_2 becomes indefinitely greater than y_1, so that it is the value of ϕ near to -1 which gives the inferior branch; and consequently the superior branch of the sinister horn belongs to the range from $-\frac{3}{4}$ to 0, and the inferior to the range from $-\frac{3}{4}$ to -1.

(69) It may further be noticed that each horn so called is a true horn, being destitute of any point of contrary flexure, except at infinity; for otherwise we should have

$$\frac{d^2y}{dx^2}=\frac{d\phi}{dx}\cdot\frac{d\frac{dy}{dx}}{d\phi}=-\frac{d\phi}{dx}(\phi+1)=\frac{(\phi+1)^2\phi^4}{8(4\phi+3)}=0,$$

which implies $\phi=0$ or $\phi=-1$, for each of which values of ϕ x and y become infinite. It will be seen hereafter that it is only for the value corresponding to $\phi=0$ that there does exist at infinity a point of inflexion.

(70) The two points where the asymptote cuts the curve will be found by writing

$$\frac{\phi^3-2\phi-1}{\phi+1}=\phi^2-\phi-1=0,$$

which gives

$$\phi=\frac{1\pm\sqrt{5}}{2}.$$

The superior sign corresponds to a point x, y in the inferior branch of the dexter horn, and the lower sign, for which $\phi>-\frac{3}{4}$, to the superior branch of the sinister horn. It is easy to see that there can be no other asymptote; for x, y only become infinite when $\phi=-1$, or $\phi=0$; so that if $\lambda x+\mu y+\nu$ is an asymptote, it must contain $(\phi+1)^2$, or ϕ^2 as a factor. The first condition is only satisfied when $\lambda:\mu:\nu::-1:2:5$; and the latter cannot be satisfied at all.

in which J, K, L constitute a non-facultative one. But the curve above traced is obviously a homographic derivative of that line $\left(\text{for G is the resultant of } \frac{K}{J^2}=\frac{\theta+6}{(\theta+4)\theta^2},\ \frac{L}{J^3}=\frac{1}{(\theta+4)\theta^3}\right)$.

Hence this latter curve will also separate systems of values of J, D, Λ corresponding to facultative from those corresponding to non-facultative points. Moreover when J is negative and D positive, it has been shown (see dial figure) that the values of D (in facultative systems) corresponding to finite values of J are *limited* in magnitude; hence, upon the same suppositions, facultative systems of J, D, Λ will correspond to the interior and contour of the curve we have been considering.

(89) Accordingly, since D is supposed positive, our sole concern will be with the curvilinear triangle CMO and the infinite sector QOX, and we have to show that for all points not exterior to those areas Λ and $\Lambda+\mu JD$ have the same sign; that is to say, $1+\mu\frac{JD}{\Lambda}$, or $1+\mu\frac{y}{x}$ is *positive.*

When y and x have opposite signs (as is the case in the triangle CMO), all negative values of μ, and when y and x have the same signs (as is the case in the sector XOQ), all positive values of μ obviously make $1+\mu\frac{y}{x}$ positive. But furthermore $\frac{y}{x}$, which is -1 for the line OC, is greater than -1 for all points in the triangle just named; and again, $\frac{y}{x}$, which is $\frac{1}{2}$ for OB (the parallel to the asymptote through O), will be less than $\frac{1}{2}$ for all points in the sector QOX. Thus, then, as regards points either in the triangle or in the sector, $\frac{y}{x}$ is always intermediate between -1 and $\frac{1}{2}$; so that when μ lies between 1 and -2, $1+\mu\frac{y}{x}$ will be always positive, and Λ and $\Lambda+\mu JD$ will bear the same sign O, so that $\Lambda+\mu JD$ may be used to replace Λ as a criterion. Q.E.D.

(90). It is apparent from the nature of the preceding demonstration that Λ may be replaced by an invariant containing not one merely, but an infinite number of arbitrary constants (limited), provided we are indifferent to the degree which the substitute for Λ may assume. To this end we have only to draw any algebraical curve $F(x, y)=0$ passing through the origin, and with its parameter subject to such conditions of inequality as will ensure the mixtilinear triangle and sector COM, XOQ lying on opposite sides of the curve. If its degree be n, the number of parameters in F left arbitrary within limits will be $\frac{n^2+3n-2}{2}$, and $\epsilon F(\Lambda, JD)$, where ϵ means one of the two quantities $+1$ or -1, may be used as a criterion in lieu of Λ. For instance, a common parabola with its axis coincident with that of x and passing through O will obviously serve as a screen between these figures; its equation will be $y^2-x=0$, and the invariant $D^2-J\Lambda$, which is of the sixteenth degree in the coefficients, will serve together with J and D to fix the nature of the roots; so in general we may obtain invariants of any degree of the form $4i$ from twelve

upwards. Thus M. HERMITE, by a method not introducing the notion of continuity, has found one of the twenty-fourth degree, which he has been so obliging as to communicate to me, viz. $(D_1-5\Delta J_2)^2+(9D-25\Delta^2)J_3^2$, where $D_1=16J_3+25\Delta J_2$; and D is his discriminant, which I cannot safely attempt to express in terms of x, y for want of a certain knowledge of the arithmetical relations between his Δ, J_2, J_3 D, and my own J, K, L; but were this transformation effected, the curve so represented must, *ex necessitate*, pass clear between the triangle and sector above referred to, or else the invariant in question could not be a criterion. I have ascertained without difficulty that it passes through the origin and represents one of the principal species of NEWTON's diverging parabolas.

(91) The curve which we have been discussing will, on reference to PLÜCKER's 'Algebraischen Curven,' p. 193, be seen to belong to his sixteenth species of curves of the fourth order having two double points; but as in reality one of these is tantamount to the union of two, it may be considered as having three, the maximum possible number of such points, and consequently comes under the operation of CLEBSCH's rule, given in the last Number of CRELLE's Journal, and accordingly its coordinates have been seen to be rational functions of a single variable. The equation connecting x, y may of course be obtained by means of a simple and obvious substitution operated upon the G of art. 41, or it may be found directly by writing

$$\frac{x+1}{8}=\xi=\frac{1}{\varphi^4+\varphi^3}\quad\frac{y-1}{4}=\eta=-\frac{2\varphi+3}{\varphi^3+\varphi^2},$$

whence we obtain

$$\varphi^4+\varphi^3-\frac{1}{\xi}=0, \quad \ldots \ldots \ldots \quad (1)$$

$$2\varphi^2+3\varphi+\frac{\eta}{\xi}=0. \quad \ldots \ldots \ldots \quad (2)$$

Calling $\varphi_1\varphi_2$ the two roots of equation (2), making

$$\left(\varphi_1^4+\varphi_1^3-\frac{1}{\xi}\right)\left(\varphi_2^4+\varphi_2^3-\frac{1}{\xi}\right)=0,$$

and substituting the values of the symmetric functions of φ_1, φ_2 found from the same equation, we obtain without difficulty

$$\eta^4-\xi\eta^3-8\xi\eta^2+36\xi^2\eta+16\xi^2-27\xi^3=0$$

for the equation in question. The curve thus denoted I propose to call the Bicorn. Its figure is given in Plate XXV., in which ξ, η are taken at right angles, but they may of course be supposed to be inclined at any angle whatever. If we now assume at pleasure any two new axes U, V in the place of the Bicorn, the coordinates u, v will be always respectively proportional to two invariants of the twelfth order of the given quintic, whose particular forms will depend upon the positions of the two new axes so taken. If one of these axes, say that of u, be made coincident with the axis of ξ, v will be proportional to JD, and u to some other invariant of the twelfth degree. When

this is the case, then in general v, as u travels from one end of infinity to the other, will sometimes have four, and sometimes two, or else sometimes two and sometimes no real values, as will be obvious by inspection of the figure. There is, however, one direction of the axis of v which will cause v in all cases to have two, and only two real values. This direction is that of the line joining the two cusps. At the node-cusp, for which $\phi=\infty$, $\xi=0$, $\eta=0$; at the other cusp, for which $\phi=-\frac{3}{4}$, $\xi=-\frac{256}{27}$, $\eta=-\frac{32}{8}$. Hence the equation of the joining line is $9\xi-8\eta=0$. Now $\frac{\mathrm{K}}{\mathrm{J}^2}=-\frac{\eta}{32}$, $\frac{\mathrm{L}}{\mathrm{J}^3}=\frac{\xi}{256}$. Hence along this line $9\mathrm{L}+\mathrm{JK}=0$; and consequently, if the axis of v be taken parallel to this line and passing through the origin, whilst u is proportional to $9\mathrm{L}+\mathrm{JK}$, v will be proportional to JD; and thus we see that for every value of $9\mathrm{L}+\mathrm{JK}$, which is HERMITE's J_3 (see foot-note (34)(e)), D at the amphigenous surface (*i. e.* when $\mathrm{G}=0$, and therefore when HERMITE's $\mathrm{I}=0$) will always have two, and only two real values. This perfectly agrees with M. HERMITE's conclusion (71), and in an unexpected manner confirms the correctness of the concordance established, in the foot-note cited, between his J_3 and my J, K, L. Had M. HERMITE employed any duodecimal invariant whatever other than J_3, a mere inspection of the Bicorn shows that a similar conclusion could not have obtained.

(92) The intersections of the curve whose equation is written in the preceding article with infinity evidently lie in the lines $\eta^2=0$, $\eta-\xi=0$. This latter is the equation to a line parallel to the asymptote which touches the highest and lowest of the four branches of the curve, and corresponds to the value -1 of ϕ. Thus we see that there is a point of inflexion corresponding to the point at infinity at which the second and third branches of the Bicorn may be conceived to unite. It is easy to show that the Bicorn has no double tangent; for we have seen that

$$\frac{dy}{dx}=-\frac{\phi^2+2\phi}{2},$$

and consequently the values of ϕ corresponding to the two supposed points of contact may be regarded as the two roots ϕ_1, ϕ_2 of the equation $\phi^2+2\phi+2\lambda=0$, and we shall have

$$-\frac{2\phi_1+3}{\phi_1^3+\phi_1^2}+\frac{2\phi_2+3}{\phi_2^3+\phi_2^2}=\lambda\left(\frac{2}{\phi_1^4+\phi_1^3}-\frac{2}{\phi_2^4+\phi_2^3}\right),$$

$$i.\ e.\ -(2\phi_1+3)(\phi_2^3+\phi_2^2)+(2\phi_2+3)(\phi_1^3+\phi_1^2)=(\phi_2^4+\phi_2^3)-(\phi_1^4+\phi_1)^3,$$

or

$$4\lambda\,.\,(-2)+4\lambda+3(4-2\lambda)+6\big(-2(4-4\lambda)+(4-2\lambda)\big)=0,$$

or

$$(-8+4-6+8-2)\lambda+12-6-8+4=0,$$

$$i.\ e.\ -4\lambda+2=0,\ \lambda=\tfrac{1}{2},\ \phi^2+2\phi+1=0,$$

and the two values of ϕ coincide, contrary to hypothesis.

It is also easy to find its class; for when $\frac{d\eta}{d\xi}$ corresponds to any point in which the

(71) Lemma 3, p. 202, Cambridge and Dublin Journal, vol. ix.

curve is met by a tangent drawn from the point whose ξ, η coordinates are a, b, we have

$$\left(\frac{2\varphi+3}{\varphi^3+\varphi^2}+b\right)+\frac{d\eta}{d\xi}\left(\frac{1}{\varphi^4+\varphi^3}-a\right)=0;$$

but

$$\frac{d\eta}{d\xi}=2\frac{dy}{dx}=-(\varphi^2+2\varphi);$$

hence

$$\frac{(2\varphi+3)-(\varphi+2)}{\varphi^3+\varphi^2}+(\varphi^2+2\varphi)a+b=0;$$

hence

$$a\varphi^4+2a\varphi^3+b\varphi^2+1=0;$$

and φ having four values, four tangents (real or imaginary) can be drawn to the Bicorn from every point in its plane. It is thus of the fourth order, fourth class, possesses a common cusp and a cusp-node, no double tangent, and one point of inflexion at infinity. These results accord with those given by PLÜCKER (Algebraischen Curven, p. 222).

(93) The canonical form of the equation to the Bicorn is $(pr+q^2)^2+pq^3=0$, as seen in PLÜCKER, p. 193, where $p=0$, $r=0$, $q=0$ will obviously be the equations to the tangent at the node-cusp, to the tangent at the common cusp, and to the line of junction of the two cusps. This leads to a remarkable transformation of the invariant G of art. (41). Thus we may write $p=\xi$, $q=\mu(9\xi-8\eta)$; and to find r, we must draw the tangent to the lower cusp, for which $\varphi=-\frac{3}{4}$, which gives

$$\xi=-\frac{256}{27},\quad \eta=-\frac{32}{3},\quad \frac{d\eta}{d\xi}=-\frac{15}{16}\text{(72)};$$

consequently we may write $r=\lambda(144\eta-135\xi+256)$, and then proceed to satisfy, by assigning suitable values to λ, μ, ν, the identity

$$\begin{aligned}&\left(\lambda(144\eta\xi-135\xi^2+256\xi)+\mu^2(8\eta-9\xi)^2\right)^2+\mu^3\xi(8\eta-9\xi)^3\\ &=\nu(\eta^4-\eta^3\xi-8\eta^2\xi+36\eta\xi^2+16\xi^2-27\xi^3)=\nu\,.\,2^{20}\mathrm{G}.\end{aligned}$$

On performing the necessary calculations it will be found that

$$\lambda=-\frac{1}{2^{12}},\quad \mu=\frac{1}{2^6},\quad \nu=\frac{1}{2^{12}}.$$

Hence we see that $\mathrm{J^3G}$ may be expressed under the form $(\mathrm{LL_1}+c\mathrm{J}_3^2)^2+e\mathrm{LJ}_3^3$, where $\mathrm{L_1}$ is a new duodecimal invariant, and c, e are two known numbers; in fact

$$\mathrm{J^3G}=\left(\mathrm{L}(18\mathrm{JK}+135\mathrm{L}^2-\mathrm{J^3L})+(\mathrm{JK}+9\mathrm{L})^2\right)^2+64\mathrm{L}(\mathrm{JK}+9\mathrm{L})^3.$$

I am indebted to my friend Dr. HIRST for these references to the immortal work of PLÜCKER.

(94) The existence has been demonstrated of a linear asymptote which is a tangent

(72) I find, by a calculation which offers no difficulty, that the value of φ at the point where this tangent cuts the curve will be given by the equation

$$-256\varphi^4-256\varphi^3+288\varphi^2+432\varphi+135=0;$$

and taking away the factor $(4\varphi+3)^3$ which belongs to the cusp, there remains $\varphi=\frac{5}{4}$, which corresponds to a point in the lower branch of the superior horn.

t infinity to the first and fourth branch. A cubic asymptote touches the intermediate ranches in the point at infinity corresponding to $\varphi=0$. For we have

$$\xi=\frac{1}{\varphi^3(1+\varphi)}=\frac{1}{\varphi^2}(1-\varphi+\varphi^2-\varphi^3\dots);$$

nd writing v for $-\eta$,

$$v=\frac{3+2\varphi}{\varphi^2(1+\varphi)}=\frac{1}{\varphi^2}(3-\varphi+\varphi^2-\varphi^3\dots),$$

$$v^{\frac{1}{2}}=\frac{3^{\frac{1}{2}}}{\varphi^3}\left(\varphi^2-\frac{1}{6}\varphi^3+\dots\right),\quad v^{\frac{3}{2}}=\frac{3^{\frac{1}{2}}}{\varphi^3}\left(3-\frac{3}{2}\varphi+\frac{13}{8}\varphi^2-\frac{27}{16}\varphi^3\dots\right).$$

Ience we may determine

A, B, C, D so that $Av^{\frac{3}{2}}+Bv+Cv^{\frac{1}{2}}+D-\xi$ shall $=\lambda\omega^n+\mu\omega^r+\dots,$
nd I find

$$A=\frac{1}{3^{\frac{1}{2}}},\qquad B=-\frac{1}{6},\qquad C=\frac{7}{72},\qquad D=-\frac{2}{9}.$$

'hus the cubic asymptote will have for its equation

$$\left(\xi+\frac{1}{6}v+\frac{2}{9}\right)^2=3v\left(\frac{v}{9}+\frac{7}{72}\right)^2,$$

/hich is a divergent cubic parabola with a conjugate point, viz. the point for which

$$v=-\frac{7}{8},\qquad \xi+\frac{1}{6}v+\frac{2}{9}=0,\qquad \text{or } \eta=\frac{7}{8},\qquad \xi=-\frac{9}{128}.$$

(95) It is obvious from the preceding article, that we may expand ξ in terms of v by descending series

$$\xi=Av^{\frac{3}{2}}+Bv+Cv^{\frac{1}{2}}+D+\frac{E}{v^{\frac{1}{2}}}+\dots$$

we may also obtain an ascending series for ξ in terms of v which will exhibit the e of the curve of the cusp-node at which point $\varphi=\infty$. Let $\varphi=\frac{1}{\omega}$, then

$$\xi=\frac{1}{\varphi^3(\varphi+1)}=\frac{\omega^4}{1+\omega}=\omega^4(1-\omega+\omega^2-\omega^3\dots),$$

$$v=\frac{2\varphi+3}{\varphi^2(\varphi+1)}=\omega^2\left(\frac{2+3\omega}{1+\omega}\right)=\omega^2(2+\omega-\omega^2+\omega^3\dots).$$

$$\begin{aligned}
v^2&=\omega^4(4+4\omega-3\omega^2+2\omega^3\dots),\\
v^{\frac{5}{2}}&=\omega^4(4\sqrt{2}\omega+5\sqrt{2}\omega^2-\tfrac{25}{8}\sqrt{2}\omega^3\dots),\\
v^3&=\omega^4(8\omega^2+12\omega^3\dots),\\
v^{\frac{7}{2}}&=\omega^4(\sqrt{2}\omega^3\dots),\\
\&c.&=\&c.
\end{aligned}$$

from which we may easily deduce

$$\xi=2\left(\frac{v}{2}\right)^{2}-\left(\frac{v}{2}\right)^{\frac{5}{2}}+\frac{7}{4}\left(\frac{v}{2}\right)^{3}-\frac{109}{32}\left(\frac{v}{2}\right)^{\frac{7}{2}}, \text{ \&c.,}$$

in which it will be observed that the indices of the powers of v are precisely complementary to those in the preceding expansion, the two series of indices together comprising all multiples of $\frac{1}{2}$ from positive to negative infinity.

(96) We now see how, supposing the curve to be given with ξ and η at any angle, the axes corresponding to $\frac{\mathrm{K}}{\mathrm{J}^2}$, $\frac{\mathrm{L}}{\mathrm{J}^3}$ may be defined: viz., the origin of coordinates will be at the cusp-node; η, along which $\frac{\mathrm{K}}{\mathrm{J}^2}$ is reckoned, will be in the direction of the tangent at that point; and ξ, along which $\frac{\mathrm{L}}{\mathrm{J}^3}$ is reckoned, will be the axis of that common parabola which at the same point has the closest contact with the given curve.

It seems desirable, with a view to a more complete comprehension of the form of the amphigenous surface, i. e. the *limiting surface* of invariantive parameters, to ascertain the nature of the systems of plane sections of it, parallel to each of the three coordinate planes. The sections parallel to J, which are curves of the fourth order, have been already satisfactorily elucidated. It remains to consider briefly the sections parallel to J and D, which will be curves of the ninth order.

(97) When L is constant, writing $\mathrm{J}=z$, $\mathrm{D}=y$, where for facility of reference we may conceive y horizontal and z vertical, and making $\mathrm{L}=\frac{k^3}{256}$, we have

$$z^3=k^3\varphi^3(\varphi+1),\quad y=z^2\frac{(\varphi+2)^2(\varphi-3)}{\varphi^2(1+\varphi)}=k^2\frac{(\varphi-3)(\varphi+2)^2}{(1+\varphi)^{\frac{1}{3}}},$$

$$\frac{\delta y}{y}=\frac{2}{3}\,\frac{(\varphi-1)(4\varphi+3)}{(\varphi+2)(\varphi-3)(\varphi+1)}\delta\varphi,\quad \frac{\delta z}{z}=\frac{1}{3}\,\frac{4\varphi+3}{\varphi(\varphi+1)}\delta\varphi,\quad \frac{\delta z}{\delta y}=\frac{1}{2k}\,\frac{(\varphi+1)^{\frac{2}{3}}}{(\varphi-1)(\varphi+2)}\delta\varphi,$$

when	$\varphi=-1$,	$z=0$,	$y=\infty$,
,,	$\varphi=-\frac{3}{4}$,	$\delta y=0$,	$\delta z=0$,
,,	$\varphi=0$,	$z=0$,	$y=-12k^2$,
,,	$\varphi=1$,	$\frac{\delta y}{\delta z}=0$,	
,,	$\varphi=+\infty$,	$z=+\infty$,	$y=+\infty$,
,,	$\varphi=-2$,	$y=0$,	$\frac{\delta z}{\delta y}=\infty$,
,,	$\varphi=-\infty$,	$z=+\infty$,	$y=+\infty$.

Hence it appears that the curve consists of three branches, two coming together at an ordinary cusp at the point corresponding to $\varphi=-\frac{3}{4}$, and the third completely separate. The nature of the sign of k does not affect the nature of the curve. If, for greater clearness, k be supposed positive, the first branch, having the negative part of

the axis of y for its asymptote, lies entirely in the $-y$, $-z$ quadrant, and is always convex to the axis of y; the second branch, joining the first at a cusp corresponding to $\varphi=-\frac{3}{4}$, is concave to the origin, cuts the axis of y negatively and of z positively, and goes off to infinity; the third branch, having the positive part of the axis of y for its asymptote, lies in the $+y$, $+z$ quadrant, is always convex to the axis of z, which it touches at a point below that where it is cut by the second branch, and also goes off to infinity, lying entirely under the second branch. A straight line, according to the direction in which it is drawn, may cut the curve in one, three, or five real points.

(98) When D is constant, writing $\mathrm{J}=z$, $\mathrm{L}=x$, we have

$$z^2=\mathrm{D}\frac{\varphi^2(\varphi+1)}{(\varphi+2)^2(\varphi-3)},\quad x=\frac{\mathrm{D}z}{(\varphi-3)\varphi(\varphi+2)^2}.$$

The form of the curve changes with the sign of D. For sections parallel to and above the plane of D, we may make

$$\mathrm{D}=c^2,\quad \tau^2=\frac{\varphi+1}{\varphi-3},\ \text{ or }\ \varphi=\frac{3\tau^2+1}{\tau^2-1};$$

then the complete equation-system to the curve will be

$$z=c\tau\frac{3\tau^2+1}{5\tau^2-1},\quad x=c^3\tau\frac{(\tau^2-1)^4}{4(5\tau^2-1)^3},$$

it being unnecessary to affect c with a double sign, since z and x change their signs with that of τ.

Also

$$\frac{\delta x}{x}=\frac{(\tau^2+1)(15\tau^2+1)\delta\tau}{\tau(\tau^2-1)(5\tau^2-1)},\qquad \frac{\delta z}{z}=\frac{(\tau^2-1)(15\tau^2+1)\delta\tau}{\tau(3\tau^2+1)(5\tau^2-1)},$$

$$\delta x=\frac{c^3}{4}\frac{(\tau^2+1)(15\tau^2+1)(\tau^2-1)^3}{(5\tau^2-1)^4}\delta\tau,\quad \delta z=c\frac{(15\tau^2+1)(\tau^2-1)}{(5\tau^2-1)^2}\delta\tau,$$

$$\frac{dx}{\delta z}=\frac{c^2}{4}\frac{(\tau^2+1)(\tau^2-1)^2}{(5\tau^2-1)^2}.$$

To the values of τ included between $+\sqrt{\frac{1}{5}}$ and $-\sqrt{\frac{1}{5}}$ will correspond one branch of the curve passing through the origin, where it has a point of contrary flexure, and extending to infinity in both directions.

When $(5\tau^2-1)$ is positive $\frac{x}{\tau}$ is always positive; and when $\tau^2=1$,

$$\delta x=0,\quad \delta z=0,\quad \frac{\delta x}{\delta z}=0.$$

Hence there will be a cusp of the second kind when $x=0$, $z=\pm c$, the axis of z being a tangent to the curve at each cusp. One pair of branches has its cusp at the point $x=0$, $z=c$, and the values of x, z increase indefinitely in the respective branches as τ passes from 1 to $+\infty$ and from 1 to $\sqrt{\frac{1}{5}}$. This pair lies in the $+x$, $+z$ quadrant, and there will be a precisely similar and similarly situated pair in the $-x$, $-z$ quadrant. Thus there will be in all one infinite $\int$-formed branch passing through the origin, and

two detached pairs of infinite branches lying in opposite quadrants([73]). The value $\frac{1}{5}$ for τ^2, it will of course be seen, corresponds to -2 for φ, and gives, as it ought to do, the position of the cusp.

(99) Finally, for sections parallel to the plane of the discriminant and lying below it, making $D=-k^2$, $t^2=\frac{1+\varphi}{3-\varphi}$, we obtain in like manner

$$z=kt\frac{3t^2-1}{5t^2+1}, \quad x=k^3t\frac{(t^2+1)^4}{4(5t^2+1)^3}, \quad \frac{\delta x}{x}=\frac{(t^2-1)(15t^2-1)}{t(t^2+1)(5t^2+1)}\delta t, \quad \frac{\delta z}{z}=\frac{(t^2+1)(15t^2-1)}{t(3t^2-1)(5t^2+1)},$$

$$\delta x=\frac{k^3}{4}\frac{(t^2-1)(15t^2-1)(t^2+1)^3}{(5t^2+1)^4}, \quad \delta z=k\frac{(15t^2-1)(t^2+1)}{(5t^2+1)^2}, \quad \frac{\delta x}{\delta z}=\frac{k^2}{4}\frac{(t^2-1)(t^2+1)^2}{(5t^2+1)^2}.$$

When $t^2=\frac{1}{15}$ there will be an ordinary cusp, and when $t^2=1$, $\frac{\delta x}{\delta z}=0$.

There will therefore be three branches,—one corresponding to the values of t between $-\sqrt{\frac{1}{15}}$ and $+\sqrt{\frac{1}{15}}$, the other two to values of t between these limits and $-$ and $+$ infinity respectively. The middle branch passes through the origin, where it undergoes an inflexion, and comes to a cusp at a finite distance from the origin in two opposite quadrants. The connected branch at each cusp crosses the axis of x, sweeps convexly towards the axis of z, arrives at a minimum distance from it, and then goes off to infinity.

The value $\frac{1}{15}$ for t^2 corresponds to $-\frac{3}{4}$ for φ, and gives, as it ought to do, the cusp-node. In fact the values $\varphi=-\frac{3}{4}$, $\varphi=-2$ correspond respectively to a cuspidal and to a cusp-nodal line in the limiting surface whose sections we have been considering.

When the cutting plane is that of D itself, the section becomes a double cubic parabola and a single cubical parabola crossing each other at the origin.

([73]) Let s be an infinitesimal, and $\theta^2=1+s$; then

$$\delta z=\frac{4}{c^2}\frac{(4+5s)^2}{(2+s)s^2}\delta x=\frac{32}{c^2}(1+2s)\frac{\delta x}{s^2}.$$

Hence at either cusp the branch the further removed from the axis of x corresponds to the values of θ^2 be tween 1 and ∞, and the inferior branch to its values between 1 and $\frac{1}{5}$; so that the order of continuity of th five branches of the curve may be read as follows:—from the infinite point in the higher branch of the upp pair to its cusp; thence to the infinite point in the connected branch, which is contiguous to the infinite poi in the opposite extremity of the middle branch; thence along this branch to its contrary infinite extremit thence to the infinite point in the upper branch of the inferior pair; along that branch to its cusp; and then finally, along the lower branch to infinity.

Description of the Plates.

PLATE XXIV.

The (ϵ, η) equation is $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5=0$, of which two roots are always imaginary; its extreme criteria are 0, 0; its middle criteria $\epsilon^4-\epsilon\eta^2$, $\eta^4-\eta\epsilon^2$,

$$p=\epsilon\eta-1, \quad \sigma=(\epsilon^3-\eta^2)(\epsilon^2-\eta^3).$$

Points (p, σ) above the discriminatrix indicate 2 pairs of associated roots in the (ϵ, η) equation.

Points (p, σ) on the discriminatrix indicate 2 equal roots in the (ϵ, η) equation.

Points (p, σ) under the discriminatrix indicate 3 solitary roots in the (ϵ, η) equation.

Points (p, σ) above the equatrix indicate ϵ, η real and unequal.

Points (p, σ) on the equatrix indicate ϵ, η equal.

Points (p, σ) under the equatrix indicate ϵ, η imaginary and conjugate.

Points (p, σ) above the loop of the indicatrix indicate middle criteria not *both* positive.

Points (p, σ) on the loop of the indicatrix indicate middle criteria of opposite signs.

Points (p, σ) under the loop of the indicatrix indicate middle criteria not *both* negative.

The discriminatrix is a closed curve, the *whole* of which is figured on the Plate, and is shaped somewhat like a harp: it has a cusp of the fourth order at the origin.

The equatrix consists of two branches coming together at a cusp at the distance 1 from the origin; the upper branch touches the horizontal axis at the origin; the lower branch, after touching the discriminant at a single point, sweeps out from and round it, cutting the vertical axis at the distance 4 below the origin. Both branches go off to infinity to the right, and lie completely under the horizontal axis. Where the lower branch touches the discriminatrix, the discriminant of the (ϵ, η) equation passes through zero without changing its sign.

The indicatrix consists of a single branch extending indefinitely in both directions. It passes from infinity below and to the left until, at the distance 1 from the origin, it touches the axis, which at the origin it crosses at an angle of 45°, after which it goes off to infinity in the positive direction. Its *loop* extends from $p=0$ to $p=-1$. The two portions of it figured in the Plate join on together, coming to a maximum at a great distance below the horizontal axis. The narrow tract marked "Region of Real parameters" is that portion of the harp-shaped space for which alone, ϵ, η being *real*, the (ϵ, η) equation can have more than one real root. The areas of each of the three regions into which the discriminatrix is divided by the equatrix and indicatrix may readily be expressed numerically in terms of algebraic and inverse circular functions only.

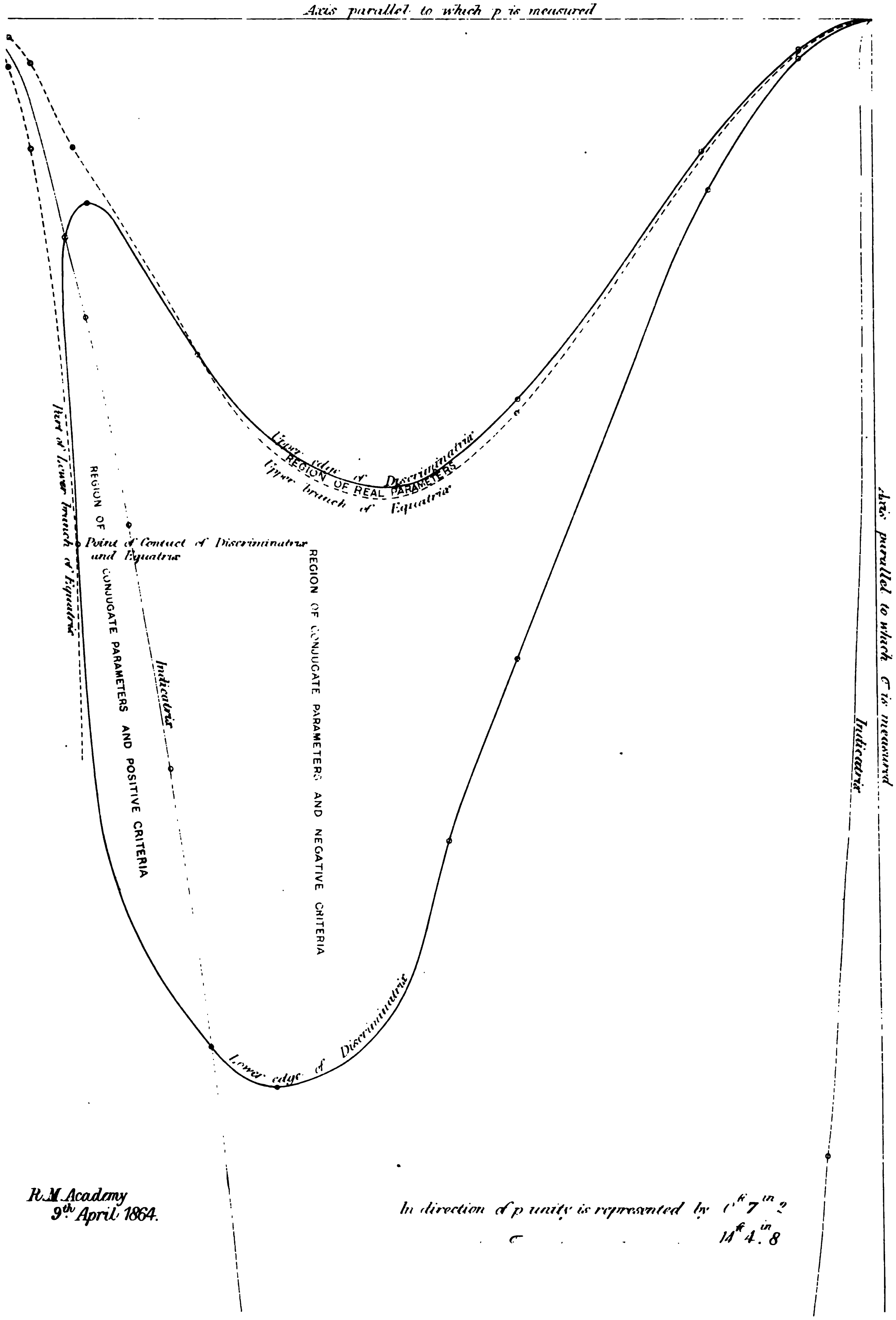
I am indebted to Gentleman Cadet S. L. JACOB, of the Royal Military Academy, for the tracing of the curves of which the above Plate is a somewhat imperfect reproduction.

PLATE XXV.

Described in text, p. 658.

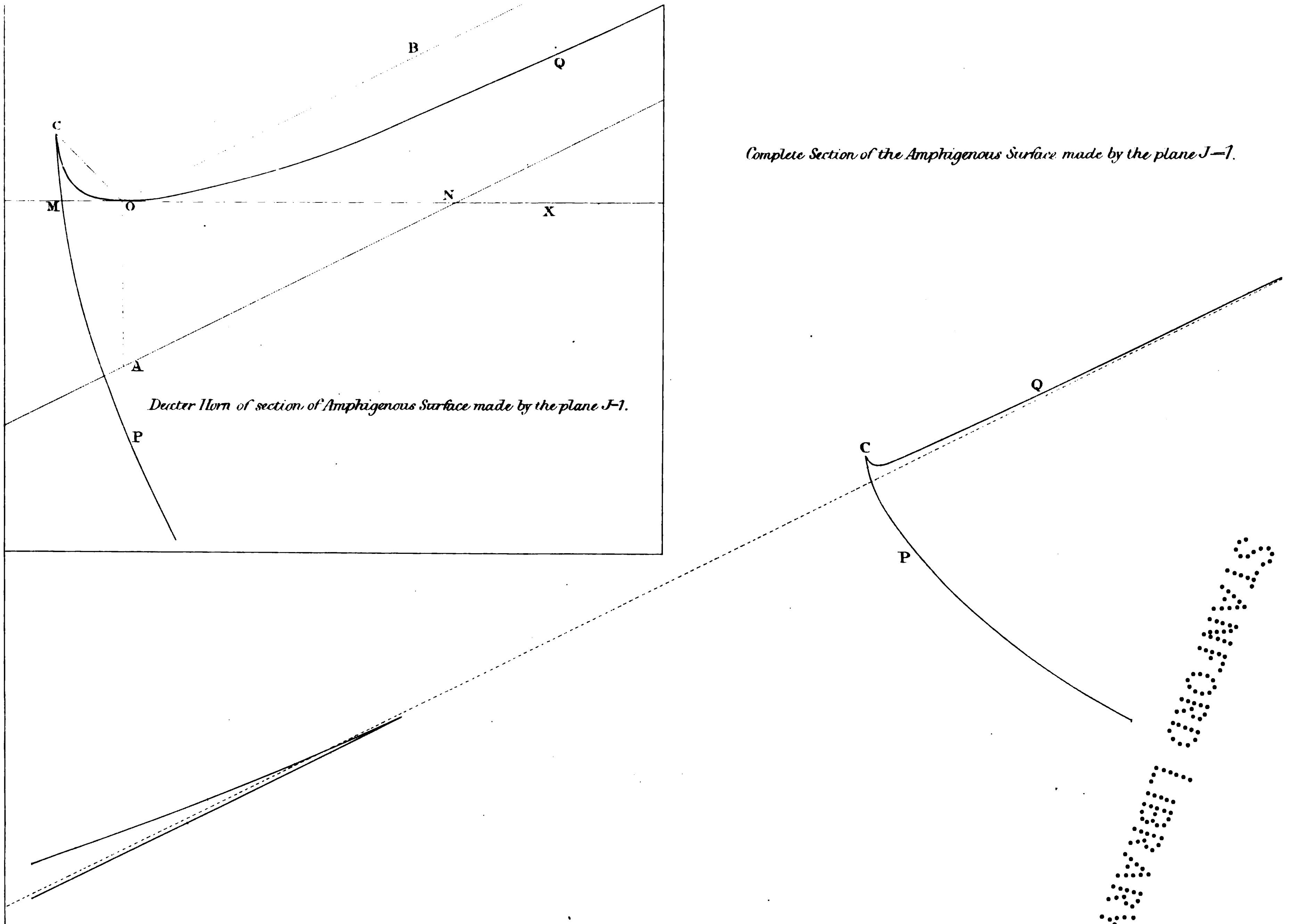
CONTENTS.

SUPPLEMENTAL REFERENCES.



R. M. Academy
9th April 1864.

In direction of p unity is represented by 0 ft 7 in. 2
σ . . . . 14 ft 4 in. 8



G. S. Pritchard del.t

J. Basire, lith.

VII. *Researches on the Partition of Numbers.* By ARTHUR CAYLEY, *Esq.*

Received April 14,—Read May 3 and 10, 1855.

I PROPOSE to discuss the following problem: "To find in how many ways a number q can be made up of the elements $a, b, c, ..$ each element being repeatable an indefinite number of times." The required number of partitions is represented by the notation

$$P(a, b, c, ..)q,$$

and we have, as is well known,

$$P(a, b, c, ..)q = \text{coefficient } x^q \text{ in } \frac{1}{(1-x^a)(1-x^b)(1-x^c)..},$$

where the expansion is to be effected in ascending powers of x.

It may be as well to remark that each element is to be considered as a separate and distinct element, notwithstanding any equalities which may exist between the numbers $a, b, c, ..$; thus, although $a=b$, yet $a+a+a+$ &c. and $a+a+b+$ &c. are to be considered as two different partitions of the number q, and so in all similar cases.

The solution of the problem is thus seen to depend upon the theory, to which I now proceed, of the expansion of algebraical fractions.

Consider an algebraical fraction $\frac{\phi x}{fx}$,

where the denominator is the product of any number of factors (the same or different) of the form $1-x^m$. Suppose in general that $[1-x^m]$ denotes the irreducible factor of $1-x^m$, *i. e.* the factor which, equated to zero, gives the prime roots of the equation $1-x^m=0$. We have

$$1-x^m = \Pi[1-x^{m'}],$$

where m' denotes any divisor whatever of m (unity and the number m itself not excluded). Hence, if a represent a divisor of one or more of the indices m, and k be the number of the indices of which a is a divisor, we have

$$fx = \Pi[1-x^a]^k.$$

Now considering apart from the others one of the multiple factors $[1-x^a]^k$, we may write $fx = [1-x^a]^k f_{,}x$.

Suppose that the fraction $\frac{\phi x}{fx}$ is decomposed into simpler fractions, in the form

$$\begin{aligned}\frac{\phi x}{fx} = {} & \mathrm{I}(x) \\ & + (x\partial_x)^{k-1}\frac{\theta x}{[1-x^a]} + (x\partial_x)^{k-2}\frac{\theta_1 x}{[1-x^a]} \cdot\cdot + \frac{\theta_{k-1}x}{[1-x^a]} \\ & + \&c.,\end{aligned}$$

where $I(x)$ denotes the integral part, and the &c. refers to the fractional terms depending upon the other multiple factors, such as $[1-x^{a}]^{k}$. The functions θx are to be considered as functions with indeterminate coefficients, the degree of each such function being inferior by unity to that of the corresponding denominator; and it is proper to remark that the number of the indeterminate coefficients in all the functions θx together is equal to the degree of the denominator fx.

The term $(x\partial_x)^{k-1}\frac{\theta x}{[1-x^{a}]}$ may be reduced to the form

$$\frac{gx}{[1-x^{a}]^{k}}+\frac{g_1 x}{[1-x^{a}]^{k-1}}+\&c.,$$

the functions gx being of the same degree as θx, and the coefficients of these functions being linearly connected with those of the function θx. The first of the foregoing terms is the only term on the right-hand side which contains the denominator $[1-x^{a}]^{k}$; hence, multiplying by this denominator and then writing $[1-x^{a}]=0$, we find

$$\frac{\phi x}{f_{\prime}x}=gx,$$

which is true when x is any root whatever of the equation $[1-x^{a}]=0$. Now by means of the equation $[1-x^{a}]=0$, $\frac{\phi x}{f_{\prime}x}$ may be expressed in the form of a rational and integral function Gx, the degree of which is less by unity than that of $[1-x^{a}]$. We have therefore $Gx=gx$, an equation which is satisfied by each root of $[1-x^{a}]=0$, and which is therefore an identical equation; gx is thus determined, and the coefficients of θx being linear functions of those of gx, the function θx may be considered as determined. And this being so, the function

$$\frac{\phi x}{fx}-(x\partial_x)^{k-1}\frac{\theta x}{[1-x^{a}]}$$

will be a fraction the denominator of which does not contain any power of $[1-x^{a}]$ higher than $[1-x^{a}]^{k-1}$; and therefore $\theta_1 x$ can be found in the same way as θx, and similarly $\theta_2 x$, and so on. And the fractional parts being determined, the integral part may be found by subtracting from $\frac{\phi x}{fx}$ the sum of the fractional parts, so that the fraction $\frac{\phi x}{fx}$ can by a direct process be decomposed in the above-mentioned form.

Particular terms in the decomposition of certain fractions may be obtained with great facility. Thus m being a prime number, assume

$$\frac{1}{(1-x^{2})(1-x^{3})\ldots(1-x^{m})}=\&c.+\frac{\theta x}{[1-x^{m}]};$$

then observing that $(1-x^{m})=(1-x)[1-x^{m}]$, we have for $[1-x^{m}]=0$,

$$\theta x=\frac{1}{(1-x)(1-x^{2})\ldots(1-x^{m-1})}.$$

Now u being any quantity whatever and x being a root of $[1-x^m]=0$, we have identically

$$[1-u^m]=(u-x)(u-x^2)..(u-x^{m-1});$$

and therefore putting $u=1$, we have $m=(1-x)(1-x^2)..(1-x^{m-1})$,
and therefore

$$\theta x=\frac{1}{m},$$

whence

$$\frac{1}{(1-x^2)(1-x^3)..(1-x^m)}=\&\text{c.}+\frac{1}{m}\,\frac{1}{[1-x^m]}.$$

Again, m being as before a prime number, assume

$$\frac{1}{(1-x)(1-x^2)..(1-x^m)}=\&\text{c.}+\frac{\theta x}{[1-x^m]},$$

we have in this case for $[1-x^m]=0$,

$$\theta x=\frac{1}{(1-x)^2(1-x^2)..(1-x^{m-1})},$$

which is immediately reduced to $\theta x=\frac{1}{m}\,\frac{1}{1-x}$. Now

$$\frac{[1-u^m]}{u-x}=\frac{[1-u^m]-[1-x^m]}{u-x}=(1+u+..+u^{m-2})+(1+u..+u^{m-3})x..+(1+u)x^{m-3}+x^{m-2};$$

or putting $u=1$,

$$\frac{m}{1-x}=\overline{m-1}+\overline{m-2}x..+x^{m-2};$$

and substituting this in the value of θx, we find

$$\frac{1}{(1-x)(1-x^2)..(1-x^m)}=\&\text{c.}+\frac{1}{m^2}\,\frac{(m-1)+(m-2)x..+x^{m-2}}{[1-x^m]}.$$

The preceding decomposition of the fraction $\frac{\phi x}{fx}$ gives very readily the expansion of the fraction in ascending powers of x. For, consider a fraction such as

$$\frac{\theta x}{[1-x^a]},$$

where the degree of the numerator is in general less by unity than that of the denominator; we have

$$1-x^a=[1-x^a]\Pi[1-x^{a'}],$$

where a' denotes any divisor of a (including unity, but not including the number a itself). The fraction may therefore be written under the form

$$\frac{\theta x\Pi[1-x^{a'}]}{1-x^a},$$

where the degree of the numerator is in general less by unity than that of the denominator, *i. e.* is equal to $\overline{a-1}$. Suppose that b is any divisor of a (including unity, but not including the number a itself), then $1-x^b$ is a divisor of $\Pi[1-x^{a'}]$, and

therefore of the numerator of the fraction. Hence representing this numerator by

$$A_0+A_1x\ldots+A_{a-1}x^{a-1},$$

and putting $a=bc$, we have (corresponding to the case $b=1$)

$$A_0+A_1+A_2\ldots+A_{a-1}=0,$$

and generally for the divisor b,

$$A_0+A_b\ldots+A_{(c-1)b}=0$$
$$A_1+A_{b+1}\ldots+A_{(c-1)b+1}=0$$
$$\cdot$$
$$\cdot$$
$$A_{b-1}+A_{2b-1}\ldots+A_{cb-1}=0.$$

Suppose now that a_q denotes a circulating element to the period a, *i. e.* write

$$a_q=1 \quad q=0 \text{ (mod. } a)$$
$$a_q=0 \text{ in every other case.}$$

A function such as

$$A_0a_q+A_1a_{q-1}\ldots+A_{a-1}a_{q-a+1}$$

will be a circulating function, or circulator to the period a, and may be represented by the notation

$$(A_0, A_1, \ldots A_{a-1}) \text{ circlor } a_q.$$

In the case however where the coefficients A satisfy, for each divisor b of the number a, the above-mentioned equations, the circulating function is what I call a prime circulator, and I represent it by the notation

$$(A_0, A_1, \ldots A_{a-1}) \text{ pcr } a_q.$$

By means of this notation we have at once

$$\text{coefficient } x_q \text{ in } \frac{\theta x}{[1-x^a]}=(A_0, A_1\ldots A_{a-1}) \text{ pcr } a_q,$$

and thence also

$$\text{coefficient } x_q \text{ in } (x\partial_x)^r\frac{\theta x}{[1-x^a]}=q^r(A_0, A_1\ldots A_{a-1}) \text{ pcr } a_q.$$

Hence assuming that in the fraction $\frac{\phi x}{fx}$ the degree of the numerator is less than that of the denominator (so that there is not any integral part), we have

$$\text{coefficient } x_q \text{ in } \frac{\phi x}{fx}=\Sigma\ q^r(A_0, A_1, \ldots A_{a-1}) \text{ pcr } a_q\ ;$$

or, if we wish to put in evidence the non-circulating part arising from the divisor $a=1$,

$$\text{coefficient } x_q \text{ in } \frac{\phi x}{fx}=Aq^{k-1}+Bq^{k-2}\ldots+Lq+M$$
$$+\Sigma\ q^r(A_0, A_1\ldots A_{a-1}) \text{ pcr } a_q\ ;$$

where k denotes the number of the factors $1-x^m$ in the denominator fx, a is any divisor (unity excluded) of one or more of the indices m; and for each value of a r extends from $r=0$ to $r=k-1$, where k denotes the number of indices m of which

a is a divisor. The particular results previously obtained show, that m being a prime number,

$$\text{coefficient } x^q \text{ in } \frac{1}{(1-x^2)(1-x^3)\ldots(1-x^m)}=\&\text{c.}+\frac{1}{m}(1,-1,0,0,\ldots) \text{ pcr } m_q,$$

and

$$\text{coefficient } x^q \text{ in } \frac{1}{(1-x)(1-x^2)\ldots(1-x^m)}=\&\text{c.}+\frac{1}{m^2}(m-1,-1,-1,\ldots) \text{ pcr } m_q.$$

Suppose, as before, that the degree of ϕx is less than that of fx, and let the analytical expression above obtained for the coefficient of x^q in the expansion in ascending powers of x of the fraction $\frac{\phi x}{fx}$ be represented by Fq, it is very remarkable that if we expand $\frac{\phi x}{fx}$ in descending powers of x, then the coefficient of x^q in this new expansion (q is here of course negative, since the expansion contains only negative powers of x) is precisely equal to $-$Fq; this is in fact at once seen to be the case with respect to each of the partial fractions into which $\frac{\phi x}{fx}$ has been decomposed, and it is consequently the case with respect to the fraction itself*. This gives rise to a result of some importance. Suppose that ϕx and fx are respectively of the degrees N and D; it is clear from the form of fx that we have $f\left(\frac{1}{x}\right)=(-)^{\text{D}}x^{-\text{D}}fx$; and I suppose that ϕx is also such that $\phi\left(\frac{1}{x}\right)=(\pm)^{\text{N}}x^{-\text{N}}\phi x$; then writing $\text{D}-\text{N}=h$, and supposing that $\frac{\phi x}{fx}$ is expanded in descending powers of x, so that the coefficient of x^q in the expansion is $-$Fq, it is in the first place clear that the expansion will commence with the term x^{-h}, and we must therefore have

$$\text{F}q=0$$

for all values of q from $q=-1$ to $q=-(h-1)$.

Consider next the coefficient of a term x^{-h-q}, where q is 0 or positive; the coefficient in question, the value of which is $-\text{F}(-h-q)$, is obviously equal to the coefficient of x^{h+q} in the expansion in ascending powers of x of $\frac{\phi\frac{1}{x}}{f\left(\frac{1}{x}\right)}$, *i. e.* to

$$(\pm)^{\text{N}}(-)^{\text{D}} \text{ coefficient } x^{h+q} \text{ in } \frac{x^h\phi x}{fx},$$

or what is the same thing, to

$$(\pm)^{\text{N}}(-)^{\text{D}} \text{ coefficient } x^q \text{ in } \frac{\phi x}{fx};$$

and we have therefore, q being zero or positive,

$$\text{F}(-h-q)=-(\pm)^{\text{N}}(-)^{\text{D}}\text{F}q.$$

In particular, when $\phi x=1$, $\quad\quad \text{F}q=0$

* The property is a fundamental one in the general theory of developments.

for all values of q from $q=-1$ to $q=-(D-1)$; and q being 0 or positive,

$$F(-D-q)=(-)^{D-1}Fq.$$

The preceding investigations show the general form of the function $P(a, b, c, \ldots)q$, viz. that

$$P(a, b, c, ..)q=Aq^{k-1}+Bq^{k-2}..+Lq+M+\Sigma\, q^r(A_0, A_1, .. A_{l-1}) \text{ per } l_q,$$

a formula in which k denotes the number of the elements $a, b, c, ..$ &c., and l is any divisor (unity excluded) of one or more of these elements; the summation in the case of each such divisor extends from $r=0$ to $r=k-1$, where k is the number of the elements $a, b, c, \ldots$ &c. of which l is a divisor; and the investigations indicate how the values of the coefficients A of the prime circulators are to be obtained. It has been moreover in effect shown, that if $D=a+b+c+..$, then, writing for shortness $P(q)$ instead of $P(a, b, c, ..)q$, we have

$$P(q)=0$$

for all values of q from $q=-1$ to $q=-(D-1)$, and that q being 0 or positive,

$$P(-D-q)=(-)^{D-1}P(q);$$

these last theorems are however uninterpretable in the theory of partitions, and they apply only to the analytical expression for $P(q)$.

I have calculated the following particular results:—

$$P(1, 2)q \quad =\frac{1}{4}\left\{2q+3 \atop +(1,-1) \text{ per } 2_q\right\}$$

$$P(1, 2, 3)q =\frac{1}{72}\left\{\begin{array}{l}6q^2+36q+47\\ \quad +9(1,-1) \text{ per } 2_q\\ +8(2,-1,-1) \text{ per } 3_q\end{array}\right\}$$

$$P(1, 2, 3, 4)q =\frac{1}{288}\left\{\begin{array}{l}2q^3+30q^2+135q+175\\ \quad +(9q+45)(1,-1) \text{ per } 2_q\\ \quad +32 \quad (1,0,-1) \text{ per } 3_q\\ \quad +36 \quad (1,0,-1,0) \text{ per } 4_q\end{array}\right\}$$

$$P(1, 2, 3, 4, 5)q=\frac{1}{86400}\left\{\begin{array}{l}30q^4+900q^3+9300q^2+38250q+50651\\ \quad +(1350q+10125) \quad (1,-1) \text{ per } 2_q\\ \quad +3200 \qquad (2,-1,-1) \text{ per } 3_q\\ \quad +5400 \qquad (1,1,-1,-1) \text{ per } 4_q\\ \quad +3456(4,-1,-1,-1,-1) \text{ per } 5_q\end{array}\right\}$$

$$P(2)q \quad =\frac{1}{2}\left\{1 \atop +(1,-1) \text{ per } 2_q\right\}$$

$$\mathrm{P}(2,3)q=\frac{1}{12}\left\{\begin{array}{l}2q+5\\ +3\ (1,-1)\text{ pcr }2_q\\ +4(1,-1,0)\text{ pcr }3_q\end{array}\right\}$$

$$\mathrm{P}(2,3,4)q=\frac{1}{288}\left\{\begin{array}{l}6q^2+54q+107\\ +(18q+81)(1,-1)\text{ pcr }2_q\\ +32\ (2,-1,-1)\text{ pcr }3_q\\ +36\ (1,-1,-1,1)\text{ pcr }4_q\end{array}\right\}$$

$$\mathrm{P}(2,3,4,5)q=\frac{1}{1440}\left\{\begin{array}{l}2q^3+42q^2+267q+497\\ +(45q+315)(1,-1)\text{ pcr }2_q\\ +160\ (1,-1,0)\text{ pcr }3_q\\ +180\ (1,0,-1,0)\text{ pcr }4_q\\ +288\ (1,-1,0,0,0)\text{ pcr }5_q\end{array}\right\}$$

$$\mathrm{P}(2,3,4,5,6)q=\frac{1}{172800}\left\{\begin{array}{l}10q^4+400q^3+5550q^2+31000q+56877\\ +(450q^2+9000q+39075)(1,-1)\text{ pcr }2_q\\ +3200q\ (1,-1,0)\text{ pcr }3_q\\ +1600\ (21,-19,-2)\text{ pcr }3_q\\ +10800\ (1,0,-1,0)\text{ pcr }4_q\\ +6912\ (4,-1,-1,-1,-1)\text{ pcr }5_q\\ +4800\ (1,-1,-2,-1,1,2)\text{ pcr }6_q\end{array}\right\}$$

$$\mathrm{P}(1,2,3,5)q=\frac{1}{720}\left\{\begin{array}{l}4q^3+66q^2+324q+451\\ +45\ (1,-1)\text{ pcr }2_q\\ +80\ (1,-1,0)\text{ pcr }3_q\\ +144(1,0,0,0,-1)\text{ pcr }5_q\end{array}\right\}$$

$$\mathrm{P}(1,2,2,3,4)q=\frac{1}{6912}\left\{\begin{array}{l}6q^4+144q^3+1194q^2+3960q+4267\\ +(54q^2+648q+1701)(1,-1)\text{ pcr }2_q\\ +256\ (2,-1,-1)\text{ pcr }3_q\\ +432\ (1,0,-1,0)\text{ pcr }4_q\end{array}\right\}$$

$$\mathrm{P}(8)q=\frac{1}{8}\left\{\begin{array}{l}1\\ +1\ (1,-1)\text{ pcr }2_q\\ +2\ (1,0,-1,0)\text{ pcr }4_q\\ +8(1,0,0,0,-1,0,0,0)\text{ pcr }8_q\end{array}\right\}$$

$$\begin{aligned}
P(7, 8)q=\frac{1}{112}\Big\{&2q+43 \\
&+7 \qquad\qquad\qquad (1, -1) \text{ pcr } 2_q \\
&+14 \qquad\qquad (1, -1, -1, 1) \text{ pcr } 4_q \\
&+16\,(3, 2, 1, 0, -1, -2, -3) \text{ pcr } 7_q \\
&+56\,(0, -1, -1, 0, 0, 1, 1, 0) \text{ pcr } 8_q\Big\},
\end{aligned}$$

which are, I think, worth preserving.

Received April 14,—Read May 3 and 10, 1855.

I proceed to discuss the following problem: "To find in how many ways a number q can be made up as a sum of m terms with the elements 0, 1, 2, ...k, each element being repeatable an indefinite number of times." The required number of partitions is represented by

$$P(0, 1, 2, ..k)^m q,$$

and the number of partitions of q less the number of partitions of $q-1$ is represented by

$$P'(0, 1, 2, ..k)^m q.$$

We have, as is well known,

$$P(0, 1, 2, ..k)^m q = \text{coefficient } x^q z^m \text{ in } \frac{1}{(1-z)(1-xz)..(1-x^k z)},$$

where the expansion is to be effected in ascending powers of z. Now

$$\frac{1}{(1-z)(1-xz)..(1-x^k z)} = 1+\frac{1-x^{k+1}}{1-x}z+\frac{(1-x^{k+1})(1-x^{k+2})}{(1-x)(1-x^2)}z^2+\text{\&c.},$$

the general term being

$$\frac{(1-x^{k+1})(1-x^{k+2})..(1-x^{k+m})}{(1-x)(1-x^2)..(1-x^m)}z^m,$$

or, what is the same thing,

$$\frac{(1-x^{m+1})(1-x^{m+2})..(1-x^{m+k})}{(1-x)(1-x^2)..(1-x^k)}z^m,$$

and consequently

$$P(0, 1, 2, ..k)^m q = \text{coefficient } x^q \text{ in } \frac{(1-x^{m+1})(1-x^{m+2})..(1-x^{m+k})}{(1-x)(1-x^2)..(1-x^k)};$$

to transform this expression I make use of the equation

$$(1+xz)(1+x^2z)..(1+x^k z)=1+\frac{x(1-x^k)}{1-x}z+\frac{x^3(1-x^k)(1-x^{k-1})}{(1-x)(1-x^2)}z^2+\text{\&c.},$$

where the general term is

$$x^{\frac{1}{2}s(s+1)}\frac{(1-x^k)(1-x^{k-1})..(1-x^{k-s+1})}{(1-x)(1-x^2)..(1-x^s)}z^s,$$

and the series is a finite one, the last term being that corresponding to $s=k$, viz. $x^{\frac{1}{2}k(k+1)}z^k$. Writing $-x^m$ for z, and substituting the resulting value of

$$(1-x^{m+1})(1-x^{m+2})..(1-x^{m+k})$$

in the formula for $P(0, 1, 2, ..k)^m q$, we have

$$P(0, 1, 2, ..k)^m q=\Sigma_s\left\{(-)^s \text{ coefficient } x^q \text{ in } \frac{x^{sm+\frac{1}{2}s.s+1}}{(1-x)(1-x^2)..(1-x^s)(1-x)(1-x^2)..(1-x^{k+s})}\right\},$$

where the summation extends from $s=0$ to $s=k$; but if for any value of s between these limits $sm+\frac{1}{2}s(s+1)$ becomes greater than q, then it is clear that the summation need only be extended from $s=0$ to the last preceding value of s, or what is the same thing, from $s=0$ to the greatest value of s, for which $q-sm-\frac{1}{2}s(s+1)$ is positive or zero.

It is obvious, that if $q>km$, then

$$P(0, 1, 2..k)^m q=0;$$

and moreover, that if $\theta \not> \frac{1}{2}km$, then

$$P(0, 1, 2, ..k)^m\theta=P(0, 1, 2..k)^m.km-\theta,$$

so that we may always suppose $q \not> \frac{1}{2}km$. I write therefore $q=\frac{1}{2}(km-\alpha)$ where α is zero or a positive integer not greater than km, and is even or odd according as km is even or odd. Substituting this value of q and making a slight change in the form of the result, we have

$$P(0, 1, 2..k)^m\tfrac{1}{2}(km-\alpha)=\Sigma_s\left\{(-)^s \text{ coeff. } x^{(\frac{1}{2}k-s)m} \text{ in } \frac{x^{\frac{1}{2}\alpha+\frac{1}{2}s.s+1}}{(1-x)(1-x^2)..(1-x^s)(1-x)(1-x^2)..(1-x^{k-s})}\right\},$$

where the summation extends from $s=0$ to the greatest value of s, for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero. But we may, if we please, consider the summation as extending, when k is even, from $s=0$ to $s=\frac{1}{2}k-1$, and when k is odd, from $s=0$ to $s=\frac{1}{2}(k-1)$, the terms corresponding to values of s greater than the greatest value for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero, being of course equal to zero. It may be noticed, that the fraction will be a proper one if $\alpha<(k-s)(k-s+1)$; or substituting for s its greatest value, the fraction will be a proper one for all values of s, if, when k is even, $\alpha<\frac{1}{4}k(k+2)$, and when k is odd, $\alpha<\frac{1}{4}(k+1)(k+3)$.

We have in a similar manner,

$$P'(0, 1, 2 \ldots k)^m q=\text{coefficient } x^q z^m \text{ in } \frac{1-x}{(1-z)(1-xz)..(1-x^k z)},$$

which leads to

$$P'(0, 1, 2..k)^m\tfrac{1}{2}(km-\alpha)=\Sigma_s\left\{(-)^s \text{ coeff. } x^{(\frac{1}{2}k-s)m} \text{ in } \frac{x^{\frac{1}{2}\alpha+s(s+1)}}{(1-x^2)..(1-x^s)(1-x)(1-x^2)..(1-x^{k-s})}\right\},$$

where the summation extends, as in the former case, from $s=0$ to the greatest value of s, for which $(\frac{1}{2}k-s)m-\frac{1}{2}\alpha-\frac{1}{2}s(s+1)$ is positive or zero, or, if we please, when k is even, from $s=0$ to $s=\frac{1}{2}k-1$, and when s is odd, from $s=0$ to $s=\frac{1}{2}(k-1)$. The condition, in order that the fraction may be a proper one for all values of s, is, when k is even, $\alpha+1<\frac{1}{4}k(k+2)$, and when k is odd, $\alpha+1<\frac{1}{4}(k+1)(k+3)$.

To transform the preceding expressions, I write when k is odd x^2 instead of x, and I put for shortness θ instead of $\frac{1}{2}k-s$ or $2(\frac{1}{2}k-s)$, and γ instead of $\frac{1}{2}\alpha+\frac{1}{2}s(s+1)$ or $\alpha+s(s+1)$; we have to consider an expression of the form

$$\text{coefficient } x^{\theta m} \text{ in } \frac{x^{\gamma}}{\mathrm{F}x},$$

where $\mathrm{F}x$ is the product of factors of the form $1-x^a$. Suppose that a' is the least common multiple of a and θ, then $(1-x^{a'})\div(1-x^a)$ is an integral function of x, equal χx suppose, and $1\div(1-x^a)=\chi x\div(1-x^{a'})$. Making this change in all the factors of $\mathrm{F}x$ which require it (*i. e.* in all the factors except those in which a is a multiple of θ), the general term becomes

$$\text{coefficient } x^{\theta m} \text{ in } \frac{x^{\gamma}\mathrm{H}x}{\mathrm{G}x},$$

where $\mathrm{G}x$ is a product of factors of the form $1-x^{a'}$, in which a' is a multiple of θ, *i. e.* $\mathrm{G}x$ is a rational and integral function of x^{θ}. But in the numerator $x^{\gamma}\mathrm{H}x$ we may reject, as not contributing to the formation of the coefficient of $x^{\theta m}$, all the terms in which the indices are not multiples of θ; the numerator is thus reduced to a rational and integral function of x^{θ}, and the general term is therefore of the form

$$\text{coefficient } x^{\theta m} \text{ in } \frac{\lambda(x^{\theta})}{\varkappa(x^{\theta})},$$

or what is the same thing, of the form

$$\text{coefficient } x^{m} \text{ in } \frac{\lambda x}{\varkappa x}.$$

Where $\varkappa x$ is the product of factors of the form $1-x^a$, and λx is a rational and integral function of x, the particular value of the fraction depends on the value of s; and uniting the different terms, we have an expression

$$\text{coefficient } x^{m} \text{ in } \mathrm{S}_s\,(-)^s\frac{\lambda x}{\varkappa x},$$

which is equivalent to

$$\text{coefficient } x^{m} \text{ in } \frac{\phi x}{f x},$$

where fx is a product of factors of the form $1-x^a$, and ϕx is a rational and integral function of x. And it is clear that the fraction will be a proper one when each of the fractions in the original expression is a proper fraction, *i. e.* in the case of $\mathrm{P}(0,1,2\,..\,k)^m\frac{1}{2}(km-\alpha)$, when for k even $\alpha<\frac{1}{4}k(k+2)$, and for k odd $\alpha<\frac{1}{4}(k+1)(k+3)$; and in the case of $\mathrm{P}'(0,1,2\,..\,k)^m\frac{1}{2}(km-\alpha)$, when for k even $\alpha+1<\frac{1}{4}k(k+2)$, and for k odd $\alpha+1<\frac{1}{4}(k+1)(k+3)$.

We see, therefore, that

$$\mathrm{P}(0,1,2\,..\,k)^m\tfrac{1}{2}(km-\alpha),$$

and

$$\mathrm{P}'(0,1,2\,..\,k)^m\tfrac{1}{2}(km-\alpha),$$

are each of them of the form

$$\text{coefficient } x^{m} \text{ in } \frac{\phi x}{f x},$$

where fx is the product of factors of the form $1-x^a$, and up to certain limiting values of α the fraction is a proper fraction. When the fraction $\frac{\phi x}{fx}$ is known, we may therefore obtain by the method employed in the former part of this Memoir, analytical expressions (involving prime circulators) for the functions P and P′.

As an example, take $$\mathrm{P}(0,1,2,3)^m\tfrac{3}{2}m,$$
which is equal to

$$\text{coefficient } x^{3m} \text{ in } \frac{1}{(1-x^2)(1-x^4)(1-x^6)}$$

$$-\text{coefficient } x^m \text{ in } \frac{1}{(1-x^2)(1-x^2)(1-x^4)}.$$

The multiplier for the first fraction is

$$\frac{(1-x^6)(1-x^{12})}{(1-x^2)(1-x^4)},$$

which is equal to $$1+x^2+2x^4+x^6+2x^8+x^{10}+x^{12}.$$
Hence, rejecting in the numerator the terms the indices of which are not divisible by 3, the first term becomes

$$\text{coefficient } x^{3m} \text{ in } \frac{1+x^6+x^{12}}{(1-x^6)(1-x^{12})(1-x^6)},$$

or what is the same thing, the first term is

$$\text{coefficient } x^m \text{ in } \frac{1+x^2+x^4}{(1-x^2)^2(1-x^4)};$$

and the second term being

$$-\text{coefficient } x^m \text{ in } \frac{x^2}{(1-x^2)^2(1-x^4)},$$

we have $$\mathrm{P}(0,1,2,3)^m\tfrac{3}{2}m=\text{coefficient } x^m \text{ in } \frac{1+x^4}{(1-x^2)^2(1-x^4)}.$$

And similarly it may be shown, that

$$\mathrm{P}(0,1,2,3)^m\tfrac{1}{2}(3m-1)=\text{coefficient } x^m \text{ in } \frac{x+x^3}{(1-x^2)^2(1-x^4)}.$$

As another example, take $$\mathrm{P}'(0,1,2,3,4,5)\tfrac{5}{2}m,$$
which is equal to

$$\text{coefficient } x^{5m} \text{ in } \frac{1}{(1-x^4)(1-x^6)(1-x^8)(1-x^{10})}$$

$$-\text{coefficient } x^{3m} \text{ in } \frac{x^2}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}$$

$$+\text{coefficient } x^m \text{ in } \frac{x^6}{(1-x^2)(1-x^4)(1-x^4)(1-x^6)}.$$

The multiplier for the first fraction is

$$\frac{(1-x^{20})(1-x^{30})(1-x^{40})}{(1-x^4)(1-x^6)(1-x^8)},$$

which is a function of x^2 of the order 36, the coefficients of which are

1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 4, 4, 6, 4, 6, 5, 7, 5, 7, 5, 7, 5, 6, 4, 6, 4, 4, 3, 4, 2, 3, 1, 2, 1, 1, 0, 1,

and the first part becomes therefore

$$\text{coefficient } x^m \text{ in } \frac{1+x^2+4x^4+5x^6+7x^8+4x^{10}+3x^{12}}{(1-x^2)(1-x^4)(1-x^6)(1-x^8)}.$$

The multiplier for the second fraction is

$$\frac{(1-x^6)(1-x^{12})(1-x^{24})}{(1-x^2)(1-x^4)(1-x^8)},$$

which is a function of x^2 of the order 14, the coefficients of which are

1, 1, 2, 1, 3, 2, 3, 1, 3, 2, 3, 1, 2, 1, 1;

and the second term becomes

$$-\text{coefficient } x^m \text{ in } \frac{2x^2+2x^4+3x^6+x^8+x^{10}}{(1-x^2)^2(1-x^4)(1-x^8)};$$

and the third term is $\quad \text{coefficient } x^m \text{ in } \dfrac{x^6}{(1-x^2)(1-x^4)^2(1-x^6)}$.

Now the fractions may be reduced to a common denominator

$$(1-x^2)(1-x^4)(1-x^6)(1-x^8)$$

by multiplying the terms of the second fraction by $\frac{1-x^6}{1-x^2}(=1+x^2+x^4)$, and the terms of the third fraction by $\frac{1-x^8}{1-x^4}(=1+x^4)$; performing the operations and adding, the numerator and denominator of the resulting fraction will each of them contain the factor $1-x^2$; and casting this out, we find

$$\mathrm{P}(0,1,2,3,4,5)^m\tfrac{5}{2}m=\text{coefficient } x^m \text{ in } \frac{1-x^6+x^{12}}{(1-x^4)(1-x^6)(1-x^8)}.$$

I have calculated by this method several other particular cases, which are given in my "Second Memoir upon Quantics;" the present researches were in fact made for the sake of their application to that theory.

Received April 20,—Read May 3 and 10, 1855.

Since the preceding portions of the present Memoir were written, Mr. SYLVESTER has communicated to me a remarkable theorem which has led me to the following additional investigations*.

Let $\frac{\phi x}{fx}$ be a rational fraction, and let $(x-x_1)^k$ be a factor of the denominator fx, then if

$$\left\{\frac{\phi x}{fx}\right\}_{x_1}$$

* Mr. SYLVESTER's researches are published in the Quarterly Mathematical Journal, July 1855, and he has there given the general formula as well for the circulating as the non-circulating part of the expression for the number of partitions.—Added 23rd February, 1856.—A. C.

denote the portion which is made up of the simple fractions having powers of $x-x_1$ for their denominators, we have by a known theorem

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \text{coefficient } \frac{1}{z} \text{ in } \frac{1}{x-x-z} \frac{\phi(x_1+z)}{f(x_1+z)}.$$

Now by a theorem of JACOBI's and CAUCHY's,

$$\text{coefficient } \frac{1}{z} \text{ in } \mathrm{F}z = \text{coefficient } \frac{1}{t} \text{ in } \mathrm{F}(\psi t)\psi' t\,;$$

whence, writing $x_1+z=x_1e^{-t}$, we have

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \text{coefficient } \frac{1}{t} \text{ in } \frac{x_1}{x_1-xe^t} \frac{\phi(x_1e^{-t})}{f(x_1e^{-t})}.$$

Now putting for a moment $x=x_1e^\theta$, we have

$$\frac{1}{x_1-xe^t} = \frac{1}{x_1(1-e^{\theta+t})} = \frac{1}{x_1(1-e^\theta)} + \partial_\theta \frac{1}{x_1(1-e^\theta)} + \dots$$

and $\partial_\theta = x\partial_x$, whence

$$\frac{1}{x_1-xe^t} = \frac{1}{x_1-x} + \frac{t}{1} x\partial_x \frac{1}{x_1-x} + \frac{t^2}{1.2}(x\partial_x)^2 \frac{1}{x_1-x} + \dots,$$

the general term of which is

$$\frac{t^{s-1}}{\Pi(s-1)}(x\partial_x)^{s-1}\frac{1}{x_1-x}.$$

Hence representing the general term of

$$\frac{x_1\phi(x_1e^{-t})}{f(x_1e^{-t})}$$

by $\chi x_1 t^{-s}$, so that

$$\chi x_1 = \text{coefficient } \frac{1}{t} \text{ in } t^{s-1}\frac{x_1\phi(x_1e^{-t})}{f(x_1e^{-t})},$$

we find, writing down only the general term

$$\left\{\frac{\phi x}{fx}\right\}_{x_1} = \dots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\frac{\chi x_1}{x_1-x} + \dots$$

where the value of χx_1 depends upon that of s, and where s extends from $s=1$ to $s=k$.

Suppose now that the denominator is made up of factors (the same or different) of the form $1-x^m$. And let a be any divisor of one or more of the indices m, and let k be the number of the indices of which a is a divisor. The denominator contains the divisor $[1-x^a]^k$, and consequently if ρ be any root of the equation $[1-x^a]=0$, the denominator contains the factor $(\rho-x)^k$. Hence writing ρ for x_1 and taking the sum with respect to all the roots of the equation $[1-x^a]=0$, we find

$$\left\{\frac{\phi x}{fx}\right\}_{[1-x^a]} = \dots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\mathrm{S}\frac{\chi\rho}{\rho-x} + \dots$$

$$= \dots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\frac{\theta x}{[1-x^a]} + \dots,$$

where $\qquad \chi\rho = \text{coefficient } \frac{1}{t} \text{ in } t^{s-1}\frac{\rho\phi(\rho e^{-t})}{f(\rho e^{-t})},$

and as before s extends from $s=1$ to $s=k$. We have thus the actual value of the function θx made use of in the memoir.

A preceding formula gives

$$\left\{\frac{\phi x}{fx}\right\}_1 = \text{coefficient } \frac{1}{t} \text{ in } \frac{1}{1-xe^t}\frac{\phi(e^{-t})}{f(e^{-t})},$$

which is a very simple expression for the non-circulating part of the fraction $\frac{\phi x}{fx}$. This is, in fact, Mr. SYLVESTER's theorem above referred to.

V. *On a Class of Differential Equations, including those which occur in Dynamical Problems.—Part I.* By W. F. Donkin, *M.A., F.R.S., F.R.A.S., Savilian Professor of Astronomy in the University of Oxford.*

Received February 23,—Read February 23, 1854.

THE Analytical Theory of Dynamics, as it exists at present, is due mainly to the labours of Lagrange, Poisson, Sir W. R. Hamilton, and Jacobi; whose researches on this subject present a series of discoveries hardly paralleled, for their elegance and importance, in any other branch of mathematics.

The following investigations in the same department do not pretend to make any important step in advance; though I should not of course have presumed to lay them before the Society, if I had not hoped they might be found to possess some degree of novelty and interest*.

Of previous publications with which I am acquainted, those most nearly on the same subject are, Sir W. R. Hamilton's two memoirs "On a General Method in Dynamics" in the Philosophical Transactions; Jacobi's Memoir in the 17th vol. of Crelle's Journal, "Ueber die Reduction der partiellen Differential-gleichungen," &c.; and M. Bertrand's "Mémoire sur l'integration des équations différentielles de la Mécanique," in Liouville's Journal (1852). The relation in which the present essay stands to the papers just named will be apparent to those who are acquainted with them, and it would be useless to attempt to make it intelligible to others.

Oxford, Feb. 21, 1854.

Section I.

1. Let $x_1, x_2, \ldots. x_n$ be n variables, connected by n relations with n other variables $y_1, y_2, \ldots y_n$; so that each variable of either set may be considered as a function of the variables of the other set. Suppose then

$$y_i = \varphi_i(x_1, x_2, \ldots. x_n),$$

[* It may be useful to specify the parts to which I should principally refer as containing what is, relatively to my own reading on the subject, new; and in the present day it can hardly be required of any one to profess more than this kind of originality. These are—the theorem (3.), art. 1. The results of arts. 2 to 4. The formulæ (19.), art. 7. The general form of the theorem (26.), art. 10. The processes and results of arts. 12 to 14. The generalization of Sir W. Hamilton's transformation of the dynamical equations, arts. 17, 18. The demonstration of Poisson's theorem, arts. 21, 22. The contents of art. 25. The method of obtaining elliptic elements, arts. 27 to 30. The contents of arts. 34 to 36. The solution of the problem of rotation, Section III.]

this equation would become identical if $x_1, x_2, \ldots x_n$, in its second member, were expressed in terms of $y_1, y_2, \ldots y_n$; hence, differentiating each side, on this hypothesis, first with respect to y_i, and then with respect to y_j, we obtain

$$1=\frac{dy_i}{dx_1}\frac{dx_1}{dy_i}+\frac{dy_i}{dx_2}\frac{dx_2}{dy_i}+\ldots+\frac{dy_i}{dx_n}\frac{dx_n}{dy_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$0=\frac{dy_i}{dx_1}\frac{dx_1}{dy_j}+\frac{dy_i}{dx_2}\frac{dx_2}{dy_j}+\ldots+\frac{dy_i}{dx_n}\frac{dx_n}{dy_j}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

where j is any index different from i. These theorems are given by JACOBI in his memoir "De Determinantibus functionalibus." They are however only particular cases of more general theorems, which may be investigated as follows.

If we represent by

$$i, j, k, \ldots .$$

$$p, q, r, \ldots .$$

any two determinate sets of m indices each, selected out of the series $1, 2, 3, \ldots n$, then the determinant formed with the m^2 differential coefficients

$$\frac{dy_i}{dx_p}, \frac{dy_i}{dx_q}, \ldots . ; \frac{dy_j}{dx_p}, \frac{dy_j}{dx_q}, \ldots ; \text{\&c.}$$

possesses properties remarkably analogous to those of a simple differential coefficient. This analogy was pointed out by JACOBI, and has been further developed by M. BERTRAND in his "Mémoire sur le Déterminant d'un système de Fonctions" (LIOUVILLE's Journal, 1851).

It appears to me that such functional determinants might be appropriately and conveniently denoted by a symbol analogous to that of a common differential coefficient; thus

$$\frac{d(y_i, y_j, y_k \ldots)}{d(x_p, x_q, x_r \ldots)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D.)$$

and I shall adopt this notation in the present paper. For example,

$$\frac{d(u, v)}{d(x, y)}$$

would represent the determinant

$$\frac{du}{dx}\frac{dv}{dy}-\frac{du}{dy}\frac{dv}{dx}.$$

[The expression (D.) is not a mere arbitrary symbol, but, like a simple differential coefficient, is a *real fraction*. For if we denote by

$$d(x_p, x_q, x_r, \ldots)$$

the determinant formed with the m^2 quantities

$$\begin{array}{l} d_1x_p, d_1x_q, d_1x_r, \ldots . \\ d_2x_p, d_2x_q, d_2x_r, \ldots \\ . \quad . \quad . \quad . \quad . \quad . \\ d_mx_p, d_mx_qd_m, x_r, \ldots . \end{array}$$

and attribute a corresponding meaning to

$$d(y_i, y_j, y_k, \ldots),$$

where $d_1, d_2, \ldots d_n$ are symbols denoting n distinct and independent sets of variations, so that

$$d_r y_i = \frac{dy_i}{dx_1} d_r x_1 + \frac{dy_i}{dx_2} d_r x_2 + \ldots + \frac{dy_i}{dx_n} d_r x_n,$$

then it follows from well-known properties of determinants (as M. BERTRAND has shown) that the complete functional determinant formed with the n^2 differential coefficients

$$\frac{dy_1}{dx_1}, \frac{dy_1}{dx_2}, \ldots, \frac{dy_2}{dx_1}, \frac{dy_2}{dx_2}, \ldots \text{ \&c.}$$

is equal to the quotient of the two determinants which I propose to denote by

$$d(y_1, y_2, y_3, \ldots y_n), \quad d(x_1, x_2, x_3, \ldots x_n),$$

and moreover that the partial functional determinant formed with the m^2 terms

$$\frac{dy_i}{dx_p}, \frac{dy_i}{dx_q}, \ldots, \frac{dy_j}{dx_p}, \frac{dy_j}{dx_q}, \ldots, \text{ \&c.}$$

is equal to the quotient of the two partial determinants

$$d(y_i, y_j, y_k, \ldots), \; d(x_p, x_q, x_r, \ldots),$$

the differentials of y_i, &c. being taken on the hypothesis that all the differentials of the x-variables are $=0$, except those of the set $x_p, x_q, x_r, \ldots$. Thus the expression (D.) is a real fraction, provided its numerator and denominator be interpreted in a manner exactly analogous to that in which the numerator and denominator of an ordinary total or partial differential coefficient are interpreted.]

This being premised, let $u_1, u_2, \ldots u_m$ be m functions of any or all of the functions $y_1, y_2, \ldots y_n$ (m being supposed not greater than n), so that u_1, u_2, &c. are functions of x_1, x_2, &c. through y_1, y_2, &c.

Let any selected sets of m indices out of the series $1, 2, \ldots n$, be denoted, for greater clearness, by $\alpha_1, \alpha_2, \ldots \alpha_m$; $\beta_1, \beta_2, \ldots \beta_m$, &c. Then the general theorem analogous to

$$du_i = \frac{du_i}{dy_1} dy_1 + \frac{du_i}{dy_2} dy_2 + \text{\&c.}$$

may be expressed as follows:—

$$d(u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_m}) = \Sigma_\beta \left\{ \frac{d(u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_m})}{d(y_{\beta_1}, y_{\beta_2}, \ldots, y_{\beta_m})} d(y_{\beta_1}, y_{\beta_2}, \ldots, y_{\beta_m}) \right\}$$

(the summation on the second side referring only to the indices β, and extending to every combination of m out of the n numbers $1, 2, \ldots n$).

In like manner, the theorem analogous to

$$\frac{du_i}{dx_j} = \frac{du_i}{dy_1} \frac{dy_1}{dx_j} + \frac{du_i}{dy_2} \frac{dy_2}{dx_j} + \ldots$$

is

$$\frac{d(u_{\alpha_1}, u_{\alpha_2}, \ldots u_{\alpha_m})}{d(x_{\gamma_1}, x_{\gamma_2}, \ldots x_{\gamma_m})} = \Sigma_\beta \left\{ \frac{d(u_{\alpha_1}, u_{\alpha_2}, \ldots u_{\alpha_m})}{d(y_{\beta_1}, y_{\beta_2}, \ldots y_{\beta_m})} \cdot \frac{d(y_{\beta_1}, y_{\beta_2}, \ldots y_{\beta_m})}{d(x_{\gamma_1}, x_{\gamma_2}, \ldots x_{\gamma_m})} \right\}.$$

These two theorems (expressed in a different notation) may be found in the memoirs above cited. But the following, which we shall have occasion to employ hereafter, has not, so far as I am aware, been explicitly stated.

Inasmuch as $\frac{dy_i}{dy_i}=1$, $\frac{dy_i}{dy_j}=0$, it follows that the determinant represented by

$$\frac{d(y_{\alpha_1}, y_{\alpha_2}, \ldots y_{\alpha_m})}{d(y_{\beta_1}, y_{\beta_2}, \ldots y_{\beta_m})} \quad \ldots \ldots \ldots \ldots \quad \text{(E.)}$$

is $=1$ if $\beta_1, \beta_2, \ldots \beta_m$ be the *same combination of indices* as $\alpha_1, \alpha_2, \ldots \alpha_m$, but is $=0$ in every other case. (For in the first case the determinant is formed with $1, 0, 0, \ldots$; $0, 1, 0, \ldots$; $0, 0, 1, \ldots$; &c., but if there be one index β_i which is not contained in the series α_1, α_2, &c., then one row of terms in the determinant will consist wholly of zeros.)

Now considering y_1, y_2, &c. as functions of x_1, x_2, &c., and again considering these latter as functions of y_1, y_2, &c. given by the inverse equations, we have, by the preceding theorem, for the value of the determinant (E.) above written, the expression $\nabla_m=$

$$\Sigma_\gamma\left\{\frac{d(y_{\alpha_1}, y_{\alpha_2}, \ldots y_{\alpha_m})}{d(x_{\gamma_1}, x_{\gamma_2}, \ldots x_{\gamma_m})}\cdot\frac{d(x_{\gamma_1}, x_{\gamma_2}, \ldots x_{\gamma_m})}{d(y_{\beta_1}, y_{\beta_2}, \ldots y_{\beta_m})}\right\}$$

(where $\alpha_1, \alpha_2, \ldots \alpha_m$; $\beta_1, \beta_2, \ldots \beta_m$ are two *determinate* sets of m out of the n indices, and the summation with respect to the indices γ extends to every combination of m out of the n). Consequently,

$$\nabla_m=1 \text{ or } \nabla_m=0, \quad \ldots \ldots \ldots \ldots \quad (3.)$$

according as the series of indices

$$\beta_1, \beta_2, \ldots \beta_m$$

is, or is not, the same combination as

$$\alpha_1, \alpha_2, \ldots. \alpha_m.$$

(I suppose, for convenience, that when the two *combinations* are the same, *the arrangement* is the same in each; otherwise the value of ∇_m may be -1.)

This is the theorem in question. If we put $m=1$, we obtain the equations (1.) and (2.) given at the beginning of this article. If we put $m=n$, the expression ∇_n reduces itself to the product of the two determinants formed respectively with the *complete* sets of differential coefficients $\frac{dy_i}{dx_j}$ &c., $\frac{dx_i}{dy_j}$ &c., the value of which product is $=1$, as is well known.

As an illustration, it may be useful to exhibit the theorem in the case of $m=2$, as expressed by the common notation. Namely,

$$\nabla_2=\Sigma\left\{\left(\frac{dy_p}{dx}\,\frac{dy_q}{dx_j}-\frac{dy_p}{dx_j}\,\frac{dy_q}{dx_i}\right)\left(\frac{dx_i}{dy_\alpha}\,\frac{dx_j}{dy_\beta}-\frac{dx_i}{dy_\beta}\,\frac{dx_j}{dy_\alpha}\right)\right\}=1, \text{ or } =0, \quad \ldots \quad (4.)$$

according as α, β are, or are not, the same as p, q. Here α, β; p, q are two determinate pairs of indices, and the summation refers to i, j, extending to every binary combination.

2. *Theorem.*—Retaining the suppositions made at the beginning of the last article, let X be a given function of $x_1, x_2, \ldots x_n$; and let us further suppose that the equations by which $y_1, y_2, \ldots y_n$ are determined as functions of x_1, &c., are

$$y_1=\frac{dX}{dx_1},\ y_2=\frac{dX}{dx_2},\ \ldots,\ y_n=\frac{dX}{dx_n},\quad \text{(5.)}$$

so that
$$\frac{dy_i}{dx_j}=\frac{dy_j}{dx_i};$$

and if we transform the equations (1.), (2.), art. 1, by this condition, we obtain the n equations

$$\frac{dy_1}{dx_1}\frac{dx_1}{dy_i}+\frac{dy_2}{dx_1}\frac{dx_2}{dy_i}+\ldots+\frac{dy_n}{dx_1}\frac{dx_n}{dy_i}=0$$
$$\frac{dy_1}{dx_2}\frac{dx_1}{dy_i}+\frac{dy_2}{dx_2}\frac{dx_2}{dy_i}+\ldots+\frac{dy_n}{dx_2}\frac{dx_n}{dy_i}=0$$
$$\cdots\cdots\cdots\cdots$$
$$\frac{dy_1}{dx_i}\frac{dx_1}{dy_i}+\frac{dy_2}{dx_i}\frac{dx_2}{dy_i}+\ldots+\frac{dy_n}{dx_i}\frac{dx_n}{dy_i}=1$$
$$\cdots\cdots\cdots\cdots$$

If these equations be added, after multiplying them respectively by

$$\frac{dx_1}{dy_j},\ \frac{dx_2}{dy_j},\ \ldots\ \frac{dx_n}{dy_j},$$

the sum of the first members reduces itself by virtue of the equations (1.), (2.), to $\frac{dx_j}{dy_i}$, whilst the second side consists of the single term $\frac{dx_i}{dy_j}$. We have then

$$\frac{dx_i}{dy_j}=\frac{dx_j}{dy_i},$$

or, in other words, *if* $x_1, x_2, \ldots x_n$ *be found from the system of equations* (5.) *in terms of* $y_1, y_2, \ldots y_n$, *the resulting expressions are the partial differential coefficients of a certain function of* $y_1, y_2, \ldots, y_n$, *so that the system inverse to* (5.) *is of the form*

$$x_1=\frac{dY}{dy_1},\ x_2=\frac{dY}{dy_2},\ \ldots,\ x_n=\frac{dY}{dy_n}.\quad \text{(6.)}$$

The relation between X and Y is easily found as follows. The equations (5.) and (6.) give

$$dX=y_1dx_1+y_2dx_2+\ldots+y_ndx_n$$
$$dY=x_1dy_1+x_2dy_2+\ldots+x_ndy_n;$$

whence, by addition, $d(X+Y)=d(x_1y_1+x_2y_2+\ldots+x_ny_n)$,

and therefore $X+Y=x_1y_1+x_2y_2+\ldots+x_ny_n \quad \text{(7.)}$

(omitting the arbitrary constant, which might of course be added).

The actual value of Y will then be

$$Y=-(X)+(x_1)y_1+(x_2)y_2+\ldots+(x_n)y_n,\quad \text{(8.)}$$

in which the brackets indicate that $x_1, x_2, \ldots x_n$ are to be expressed in terms of $y_1, y_2, \ldots y_n$, so that Y may be a function of the latter variables only. It is easy to show *à posteriori* that the expression (8.) verifies the equations (6.), but I pass on to some further considerations. (See note at the end of Section II.)

3. Suppose the function X involves explicitly, besides the variables x_1, x_2, &c., any other quantity p, so that the expressions (x_1), (x_2), &c. (or the values of $x_1, x_2, \ldots$ in terms of y_1, y_2, &c.) will also involve p explicitly, and we shall have

$$\frac{d(\mathrm{X})}{dp}=\frac{d\mathrm{X}}{dp}+\frac{d\mathrm{X}}{dx_1}\frac{d(x_1)}{dp}+\frac{d\mathrm{X}}{dx_2}\frac{d(x_2)}{dp}+\ldots$$
$$=\frac{d\mathrm{X}}{dp}+y_1\frac{d(x_1)}{dp}+y_2\frac{d(x_2)}{dp}+\ldots$$

Now, differentiating the equation (8.) with respect to p (so far as it contains p *explicitly*), we obtain

$$\frac{d\mathrm{Y}}{dp}=-\frac{d(\mathrm{X})}{dp}+y_1\frac{d(x_1)}{dp}+y_2\frac{d(x_2)}{dp}+\ldots$$

which the equation above written reduces simply to

$$\frac{d\mathrm{X}}{dp}+\frac{d\mathrm{Y}}{dp}=0. \quad \ldots \ldots \quad (9.)$$

In the particular case in which X is a homogeneous function of $x_1, x_2, \ldots x_n$, and of m dimensions with respect to those variables, the equations (8.) and (9.) become

$$\left.\begin{aligned}&\mathrm{Y}=(m-1)(\mathrm{X})\\&\frac{d\mathrm{X}}{dp}+(m-1)\frac{d(\mathrm{X})}{dp}=0\end{aligned}\right\} \quad \ldots \ldots \quad (10.)$$

and it is easily seen that Y is also homogeneous and of $\frac{m}{m-1}$ dimensions in $y_1, y_2, \ldots y_n$.

4. The theorems (8.) and (9.) are cases of more general ones which are easily proved in a perfectly similar way, and which I shall therefore only enunciate. If, by means of the equations (5.), art. 2, we express a set of n out of the $2n$ variables, consisting of r x's and $n-r$ y's, of which no two indices are the same, for example,

$$x_1, x_2, \ldots x_r, y_{r+1}, \ldots, y_n \quad \ldots \ldots \quad (\alpha.)$$

in terms of the remaining n variables,

$$y_1, y_2, \ldots y_r, x_{r+1}, \ldots, x_n; \quad \ldots \ldots \quad (\beta.)$$

then, taking $\qquad \mathrm{Q}=-(\mathrm{X})+(x_1)y_1+(x_2)y_2+\ldots+(x_r)y_r$

(in which the brackets indicate that the variables of the set (α.) are to be expressed in terms of those of the set (β.), so that Q is a function of the latter set), we shall have

$$\frac{d\mathrm{Q}}{dy_i}=x_i \text{ from } i=0 \text{ to } i=r,$$

$$\frac{d\mathrm{Q}}{dx_j}=-y_j \text{ from } j=r+1 \text{ to } j=n,$$

and $$\frac{dX}{dp}+\frac{dQ}{dp}=0, \text{ as before*};$$

but the equations corresponding to (10.) will not subsist unless X be homogeneous with respect to the r variables $x_1, x_2, \dots x_r$.

5. Let us now suppose that the function X contains, explicitly, besides the n variables $x_1, x_2, \dots x_n$, another variable t, and also n constants $a_1, a_2, \dots a_n$; and that these last are contained in such a way that the n equations

$$\frac{dX}{da_1}=b_1, \frac{dX}{da_2}=b_2, \dots \frac{dX}{da_n}=b_n \quad \dots \dots \dots \dots \quad (11.)$$

would be algebraically sufficient to determine $a_1, a_2, \dots a_n$ in terms of b_1, b_2, &c., x_1, &c. Then taking $$X_b=-(X)+(a_1)b_1+(a_2)b_2+\dots+(a_n)b_n$$

(the brackets indicating that a_1, a_2, &c. are to be expressed as above supposed), we shall have, by the theorems of arts. 2 and 3,

$$\frac{dX_b}{db_1}=a_1, \frac{dX_b}{db_2}=a_2, \dots \frac{dX_b}{db_n}=a_n; \quad \dots \dots \dots \quad (12.)$$

and also, for all values of i,

$$\frac{dX_b}{dx_i}=-\frac{dX}{dx_i}=-y_i; \quad \dots \dots \dots \dots \quad (13.)$$

to which we may add

$$\frac{dX_b}{dt}=-\frac{dX}{dt}=\frac{dY}{dt}. \quad \dots \dots \dots \dots \quad (14.)$$

Now assuming the $2n$ equations (5.) and (11.), namely (for all values of i),

$$\frac{dX}{dx_i}=y_i, \ \frac{dX}{da_i}=b_i,$$

we may suppose each of the $2n$ variables $x_1, x_2, \dots y_1, y_2, \dots$ to be expressed by means of them as a function of the $2n$ constants a_1, &c., b_1, &c., and t; or, conversely, each of the $2n$ constants to be expressed as a function of the variables x_1, &c., y_1, &c., and t. On the former hypothesis each of the variables $x_1, \dots y_1, \dots$ is given as an explicit, and on the latter as an implicit function of the single variable t, which we will consider as independent; and *total differentiation with respect to* t will throughout this paper be denoted by *accents*, which will be used for no other purpose. Thus, p being any function of all the variables, we shall have

$$p'=\frac{dp}{dt}+\frac{dp}{dx_1}x'_1+\dots+\frac{dp}{dy_1}y'_1+\dots$$

For the rest, we may, when necessary, distinguish the meanings of the various partial

* Although these theorems, as stated in the text, are more general in form than those of the preceding article, they may, under another point of view, be considered as particular cases of them, and may in this way be best established.

differential coefficients employed, by referring to the hypotheses on which they are taken, and which I shall denote as follows:—

Hyp. I.—The $2n$ variables $x_1, x_2, \dots y_1, y_2, \dots$ expressed as functions of $a_1, a_2, \dots b_1, b_2, \dots$ and t.

Hyp. II.—The $2n$ constants $a_1, a_2, \dots b_1, b_2, \dots$ expressed as functions of $x_1, x_2, \dots y_1, y_2, \dots$ and t.

Hyp. III.—The n variables $y_1, y_2, \dots y_n$ expressed as functions of the n variables $x_1, x_2, \dots x_n$, the n constants $a_1, a_2, \dots a_n$, and t (as by equations (5.)).

Hyp. IV.—The n constants $b_1, b_2, \dots b_n$ expressed as functions of the n variables $x_1, \dots x_n$, the n constants $a_1, \dots a_n$, and t (as by equations (11.)).

6. Differentiating totally the equation (11.),

$$\frac{dX}{da_i}=b_i$$

with respect to t, we obtain (observing that $\frac{d^2X}{da_i dx_j}=\frac{dy_j}{da_i}$ by virtue of the conditions (5.)),

$$\frac{d^2X}{da_i dt}+\frac{dy_1}{da_i}x'_1+\frac{dy_2}{da_i}x'_2+\dots+\frac{dy_n}{da_i}x'_n=0$$

$\left(\text{where } \frac{dy_1}{da_i}, \text{ \&c. are taken on } Hyp. \text{ III., art. 5.}\right)$.

Now let (Z) be a function of $x_1, \dots x_n, t, a_1, \dots a_n$, defined by the equation

$$(Z)=-\frac{dX}{dt}, \qquad (15.)$$

the above equation then becomes

$$\frac{d(Z)}{da_i}=\frac{dy_1}{da_i}x'_1+\frac{dy_2}{da_i}x'_2+\dots+\frac{dy_n}{da_i}x'_n.$$

If this equation be multiplied by $\frac{da_i}{dy_j}$, and the result on each side summed with respect to i, it will be seen that the coefficients of x'_1, x'_2, &c. on the second side all vanish except that of x'_j, which reduces itself to 1 (see art. 1, equations (1.) (2.)); so that we have

$$\frac{d(Z)}{da_1}\frac{da_1}{dy_j}+\frac{d(Z)}{da_2}\frac{da_2}{dy_j}+\dots+\frac{d(Z)}{da_n}\frac{da_n}{dy_j}=x'_j.$$

Now the expression on the left of this equation is equivalent to

$$\frac{dZ}{dy_j},$$

if by Z (without brackets) we denote the result of substituting for $a_1, a_2, \dots a_n$ in (Z), their values in terms of all the variables (*Hyp.* II.), so that Z is a function of the *variables only*. We have then, finally (writing i instead of j),

$$x'_i=\frac{dZ}{dy_i}. \qquad (16.)$$

Again, we have (*Hyp.* III.) $\quad y'_i=\frac{dy_i}{dt}+\frac{dy_i}{dx_1}x'_1+\frac{dy_i}{dx_2}x'_2+\dots$

(see equations (6.) and (9.), putting a_i for p in the latter) exactly in the same way, it is plain that the result may be deduced from (18.) by interchanging x and y, and changing the sign of b; thus

$$\frac{db_j}{dx_k}=\frac{dy_k}{da_j}.$$

Lastly, from the equations

$$\frac{dX_b}{db_i}=a_i,\ \frac{dX_b}{dx_i}=-y_i \text{ (see (12.) and (13.))},$$

we should find in a similar manner

$$\frac{da_j}{dy_k}=\frac{dx_k}{db_j};$$

and from the analogous equations (the existence of which is obvious)

$$\frac{dY_b}{dy_i}=-x_i,\ \frac{dY_b}{db_i}=-a_i$$

we should obtain $$\frac{da_j}{dx_k}=-\frac{dy_k}{db_j}.$$

Collecting these results, and changing the indices, we have the system

$$\left.\begin{array}{ll}\dfrac{dx_i}{da_j}=-\dfrac{db_j}{dy_i}, & \dfrac{dx_i}{db_j}=\dfrac{da_j}{dy_i}\\ \dfrac{dy_i}{da_j}=\dfrac{db_j}{dx_i}, & \dfrac{dy_i}{db_j}=-\dfrac{da_j}{dx_i}\end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (19.)$$

in each of which equations the first member refers to *Hyp.* I., and the second to *Hyp.* II. (art. 5.); and it is to be remembered that there is no relation between the indices of the variables and those of the constants, so that the case of $i=j$ has no peculiarity*.

8. Let δ, Δ be symbols denoting two distinct sets of arbitrary and independent variations attributed to the $2n$ constants; then the equations

$$\frac{dX}{dx_i}=y_i,\ \frac{dX}{da_i}=b_i$$

give $$\delta X=\Sigma_i(y_i\delta x_i+a_i\delta b_i);$$

and if the operation Δ be performed on each side, we have

$$\begin{aligned}\Delta\delta X=&\Sigma_i(\Delta y_i\delta x_i+\Delta a_i\delta b_i)\\&+\Sigma_i(y_i\Delta\delta x_i+a_i\Delta\delta b_i).\end{aligned}$$

* It is remarkable that each of the equations (19.) is also true on a different and separate hypothesis, as is apparent on inspection of the four different sets of equations,

$$\frac{dX}{dx_i}=y_i,\ \frac{dX_b}{dx_i}=-y_i,\ \frac{dY}{dy_i}=x_i,\quad \frac{dY_b}{dy_i}=-x_i$$

$$\frac{dX}{da_i}=b_i,\ \frac{dX_b}{db_i}=a_i,\quad \frac{dY}{da_i}=-b_i,\ \frac{dY_b}{db_i}=-a_i$$

(see the preceding articles).

according as h, k are or are not a *conjugate pair*, i. e. of the form a_j, b_j. (The value $+1$ belongs to $h=a_j$, $k=b_j$, and -1 to the converse.)

According to the notation proposed at the beginning of this paper, the above formula may be written

$$\Sigma_i \frac{d(h, k)}{d(y_i, x_i)} = \Sigma_i \frac{d(y_i, x_i)}{d(h, k)} = \pm 1, \text{ or } = 0.$$

By a usual and convenient abbreviation, the sum

$$\Sigma_i \frac{d(h, k)}{d(y_i, x_i)}$$

may be denoted by the symbol* $[h, k]$. We have then, by (22.),

$$[a_i, b_i] = -[b_i, a_i] = 1 \qquad [a_i, b_j] = [a_i, a_j] = [b_i, b_j] = 0, \quad \ldots \ldots \quad (23.)$$

j being different from i; and, obviously,

$$[a_i, a_i] = [b_i, b_i] = 0.$$

Now let f, g be any two functions whatever of the $2n$ constants a_1, &c. b_1, &c.; when the latter are expressed in terms of the variables (*Hyp.* II.), f, g become also functions of the variables; and if h, k represent, as above, any pair whatever of a_1, &c., b_1, &c., we have (see art. 1.)

$$\frac{d(f, g)}{d(y_i, x_i)} = \Sigma \left\{ \frac{d(f, g)}{d(h, k)} \cdot \frac{d(h, k)}{d(y_i, x_i)} \right\},$$

the summation referring to h, k, and extending to every binary combination.

If, now, we sum each side of this equation *with respect to* i, we obtain

$$[f, g] = \Sigma \left\{ [h, k] \cdot \frac{d(f, g)}{d(h, k)} \right\} \quad \ldots \ldots \ldots \quad (24.)$$

(the summation referring as before to h, k). But, by (23.), $[h, k]$ is 0 unless h, k be a conjugate pair, and then it is ± 1; so that (24.) becomes simply

$$[f, g] = \Sigma_i \frac{d(f, g)}{d(a_i, b_i)}, \quad \ldots \ldots \ldots \quad (25.)$$

an equation which, written at length in the common notation, is

$$\Sigma_i \left(\frac{df}{dy_i} \frac{dg}{dx_i} - \frac{df}{dx_i} \frac{dg}{dy_i} \right) = \Sigma_i \left(\frac{df}{da_i} \frac{dg}{db_i} - \frac{df}{db_i} \frac{dg}{da_i} \right).$$

The expression on the right being a function of the constants a_1, &c., b_1, &c. only, the equation (25.) expresses obviously the following theorem.

If
$$f = \varphi(x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n, t)$$
$$g = \psi(x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n, t)$$

be any two integrals of the system of simultaneous equations (16.), (17.), art. 6, then

* POISSON employs the notation (h, k), which would have led to confusion if adopted here. LAGRANGE (in the *Méc. Anal.*) uses $[h, k]$, but with a different signification. See below, note to art. 34.

other quantities whatever, except a_1, &c. It is assumed that these equations are algebraically sufficient to determine each of the n variables $y_1, \dots y_n$, as a function of the other n variables $x_1, \dots x_n$ and the constants. Then the theorem in question is as follows:—

If, by means of the equations (a.), the n variables $y_1, \dots y_n$ be expressed as functions of x_1, &c., then in order that the conditions

$$\frac{dy_i}{dx_j}=\frac{dy_j}{dx_i}$$

may subsist identically, it is necessary and sufficient that the expression $[a_i, a_j]$ (defined as in art. 9.) shall vanish for every binary combination of the n equations. This may be proved as follows:—

Putting h, k for any two of the constants a_1, a_2, &c., let $h=\phi(x_1$, &c., y_1, &c.) represent one of the equations (a.) above written. If in this equation the values of $y_1, \dots y_n$ be expressed, as above supposed, in terms of x_1, &c., a_1, &c., it becomes identical. Differentiating it, on this hypothesis, with respect to x_i, we obtain

$$\frac{dh}{dx_i}+\frac{dh}{dy_1}\frac{dy_1}{dx_i}+\frac{dh}{dy_2}\frac{dy_2}{dx_i}+\dots+\frac{dh}{dy_n}\frac{dy_n}{dx_i}=0\,;$$

and in like manner

$$\frac{dk}{dx_i}+\frac{dk}{dy_1}\frac{dy_1}{dx_i}+\frac{dk}{dy_2}\frac{dy_2}{dx_i}+\dots+\frac{dk}{dy_n}\frac{dy_n}{dx_i}=0\,;$$

and if we multiply the first of these equations by $\frac{dk}{dy_i}$ and the second by $\frac{dh}{dy_i}$ and subtract, there results an equation which may be written as follows:—

$$\frac{dh}{dy_i}\frac{dk}{dx_i}-\frac{dh}{dx_i}\frac{dk}{dy_i}=\Sigma_j\left\{\frac{dy_j}{dx_i}\left(\frac{dh}{dy_j}\frac{dk}{dy_i}-\frac{dh}{dy_i}\frac{dk}{dy_j}\right)\right\}\,;$$

or, putting now a_p, a_q instead of h, k, and employing the same notation as before,

$$\frac{d(a_p, a_q)}{d(y_i, x_i)}=\Sigma_j\left\{\frac{dy_j}{dx_i}\cdot\frac{d(a_p, a_q)}{d(y_j, y_i)}\right\}.$$

If now the terms on each side be summed with respect to i, the result on the first side is $[a_p, a_q]$; and observing that on the second side the term multiplied by $\frac{dy_i}{dx_j}$ will only differ in *sign* from that multiplied by $\frac{dy_j}{dx_i}$, we shall have

$$[a_p, a_q]=\Sigma_i\Sigma_j\left\{\left(\frac{dy_j}{dx_i}-\frac{dy_i}{dx_j}\right)\frac{d(a_p, a_q)}{d(y_j, y_i)}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27.)$$

the summation on the right extending to all binary combinations i, j. Suppose this equation to be written at length, and then after multiplying each side by

$$\frac{d(y_r, y_s)}{d(a_p, a_q)},$$

let the sum be taken with respect to all the binary combinations p, q. It follows

from the theorems of art. 1, that the coefficient of

$$\frac{dy_r}{dx_s}-\frac{dy_s}{dx_r}$$

on the right will reduce itself to unity, and that of each of the remaining terms to zero; so that we shall have, writing now j, i for r, s,

$$\frac{dy_j}{dx_i}-\frac{dy_i}{dx_j}=\Sigma_p\Sigma_q\left\{\left[a_p, a_q\right]\cdot\frac{d(y_j, y_i)}{d(a_p, a_q)}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28.)$$

In order then that the expression $\frac{dy_j}{dx_i}-\frac{dy_i}{dx_j}$ should vanish identically for every binary combination of indices, it follows from (28.) that it is *sufficient*, and from (27.) that it is *necessary*, that each of the $\frac{n(n-1)}{2}$ terms $\left[a_p, a_q\right]$ should vanish, and *vice versâ*. It will be observed that the terms $\left[a_p, a_q\right]$ cannot vanish otherwise than identically, since they do not contain any of the constants a_1, a_2, &c., and it is by hypothesis impossible to eliminate all these constants from the equations (a.). It follows then that when the conditions $\left[a_p, a_q\right]=0$ subsist, the values of $y_1, \ldots y_n$ expressed as above, are identically the partial differential coefficients of a function of $x_1, \ldots x_n, a_1, \ldots a_n$.

We have thus established the theorem enunciated at the beginning of this article.

13. The preceding theorem may be made somewhat more general as follows:—

If we divide the $2n$ variables into any two sets of n each, so that no two in the same set are *conjugate* (as for instance

$$x_1, x_2, \ldots x_r, y_{r+1}, \ldots y_n$$

$$y_1, y_2, \ldots y_r, x_{r+1}, \ldots x_n),$$

and denote one set by

$$\xi_1, \xi_2, \ldots \xi_n,$$

and the other by

$$\pm\eta_1, \pm\eta_2, \ldots \pm\eta_n,$$

taking the + or − sign according as η_i represents y_i or x_i, it is obvious that the expression $\Sigma_i \frac{d(a_p, a_q)}{d(\eta_i, \xi_i)}$ is identical with $\left[a_p, a_q\right]$; and therefore whenever all the terms $\left[a_p, a_q\right]$ vanish, if the set $\eta_1, \eta_2, \ldots \eta_n$ can be expressed by means of the equations (a.) of the last article, in terms of $\xi_1, \xi_2, \ldots \xi_n, a_1, a_2, \ldots a_n$, their values will be the partial differential coefficients with respect to $\xi_1, \xi_2, \ldots \xi_n$, of a function of these variables and of the constants.

14. *Theorem.*—If of the system of $2n$ simultaneous differential equations of the first order

$$\left.\begin{aligned} x'_1=\frac{dZ}{dy_1}, \ldots, x'_n=\frac{dZ}{dy_n} \\ y'_1=-\frac{dZ}{dx_1}, \ldots, y'_n=-\frac{dZ}{dx_n}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

(where Z denotes any function of $x_1, \ldots x_n, y_1, \ldots y_n$ and t, and accents denote as usual

total differentiation with respect to t) there be given n integrals, involving n arbitrary constants $a_1, \dots a_n$, as

$$a_i = \varphi_i(x_1, x_2, \dots x_n, y_1, y_2, \dots y_n, t),$$

the remaining integrals may be found, whenever the $\frac{n(n-1)}{2}$ conditions $[a_i, a_j]=0$ are satisfied.

For let $y_1, y_2, \dots y_n$ be expressed, by means of the given integrals, in terms of

$$x_1, \dots x_n, a_1, \dots a_n, t.$$

Their values so expressed will satisfy (art. 12.) the conditions

$$\frac{dy_j}{dx_i} - \frac{dy_i}{dx_j} = 0. \quad \dots \dots \dots \quad (b.)$$

Let (Z) represent the result of substituting in Z these values of $y_1, \dots, y_n$, so that (Z) is a given function of $x_1, \dots x_n, a_1, \dots a_n, t$. We shall have

$$\frac{d(\mathrm{Z})}{dx_i} = \frac{d\mathrm{Z}}{dx_i} + \frac{d\mathrm{Z}}{dy_1}\frac{dy_1}{dx_i} + \frac{d\mathrm{Z}}{dy_2}\frac{dy_2}{dx_i} + \dots$$

which the equations (I.) and (b.) reduce to

$$\frac{d(\mathrm{Z})}{dx_i} = -y'_i + \frac{dy_i}{dx_1}x'_1 + \frac{dy_i}{dx_2}x'_2 + \dots.$$

but
$$y'_i = \frac{dy_i}{dt} + \frac{dy_i}{dx_1}x'_1 + \frac{dy_i}{dx_2}x'_2 + \dots.$$

consequently
$$\frac{d(\mathrm{Z})}{dx_i} = -\frac{dy_i}{dt}. \quad \dots \dots \dots \quad (c.)$$

Looking now at the assemblage of equations (b.), (c.), we see that they express the following proposition:—

The values of $\quad y_1, y_2, \dots, y_n, -(\mathrm{Z}),$

are the partial differential coefficients with respect to $x_1, x_2, \dots x_n, t$, of one and the same function. Let this function be called X; we have then

$$\frac{d\mathrm{X}}{dx_i} = y_i, \ \frac{d\mathrm{X}}{dt} = -(\mathrm{Z})\,; \quad \dots \dots \dots \quad (\mathrm{II.})$$

and since $y_1, \dots, y_n$, (Z) are given functions of x_1, &c., a_1, &c., t, the function X can be found by simple integration.

Let us then suppose X to be known, and let us take the total differential coefficient with respect to t, of $\frac{d\mathrm{X}}{da_i}$; we shall have

$$\left(\frac{d\mathrm{X}}{da_i}\right)' = \frac{d^2\mathrm{X}}{da_i dt} + \frac{d^2\mathrm{X}}{da_i dx_1}\cdot x'_1 + \frac{d^2\mathrm{X}}{da_i dx_2}\cdot x'_2 + \dots.$$

which, by virtue of (I.) and (II.), becomes

$$\left(\frac{d\mathrm{X}}{da_i}\right)' = -\frac{d(\mathrm{Z})}{da_i} + \frac{d\mathrm{Z}}{dy_1}\frac{dy_1}{da_i} + \frac{d\mathrm{Z}}{dy_2}\frac{dy_2}{da_i} + \dots$$

but $$\frac{d(Z)}{da_i}=\frac{dZ}{dy_1}\frac{dy_1}{da_i}+\frac{dZ}{dy_2}\frac{dy_2}{da_i}+\dots$$

(since (Z) is derived from Z by introducing the values of $y_1, \dots y_n$, in terms of x_1, &c., $a_1, \dots a_n$), hence the second member of the preceding equation vanishes, and we have

$$\left(\frac{dX}{da_i}\right)'=0,$$

so that $\frac{dX}{da_i}$ is constant, and we may write

$$\frac{dX}{da_i}=b_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

and b_i is an independent arbitrary constant, as it is easy to prove; it is however unnecessary to do so here, because we have in fact already proved it in showing that the elimination of $a_1, \dots a_n, b_1, \dots b_n$, from the system of equations (II.), (III.), leads to the differential equations (I.) (see art. 6.). The n equations (III.) give therefore the remaining n integrals of the system (I.), of which (II.) and (III.) together are the complete solution.

The system of equations (II.), (III.) being the same as that discussed in the preceding articles, all the conclusions there obtained will continue to subsist.

15. Suppose the expression for Z (see the last article) in terms of the variables is

$$Z=f(x_1, x_2, \dots x_n, y_1, y_2, \dots y_n, t),$$

Z is changed into (Z) by the substitution of $\frac{dX}{dx_1}$ for y_1, &c.; and since $\frac{dX}{dt}$ is (identically) $=-$(Z), the equation

$$\frac{dX}{dt}+f\left(x_1, x_2, \dots x_n, \frac{dX}{dx_1}, \dots, \frac{dX}{dx_n}, t\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(X.)}$$

is a partial differential equation satisfied by the function X.

We have thus arrived, by an inverse route, at the point from which Sir W. HAMILTON's theory, as improved by JACOBI, sets out.

JACOBI, namely, has shown (by a demonstration immediately applying only to a particular form of the equation (X.), but easily extended), that if X be *any "complete" solution* of the equation (X.), that is, a solution involving (besides the constant which may be merely added to X) n arbitrary constants $a_1, a_2, \dots a_n$, in such a way that they cannot be all eliminated from the $n+1$ equations obtained by differentiating X with respect to $x_1, \dots x_n, t$, without employing *all* those equations, then X possesses the properties of Sir W. HAMILTON's "Principal Function," or in other words, gives all the integrals of the system (I.) by means of the system (II.), (III.). It will be desirable briefly to indicate the mode in which this demonstration may be made to apply to the general form (X.).

Assuming that a complete solution X, of that equation, is given, put $\frac{dX}{dx_i}=y_i$; then

differentiating the equation (X.) with respect to x_i, and employing the equations

$$\frac{dy_p}{dx_q}=\frac{dy_q}{dx_p},$$

we have

$$\frac{dy_i}{dt}+\frac{df}{dx_i}+\frac{df}{dy_1}\frac{dy_i}{dx_1}+\frac{df}{dy_2}\frac{dy_i}{dx_2}+\ldots=0\,;$$

on the other hand, taking the differential coefficient of y_i with respect to t, without assuming anything as to the nature of the relations between t and the other variables, we find

$$y_i'=\frac{dy_i}{dt}+\frac{dy_i}{dx_1}x_1'+\frac{dy_i}{dx_2}x_2'+\ldots$$

and adding to this the preceding equation,

$$y_i'+\frac{df}{dx_i}=\frac{dy_i}{dx_1}\left(x_1'-\frac{df}{dy_1}\right)+\frac{dy_i}{dx_2}\left(x_2'-\frac{df}{dx_2}\right)+\ldots$$

from which it follows that the n assumptions

$$x_i'=\frac{df}{dy_i}$$

would involve the n further equations

$$y_i'=-\frac{df}{dx_i}.$$

Again, the n assumptions

$$\frac{dX}{da_i}=b_i$$

would give, by combining the n equations obtained by differentiating totally with respect to t, viz.

$$\frac{d^2X}{da_idt}+\frac{d^2X}{da_idx_1}x_1'+\frac{d^2X}{da_idx_2}x_2'+\ldots.=0,$$

with the n others obtained by differentiating the equation (X.) with respect to a_i, viz.

$$\frac{d^2X}{da_idt}+\frac{df}{dy_1}\frac{d^2X}{da_idx_1}+\frac{df}{dy_2}\frac{d^2X}{da_idx_2}+\ldots=0,$$

the n following, namely,

$$\frac{d^2X}{da_idx_1}\left(x_1'-\frac{df}{dy_1}\right)+\frac{d^2X}{da_idx_2}\left(x_2'-\frac{df}{dy_2}\right)+\ldots.=0,$$

from which it follows *either* that $x_i'=\frac{df}{dy_i}$, *or* that the determinant formed with the n^2 expressions $\frac{d^2X}{da_idx_j}$, or $\frac{d}{da_i}\left(\frac{dX}{dx_j}\right)$, vanishes; but this last condition would express, as is well known, the possibility of eliminating the n constants a_1, a_2, ... a_n from the n equations

$$\frac{dX}{dx_j}=\Psi_j(x_1,\ \&c.,\ a_1,\ \&c.,\ t),$$

which would contradict the assumption that X is a *complete* solution of the equation (X.).

similar transformation to the equations (T.), in the case in which no limitation is imposed upon the form of the function T, as I shall now proceed to show.

18. Putting $T+U=W$, we shall have (since U does not contain x'_1, &c.)

$$\left(\frac{dW}{dx'_i}\right)'=\frac{dW}{dx_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (W.)$$

Let $\frac{dW}{dx'_i}=y_i$; then if we take

$$Z=-(W)+(x'_1)y_1+(x'_2)y_2+\ldots+(x'_n)y_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (V.)$$

(where, in the terms enclosed in brackets, x'_1, x'_2, &c. are to be expressed in terms of y_1, y_2, &c., x_1, x_2, &c.), we shall have, by the theorems of the former articles (see equations (6.), (8.), (9.) of arts. 2 and 3, putting x'_i instead of x_i and x_i instead of p, in those equations),

$$x'_i=\frac{dZ}{dy_i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha.)$$

and

$$\frac{dW}{dx_i}=-\frac{dZ}{dx_i},$$

so that the equation (W.) becomes

$$y'_i=-\frac{dZ}{dx_i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta.)$$

and (α.), (β.) are of the form in question* ((I.), art. 14.). Thus, so far as the application of any methods of integration, founded upon the preceding principles, and the theories of Sir W. Hamilton and Jacobi, to the system (T.), art. 16, is concerned, there is no restriction to the form of the function T. This extension is probably at present of no practical importance, but may perhaps be thought of some interest in a purely analytical point of view.

19. Returning now to the suppositions and conclusions of art. 14, let us further suppose that Z does not contain t explicitly, so that

$$Z'=\Sigma_i\left(\frac{dZ}{dx_i}x'_i+\frac{dZ}{dy_i}y'_i\right)=0$$

by virtue of the system (I.); in this case

$$Z=h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (h.)$$

is one of the integrals of the system, and if we suppose this to be one of the n given integrals from which the principal function X is to be found, so that

$$h,\ a_1,\ a_2,\ \ldots\ a_{n-1}$$

are now the n arbitrary constants, and the conditions

$$[a_i,\ a_j]=0,\ \ [h,\ a_i]=0$$

subsist, it is plain that we shall have

$$(Z)=h,$$

* In the case in which T is homogeneous and of the second degree, in x'_1, x'_2, ... x'_n, it is obvious that the expression for Z reduces itself to $2(T)-W$, or $(T)-U$.

form (I.), or for the transformation of known solutions into forms convenient for the application of the method of variation of elements—by making the discovery of principal functions depend upon that of n integrals satisfying given conditions, rather than upon the solution of a partial differential equation. Having now prepared the way for this inquiry, I shall proceed with it in the following section.

Section II.

21. *Theorem.*—If p, q, r be any three functions whatever of the $2n$ variables $x_1, \ldots, x_n, y_1, \ldots y_n$, then

$$[[p,q],r]+[[q,r],p]+[[r,p],q]=0. \quad \ldots \ldots \quad (30.)$$

(The symbols have the same signification as in the last section. See art. 9.)

This may be proved as follows. It is evident that if the above expression were developed, each term would consist of a *second* differential coefficient of one of the functions p, q, r, multiplied by a *first* differential coefficient of each of the other two.

Consider then the terms in which p is twice differentiated; these will be of the three forms

$$\frac{d^2p}{dx_idy_j}\cdot\frac{dq}{dx_j}\cdot\frac{dr}{dy_i}, \quad \frac{d^2p}{dx_idx_j}\cdot\frac{dq}{dy_i}\cdot\frac{dr}{dy_j}, \quad \text{and } \frac{d^2p}{dy_idy_j}\cdot\frac{dq}{dx_i}\cdot\frac{dr}{dx_j},$$

each of which will arise from the first and third terms of (30.) only. (It is to be observed that i may $=j$.)

Now if we examine each of these forms, we see easily that for every term arising from the *first* term of (30.), there is a similar term with the *opposite sign* arising from the *third* term of (30.); and since a similar proposition would be true of the terms in which q, r, respectively, are twice differentiated, the whole expression on the left of the equation (30.) vanishes identically. The theorem is therefore established.

It is obvious that p, q, r may contain, explicitly, any other quantities (as t) besides the $2n$ variables with respect to which the differentiations are performed.

Let ξ represent, either, one of the $2n$ variables x_1, &c., y_1, &c., or any other quantity whatever, explicitly contained in p and q. It is evident that we shall have

$$\frac{d}{d\xi}[p,q]=\left[\frac{dp}{d\xi},q\right]+\left[p,\frac{dq}{d\xi}\right]. \quad \ldots \ldots \quad (31.)$$

22. Resuming now the consideration of the $2n$ simultaneous differential equations discussed in the first section, namely,

$$x'_i=\frac{dZ}{dy_i}, \quad y'_i=-\frac{dZ}{dx_i}, \quad \ldots \ldots \quad (\text{I.})$$

we shall be enabled, by means of the theorems (30.), (31.) of the last article, to give a very simple and direct proof of the proposition indirectly demonstrated in art. 9.

For let u be any function whatever of the variables x_1, &c., y_1, &c., t; then

$$u'=\frac{du}{dt}+\Sigma_i\left(\frac{du}{dx_i}x'_i+\frac{du}{dy_i}y'_i\right),$$

and if the values of x'_i, y'_i given by (I.) be substituted in this expression, it becomes

$$u'=\frac{du}{dt}+[\mathrm{Z}, u]. \quad \ldots \ldots \ldots \quad (32.)$$

Let $u=[p, q]$, then (making use of (31.))

$$[p, q]'=\left[\frac{dp}{dt}, q\right]+\left[p, \frac{dq}{dt}\right]+\left[\mathrm{Z}, [p, q]\right].$$

Now suppose that, by virtue of the differential equations (I.), the values of p and q are constant; or, in other words, that

$$p=\varphi(x_1, \text{\&c.}, y_1, \text{\&c.}, t)$$
$$q=\psi(x_1, \text{\&c.}, y_1, \text{\&c.}, t)$$

are any two integrals whatever of the system (I.); p, q representing two arbitrary constants. The equation $p'=0$ gives (see (32.))

$$\frac{dp}{dt}+[\mathrm{Z}, p]=0,$$

or $$\frac{dp}{dt}=-[\mathrm{Z}, p],$$

hence $$\left[\frac{dp}{dt}, q\right]=-\left[[\mathrm{Z}, p], q\right]=\left[q, [\mathrm{Z}, p]\right].$$

In like manner $$\left[p, \frac{dq}{dt}\right]=\left[p, [q, \mathrm{Z}]\right].$$

Thus the expression given above for $[p, q]'$ becomes

$$[p, q]'=\left[p, [q, \mathrm{Z}]\right]+\left[q, [\mathrm{Z}, p]\right]+\left[\mathrm{Z}, [p, q]\right],$$

which is identically equal to 0, by the theorem (30.). Consequently, for any two integrals p and q,

$$[p, q]=\text{constant}. \quad \ldots \ldots \ldots \quad (33.)$$

This theorem, as has been already mentioned, was discovered, in the case of the dynamical equations, by POISSON; and the fact that he was able to arrive at it through so long and complex a process as that which he has given in his first memoir on the Variation of Arbitrary Constants*, must be looked upon as a remarkable instance of his analytical skill. I am not acquainted with any attempt to simplify the demonstration, except that of Sir W. HAMILTON†; in fact it is probable that no material simplification was attainable without the help of the transformation of the differential equations to the form (I.), towards which POISSON (as JACOBI has remarked) only made a first step. Sir W. HAMILTON's demonstration may certainly be considered simple as compared with that of POISSON. That which I have given above will, I hope, be regarded as a further improvement.

23. In what follows I shall use such expressions as "the integral c," as an abbreviation for "the equation $c=\varphi(x_1, \text{\&c.}, y_1, \text{\&c.}, t)$."

* Journ. de l'École Polytechnique, tom. viii. † Philosophical Transactions, 1835, p. 108–9.

It is of course understood that the function on the right contains neither c nor any other arbitrary constant explicitly.

Let then f, g be any two given integrals of the system (I.). It has been shown that we shall always have

$$[f, g] = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (K.)$$

But this equation may be true either (1) identically, or (2) not identically. In the first case the expression $[f, g]$ may either be identically $=0$, or it may reduce itself identically to a *determinate constant*, which might always be made unity by multiplying one of the integrals by a factor. (In the case of a "*normal system*" of integrals (art. 20.), it has been seen that every binary combination gives either 0 or 1.) But if the above equation (K.) be not identically true, so that $[f, g]$ obtains a constant value only by virtue of the differential equations, then the constant on the right of (K.) is an *arbitrary constant*, and that equation is itself an integral. But here again there are two cases; for the function $[f, g]$ may be only a combination of the functions on the right of the two integrals f, g; and then (K.) is not a new integral, but only a combination of the two given ones; or, on the other hand, $[f, g]$ may be a function independent of f, g; and then (K.) is really a new integral, which cannot be produced by merely combining the other two. Thus it appears that POISSON's theorem may in some cases *lead to the discovery of new integrals*, when two are known. On this subject, and others connected with it, I refer to the interesting memoir of M. BERTRAND in LIOUVILLE's Journal (1852), "Sur l'intégration des équations différentielles de la Mécanique."

24. Let $c_1, c_2, \ldots c_m$ be any m integrals, and let f, g be any two functions of the m constants $c_1, c_2, \ldots, c_m$, so that f, g are also two integrals; and considering f, g as functions of $c_1, \ldots, c_m$, and, through them, of the variables, we have exactly as in art. 9, equation (24.),

$$[f, g] = \Sigma\left\{\frac{d(f, g)}{d(c_i, c_j)} \cdot [c_i, c_j]\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (L.)$$

the summation extending to all binary combinations of the m constants c_1, &c. If then we suppose $k_1, k_2, \ldots, k_m$ to be m functions (such as h, k) of the m constants $c_1, \ldots, c_m$, we shall have for any pair k_p, k_q,

$$[k_p, k_q] = \Sigma\left\{\frac{d(k_p, k_q)}{d(c_i, c_j)} \cdot [c_i, c_j]\right\} . \quad . \quad . \quad . \quad . \quad . \quad (34.)$$

(the summation referring as before to i, j); and the inverse equations (obtained either by considering c_1, &c. as functions of k_1, &c., and reasoning in the same way, or by multiplying the above equation by $\frac{d(c_r, c_s)}{d(k_p, k_q)}$ and summing with respect to p, q) will be

$$[c_i, c_j] = \Sigma\left\{\frac{d(c_i, c_j)}{d(k_p, k_q)} \cdot [k_p, k_q]\right\} . \quad . \quad . \quad . \quad . \quad . \quad (35.)$$

(the summation referring to p, q).

This inversion can only fail in the case in which the equations expressing k_1, &c. in

terms of c_1, &c. are not all independent; a supposition which we exclude, in order that $k_1, \ldots k_m$ may represent m distinct integrals.

The equations above written lead obviously to the following conclusions:—

(1.) If f be a given function of the m constants $c_1, \ldots c_m$; then the determination of another function g, such that $[f, g]=0$, depends in general upon the solution of a linear partial differential equation of the first order.

(2.) It is impossible that the conditions $[k_i, k_j]=0$ can exist for every binary combination of $k_1, \ldots, k_m$, unless $[c_i, c_j]=0$ for every binary combination of $c_1, \ldots, c_m$.

25. As an illustration of the first of these conclusions, we may take a case which actually occurs in many dynamical problems. Let c_1, c_2, c_3 be three integrals, such that

$$[c_2, c_3]=c_1, \ [c_3, c_1]=c_2, \ [c_1, c_2]=c_3, \quad \ldots \ldots \ldots \quad (c.)$$

and let it be required to find a function g of c_1, c_2, c_3, such that $[c_1, g]=0$. The equation (L.) of the last article gives, if we put $f=c_1$, and introduce the conditions (c.),

$$c_3\frac{dg}{dc_2}-c_2\frac{dg}{dc_3}=0.$$

The solution of which is

$$g=\psi(c_2^2+c_3^2), \quad \ldots \ldots \ldots \ldots \quad (g.)$$

ψ being an arbitrary function (which may evidently also contain c_1 in an arbitrary manner).

If, instead of $f=c_1$, we put $f=\varphi(c_1^2+c_2^2+c_3^2)$, it will be found that the expression on the right of the equation (L.) vanishes identically; so that in this case, if g be any arbitrary function of c_1, c_2, c_3, the condition $[f, g]=0$ will be satisfied.

26. If $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$ be a system of *normal* elements (art. 20.), we have (equation (25.), art. 9.)

$$[f, g]=\Sigma_i\frac{d(f, g)}{d(a_i, b_i)},$$

where f, g represent any two functions of the elements, or in other words, any two integrals whatever. If in the above equation we put successively $f=a_i$, $f=b_i$, we obtain

$$[a_i, g]=\frac{dg}{db_i}, \ [b_i, g]=-\frac{dg}{da_i}. \quad \ldots \ldots \ldots \ldots \quad (36.)$$

In the case where the principle of *vis viva* subsists, we may suppose the constant of *vis viva*, h, to be one of the elements. In this case (see (29.), art. 19.) the element conjugate to h is τ, and t appears in none of the integrals explicitly, except one, namely, the integral conjugate to h, which is

$$\tau=-t+\frac{d\mathrm{V}}{dh}.$$

If, then, g be any integral whatever, *not containing* t *explicitly*, it cannot contain τ, since any combination of the normal integrals involving τ, will involve it in the form

$\tau+t$. Consequently, for every such integral we shall have, by (36.),

$$[g, h]=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37.)$$

since
$$\frac{dg}{d\tau}=0.$$

This particular consequence of the formula (36.) follows also immediately from (32.), art. 22, since the equation $g'=0$ gives, by (32.), $[Z, g]=0$, and in this case $Z=h$, so that $[Z, g]=[h, g]$. In this manner the theorem expressed by (37.) has been already obtained by M. BERTRAND.

Examples of the preceding Methods.

27. I shall now exemplify the principles which have been explained, by applying them to two of the most familiar as well as important problems of dynamics. First then let it be required to obtain in a normal form the integrals of the differential equations which determine the motion of a material point, acted on by a force emanating from a fixed centre and depending only on the distance.

Taking the centre of force as the origin of a system of rectangular coordinates, let m be the mass, and x, y, z the coordinates of the moving point. Then

$$T=\frac{1}{2}m(x'^2+y'^2+z'^2),$$

and U (see art. 17.) is a given function of r, say $\varphi(r)$, where $r^2=x^2+y^2+z^2$. Let us put

$$\frac{dT}{dx'}=u,\ \frac{dT}{dy'}=v,\ \frac{dT}{dz'}=w,$$

so that, referring to the notation used in the preceding pages, we have

$$x, y, z \text{ instead of } x_1, x_2, x_3$$
$$u, v, w \text{ instead of } y_1, y_2, y_3.$$

Moreover, $\qquad u=mx',\ v=my',\ w=mz'.$

Hence we obtain $\qquad Z=(T)-U=m^{-1}(u^2+v^2+w^2)-\varphi(r),$

so that the integral of *vis viva*, or $Z=h$, becomes

$$\frac{1}{m}(u^2+v^2+w^2)-\varphi(r)=h\,;$$

and the three integrals which express the conservation of areas become

$$yw-zv=c_1$$
$$zu-xw=c_2$$
$$xv-yu=c_3.$$

These integrals are immediately seen to satisfy the conditions

$$[c_2, c_3]=-c_1,\ [c_3, c_1]=-c_2,\ [c_1, c_2]=-c_3\,;$$

from which it follows (see art. 25, the result of which is obviously unaffected by the negative signs), that if we take $k=(c_1^2+c_2^2+c_3^2)^{\frac{1}{2}}$, the condition $[c_3, k]=0$ will be satisfied

(as is easily found to be true); and since neither of the integrals c_3, k contain t explicitly, the conditions $[h, c_3]=0$, $[h, k]=0$ will subsist also (art. 26.). Hence it follows that if we solved algebraically the three integrals h, c_3, k so as to express u, v, w in terms of x, y, z, their values would be the partial differential coefficients of a function V, from which the three remaining integrals could be found (arts. 12 and 19.).

But it is more convenient to adopt a different system of coordinates. Reverting then to the primitive form of the three integrals which we have chosen, and writing c instead of c_3, we have

$$\mathrm{T}-\mathrm{U}=h. \quad \text{(i.)}$$

$$m(xy'-yx')=c. \quad \text{(ii.)}$$

$$m^2(r^2(x'^2+y'^2+z'^2)-r^2r'^2)=k^2. \quad \text{(iii.)}$$

28. Let us now employ, instead of x, y, z, the three coordinates ϱ, θ, z; where z is the same as before, ϱ is the projection of r on the plane of xy, and θ is the angle between ϱ and the positive axis of x. We shall thus have

$$\varrho^2+z^2=r^2,\ x=\varrho\cos\theta,\ y=\varrho\sin\theta,$$

and

$$\mathrm{T}=\frac{1}{2}m(\varrho'^2+\varrho^2\theta'^2+z'^2).$$

Let

$$\frac{d\mathrm{T}}{d\varrho'}=u,\ \frac{d\mathrm{T}}{d\theta'}=v,\ \frac{d\mathrm{T}}{dz'}=w$$

(where u and v have now a new signification), then

$$\varrho'=\frac{u}{m},\ \theta'=\frac{v}{m\varrho^2},\ z'=\frac{w}{m};$$

and the three integrals at the end of the last article become, after obvious reductions,

$$\frac{1}{2m}\left(u^2+\frac{v^2}{\varrho^2}+w^2\right)=h+\varphi(r). \quad \text{(i.)}$$

$$v=c. \quad \text{(ii.)}$$

$$(\varrho w-zu)^2+\frac{r^2}{\varrho^2}v^2=k^2. \quad \text{(iii.)}$$

The conditions $[h, c]=0$, $[h, k]=0$, $[c, k]=0$ continue to subsist with reference to the new variables; the two former *necessarily*, because (ii.) and (iii.) do not contain t (art. 26.), and the third *actually*, as is seen on trial (*not accidentally*, as will be shown hereafter).

We know, therefore, that the values of u, v, w, found from these equations, will be the partial differential coefficients with respect to ϱ, θ, z of a function V of these latter variables.

The two first give $$u^2+w^2=2m(h+\varphi(r))-\frac{c^2}{\varrho^2};$$

and if we multiply this by $\varrho^2+z^2=r^2$, and subtract (iii.), we obtain (introducing the condition (ii.)

$$(\varrho u+zw)^2=2mr^2(h+\varphi(r))-k^2.$$

Lastly, if this be combined with (iii.), the following expressions are found for u and w:

$$u=\frac{\varrho}{r^2}\left\{2mr^2(h+\varphi(r))-k^2\right\}^{\frac{1}{2}}-\frac{z}{r^2}\left\{k^2-\frac{r^2}{\varrho^2}c^2\right\}^{\frac{1}{2}}$$

$$w=\frac{z}{r^2}\left\{2mr^2(h+\varphi(r))-k^2\right\}^{\frac{1}{2}}+\frac{\varrho}{r^2}\left\{k^2-\frac{r^2}{\varrho^2}c^2\right\}^{\frac{1}{2}}$$

(in which it is to be remembered that $r^2=z^2+\varrho^2$), and if to these we join the equation (ii.), the values of u, v, w are explicitly given in terms of the conjugate variables ϱ, θ, z. We have then (art. 19.)

$$V=\int(ud\varrho+vd\theta+wdz);$$

or, substituting the above values,

$$V=c\theta+\int\left\{\frac{\varrho d\varrho+zdz}{r^2}(2mr^2(h+\varphi(r))-k^2)^{\frac{1}{2}}+\frac{\varrho dz-zd\varrho}{r^2}\left(k^2-\frac{r^2}{\varrho^2}c^2\right)^{\frac{1}{2}}\right\}.$$

The term under the integral sign is easily seen to be, (as we know *à priori* it must be) a complete differential. It is convenient however to transform it thus. First, we have $\varrho d\varrho+zdz=rdr$; next, let the *latitude* of the body (or the angle between r and the plane of x, y) be λ; then $\tan\lambda=\frac{z}{\varrho}$, and

$$\varrho dz-zd\varrho=r^2d\lambda,\ \frac{r^2}{\varrho^2}=\sec^2\lambda.$$

Making these substitutions, the expression for V becomes

$$V=c\theta+\int\frac{dr}{r}(2mr^2(h+\varphi(r))-k^2)^{\frac{1}{2}}+\int d\lambda(k^2-c^2\sec^2\lambda)^{\frac{1}{2}}.$$

The integration in the second term cannot be effected till the form of the function $\varphi(r)$ is given: that of the third term may be more conveniently performed after the differentiations with respect to c and k, as in the next article.

29. The remaining integrals* of the problem are (art. 19.)

$$\frac{dV}{dk}=\alpha,\ \frac{dV}{dc}=\beta,\ \frac{dV}{dh}=t+\tau.$$

Performing the operations indicated, and observing that

$$\int\frac{d\lambda}{\sqrt{k^2-c^2\sec^2\lambda}}=\frac{1}{k}\sin^{-1}\left(\frac{k\sin\lambda}{\sqrt{k^2-c^2}}\right)$$

and

$$\int\frac{\sec^2\lambda d\lambda}{\sqrt{k^2-c^2\sec^2\lambda}}=\frac{1}{c}\sin^{-1}\left(\frac{c\tan\lambda}{\sqrt{k^2-c^2}}\right),$$

* It would perhaps be better to use the term "integral equations" here, in order to reserve the term "integral" for the case of an equation involving only one arbitrary constant (see art. 23.). The equations $\frac{dV}{dk}=\alpha$, &c. become "integrals" in this sense, when for k, c, and h, on the left, are substituted the functions of the variables to which they are respectively equal (from (i.), (ii.), (iii.)). An "integral" in this limited meaning is what is commonly called a "first integral," when the problem is considered as the solution of n differential equations of the second order. And any equation obtained by combining "integrals" so as to eliminate a set of n of the variables $x_1, x_2, \dots x_n, y_1, y_2, \dots y_n$, of which no two are conjugate, corresponds to what is commonly called a "final integral."

we obtain for the final integrals,

$$m\int rdr\{2mr^2(h+\varphi(r))-k^2\}^{-\frac{1}{2}}=t+\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)}$$

$$\theta-\sin^{-1}\left(\frac{c\tan\lambda}{\sqrt{k^2-c^2}}\right)=\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)}$$

$$-k\int\frac{dr}{r}\left\{2mr^2(h+\varphi(r))-k^2\right\}^{-\frac{1}{2}}+\sin^{-1}\left(\frac{k\sin\lambda}{\sqrt{k^2-c^2}}\right)=\alpha. \quad . \quad \text{(vi.)}$$

Let $\frac{c}{k}=\cos\iota$; then $\frac{c}{\sqrt{k^2-c^2}}=\cot\iota$, and the equation (v.) becomes

$$\tan\lambda=\tan\iota.\sin(\theta-\beta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v.a)}$$

which expresses that the orbit is in a plane whose inclination to the plane of x, y is ι. Also β is evidently the longitude of the node, reckoned from the axis of x.

The last term on the left of (vi.) becomes

$$\sin^{-1}\left(\frac{\sin\lambda}{\sin\iota}\right).$$

Now if ϑ be the "argument of latitude" or the angle between the node and the radius vector r, we have evidently $\sin\vartheta=\frac{\sin\lambda}{\sin\iota}$, so that the above term is simply ϑ, and the integral (vi.) becomes

$$\vartheta-\alpha=k\int\frac{dr}{r}\left\{2mr^2(h+\varphi(r))-k^2\right\}^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad \text{(vi.a)}$$

30. To apply the above expressions to the case of the undisturbed motion of a planet, we have only to put $\varphi(r)=\frac{\mu m}{r}$, where m is now the mass of the planet, and μ the sum of the masses of the sun and planet, the origin of coordinates being placed at the sun. It would be useless to give the well-known expressions to which the integrations now lead, my object being merely to obtain a set of *normal elements*. Now in this case we have (by well-known theorems), if a be the semiaxis major, e the excentricity, and ι as before the inclination,

$$h=\frac{-\mu}{2a}, \quad k=\sqrt{\mu a(1-e^2)},$$

and therefore $\qquad c=\sqrt{\mu a(1-e^2)}.\cos\iota.$

Also, if we take for the inferior limit of the integrations in (iv.) and (vi.a) the minimum value of r, or the perihelion distance, it is plain that α will be the longitude of the node, reckoned from the perihelion in the plane of the orbit, and $-\tau$ the time of perihelion passage. Thus we have the following six elements, arranged in conjugate pairs:—

$-\frac{\mu}{2a}$,	$-$(time of perihelion passage)
$\sqrt{\mu a(1-e^2)}$,	(angle between node and perihelion)
$\sqrt{\mu a(1-e^2)}.\cos\iota$,	(longitude of node).

It is obvious that we may change the signs of the first pair. And generally, that if f, g be any two conjugate elements, we may substitute for them λf, $\frac{1}{\lambda}g$, where λ is any *determinate* constant, i. e. not a function of the elements*.

The above elements coincide with those given by JACOBI. My object has been merely to illustrate a mode of obtaining them which seems capable of useful applications.

31. As a second example I shall apply the method to the case of the motion of a solid body about a fixed point.

Let the fixed point be taken for the origin, and the principal axes of the body through that point for the axes of x, y, z. Let ξ, η, ζ refer to the same origin and to axes fixed in space; a, b, c being the direction-cosines of the axis of x referred to the fixed axes of ξ, η, ζ, and a', b', c'; a'', b'', c'' being respectively the direction-cosines of the axes of y and z. Let θ be the inclination of the plane of x, y (or "equator") to that of ξ, η (or "ecliptic"); ψ the longitude of the node, reckoned from the axis of ξ, and φ the right-ascension of the axis of x. Then if A, B, C be the Moments of Inertia, and p, q, r the angular velocities, about the axes of x, y, z, the expression for the *vis viva* is $\mathrm{T}=\frac{1}{2}(\mathrm{A}p^2+\mathrm{B}q^2+\mathrm{C}r^2)$, where

$$p=-\theta'\cos\varphi-\psi'\sin\varphi\sin\theta$$
$$q=\theta'\sin\varphi-\psi'\cos\varphi\sin\theta$$
$$r=\varphi'+\psi'\cos\theta.$$

Let u, v, w be the variables conjugate respectively to θ, φ, ψ, so that

$$u=\frac{d\mathrm{T}}{d\theta'},\quad v=\frac{d\mathrm{T}}{d\varphi'},\quad w=\frac{d\mathrm{T}}{d\psi'},$$

the following expressions will be found without difficulty:

$$\mathrm{A}p=-u\cos\varphi+\frac{\sin\varphi}{\sin\theta}(v\cos\theta-w)$$

$$\mathrm{B}q=u\sin\varphi+\frac{\cos\varphi}{\sin\theta}(v\cos\theta-w)$$
$$\mathrm{C}r=v.$$

Considering at present only the case in which no forces act, we have the integral of *vis viva* $\mathrm{T}=h$, which becomes

$$\frac{1}{\mathrm{A}}\left(-u\cos\varphi+\frac{\sin\varphi}{\sin\theta}(v\cos\theta-w)\right)^2$$
$$+\frac{1}{\mathrm{B}}\left(u\sin\varphi+\frac{\cos\varphi}{\sin\theta}(v\cos\theta-w)\right)^2$$
$$+\frac{1}{\mathrm{C}}v^2=2h. \quad \ldots\ldots\ldots \quad \text{(i.)}$$

* More generally, we may substitute for f, g any two functions of them, p, q, such that

$$\frac{dp}{dg}\frac{dq}{df}-\frac{dp}{df}\frac{dq}{dg}=1,$$

a condition which requires the solution of a linear partial differential equation for the determination of one function, if the other be assumed. But on the subject of the transformation of elements see below, arts. 34, 35.

The three integrals which express the conservation of areas, namely,

$$\mathrm{A}ap+\mathrm{B}a'q+\mathrm{C}a''r=e,$$
$$\mathrm{A}bp+\mathrm{B}b'q+\mathrm{C}b''r=f,$$
$$\mathrm{A}cp+\mathrm{B}c'q+\mathrm{C}c''r=g,$$

become, after simple reductions,

$$-u\cos\psi-\frac{\sin\psi}{\sin\theta}(v-w\cos\theta)=e$$
$$-u\sin\psi+\frac{\cos\psi}{\sin\theta}(v-w\cos\theta)=f$$
$$w=g.$$

Let $e^2+f^2+g^2=k^2$; we have, adding the squares of these three equations,

$$u^2+w^2+\frac{(v-w\cos\theta)^2}{\sin^2\theta}=k^2, \quad \text{(ii.)}$$

and we may take the three equations (i.), (ii.), and

$$w=g \quad \text{(iii.)}$$

as three normal integrals; the conditions

$$[g, h]=0, \quad [h, k]=0, \quad [k, g]=0$$

being obviously satisfied.

These three equations determine u, v, w as functions of θ, ϕ, ψ; and supposing the three former variables to be explicitly expressed in terms of the latter, we should have at once the three partial differential coefficients $\frac{d\mathrm{V}}{d\theta}$, $\frac{d\mathrm{V}}{d\phi}$, $\frac{d\mathrm{V}}{d\psi}$; the determination of V would therefore depend upon simple integration, and the remaining integrals would be given by means of the three equations

$$\frac{d\mathrm{V}}{dh}=t+\tau, \quad \frac{d\mathrm{V}}{dg}=c_1, \quad \frac{d\mathrm{V}}{dk}=c_2,$$

τ, c_1, c_2 being new arbitrary constants.

In the general case, however, the algebraical solution of the equations (i.), (ii.), (iii.) is impracticable, since the elimination of v and w leads to an equation of the fourth degree in u; nor does it seem possible to evade the difficulty by choosing a different combination of integrals, since it may be shown that the necessary conditions cannot be satisfied unless two at least of the combinations chosen are of the second degree in u, v, w.

32. Mr. CAYLEY has given* a solution of this problem, which, though differing totally in form and method from the above, resembles it in arriving exactly at a corresponding point. For in Mr. CAYLEY's equations (27.), (28.), Φ and ∇ are to be expressed as functions of v; but this requires the algebraical solution of the system (18.) for p, q, r, and is therefore impracticable. (The two equations (i.), (ii.) of the

* Cambridge and Dublin Mathematical Journal, vol. i. p. 167.

last article are merely transformations of the two first of Mr. CAYLEY's (18.); and (iii.), though not identical with the third, is of the same degree; so that the algebraical difficulty is precisely the same in both methods.)

33. If we suppose A=B, the algebraical difficulty disappears, and the solution of the problem can be explicitly completed. But on account of the importance and interest of this case I shall make it the subject of a separate section, in which it will also be shown that the solution of the general case may be made to depend upon it, by means of the variation of elements. (See Section III.)

34. Suppose any complete normal solution of the system of differential equations (I.), art. 14, be known, i. e. a solution involving the $2n$ elements

$$a_1, a_2, \ldots a_n, \quad b_1, b_2, \ldots b_n$$

which satisfy the conditions (23.), art. 9; then an infinite number of other sets of normal elements can always be found.

For if we determine the $2n$ quantities $\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n$, as functions of a_1, &c., b_1, &c. by the $2n$ equations

$$\frac{dA}{da_i}=b_i, \quad \frac{dA}{d\alpha_i}=\beta_i,$$

where A is any arbitrary function of

$$a_1, a_2, \ldots a_n, \alpha_1, \alpha_2, \ldots \alpha_n,$$

it is obvious that the whole of the reasoning by which the formulæ (19.), art. 7, were established may be repeated, merely putting A in place of X, and α, β instead of x, y. And repeating in like manner the reasoning of art. 9, *mutatis mutandis*, it will follow that if f, g represent any two of the $2n$ quantities α_1, &c., β_1, &c., the expression

$$\Sigma_i \frac{d(f, g)}{d(b_i, a_i)}$$

will be equal to unity if f, g be a pair of the form α_j, β_j, and will vanish in every other case. But it was also shown ((25.) art. 9) that the above expression is equivalent to $-[f, g]$; it follows then that

$$[\alpha_j, \beta_j]=-1, \quad [\alpha_i, \alpha_j]=[\alpha_i, \beta_j]=[\beta_i, \beta_j]=0^{*};$$

or, in other words, that $\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n$, are a new set of normal elements.

This method however can hardly be of much use in practice, because we cannot (at least without the solution of partial differential equations) determine what form

* I shall have occasion to refer afterwards to M. DESBOVES' Memoir in LIOUVILLE's Journal, vol. xiii., "Démonstration de deux théorèmes de M. JACOBI." But it may be observed here that the proposition in the text is not the same as that expressed by the same notation in the memoir alluded to, p. 400. For M. DESBOVES uses the symbols $[a_i, a_j]$ in a different sense. His theorem, in the notation of the present paper, is

$$\Sigma_i \frac{d(a_i, b_i)}{d(f, g)}=1, \text{ or } =0,$$

according as f, g are of the form α_j, β_j or not, which is easily established without the help of relations analogous

of the function A will cause any of the new elements to be *given* functions of the old. But the problems most likely to occur may be solved in another way, as follows.

35. Assuming for the set $\alpha_1, \alpha_2, \ldots \alpha_n$, given functions of the set $a_1, a_2, \ldots a_n$ *only*, it is required to find $\beta_1, \ldots \beta_n$.

(It will be observed that the conditions $[\alpha_i, \alpha_j]=0$ are necessarily satisfied in this case by virtue of (25.), art. 9, since α_1, &c. do not involve b_1, &c.)

It is plain, that if the principal function X had been found from the n integrals $a_1, a_2, \ldots a_n$ (as in art. 14.), it would be changed into that which would be found from the n integrals $\alpha_1, \alpha_2, \ldots \alpha_n$, merely by introducing the expressions for $a_1, \ldots a_n$ in terms of $\alpha_1, \ldots \alpha_n$; which expressions would be found by algebraical inversion of the assumed equations which give the latter set as functions of the former. Let $\overline{\mathrm{X}}$ represent the function X thus transformed; we have then

$$\begin{aligned}\beta_i&=\frac{d\overline{\mathrm{X}}}{d\alpha_i}=\frac{d\mathrm{X}}{da_1}\frac{da_1}{d\alpha_i}+\frac{d\mathrm{X}}{da_2}\frac{da_2}{d\alpha_i}+\ldots\\&=b_1\frac{da_1}{d\alpha_i}+b_2\frac{da_2}{d\alpha_i}+\ldots+b_n\frac{da_n}{d\alpha_i}. \quad (38.)\end{aligned}$$

Thus β_i is determined as a function of the old elements, since $\frac{da_1}{d\alpha_i}$, &c. may be expressed in terms of the latter. In like manner we should have a set of inverse equations

$$b_i=\beta_1\frac{d\alpha_1}{da_i}+\beta_2\frac{d\alpha_2}{da_i}+\ldots+\beta_n\frac{d\alpha_n}{da_i}, \quad (39.)$$

which may be used instead of (38.).

It is apparent that β_1, &c. will involve in general the elements a_1, &c. as well as b_1, &c.

Conversely, if we assumed for β_1, &c. given functions of the set $b_1, \ldots b_n$ alone, we

to (19.), but would not answer our present purpose. I regret to use symbols with a meaning different from that which custom has to some extent sanctioned; but there seemed to be only a choice of difficulties.

Mr. SPOTTISWOODE has suggested to me the employment of the symbols (analogous to Mr. SYLVESTER's "umbral" notation)

$$\begin{bmatrix} u, v, w, \ldots. \\ \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots. \end{bmatrix}, \quad \begin{bmatrix} x, & y, & z, \ldots \\ d_1, & d_2, & d_3 \ldots \end{bmatrix}$$

instead of those which I have used, namely,

$$\frac{d(u, v, w, \ldots)}{d(x, y, z, \ldots)}, \quad d(x, y, z, \ldots.).$$

If these were adopted, the two forms (p, q), $[p, q]$ might be used without confusion in their usual significations. See note to art. 9. But although the "umbral" forms are more suggestive of the properties which belong to the above expressions as *determinants*, the other forms bring more into view the analogies which connect them with the differential calculus; and therefore, for the purposes of this paper, I have preferred them. And it is perhaps better, for the present, that different notations should be *tried*, than that any attempt should be made to fix upon a definitive system for subjects so recent as those connected with the theory of determinants.

should have, for determining α_1, &c., either of the systems

$$\alpha_i = \Sigma_j\left(a_j\frac{db_j}{d\beta_i}\right), \quad a_i = \Sigma_j\left(\alpha_j\frac{d\beta_j}{db_i}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40.)$$

We might obtain in this way an indefinite variety of sets of elements for the case of elliptic motion, beginning with those given at the end of art. 30. But it will be better to defer this illustration till after the discussion of the Method of the Variation of Elements, which will form the subject of a future Section.

36. It results, from the investigations of this and the preceding Sections, that if a set of n integrals $a_1, a_2, \dots a_n$ be given, satisfying the $\frac{n(n-1)}{2}$ conditions $[a_i, a_j]=0$, the determination of n more integrals $b_1, \dots b_n$, constituting, with the given ones, a complete normal set, is a *determinate problem*, admitting of a unique solution, and always reducible (setting aside algebraical difficulties) to quadratures.

But if, out of a complete normal set, n be given of which one or more pairs are *conjugate*, then the completion of the set is no longer a determinate problem, since the remaining n integrals, containing also one or more conjugate pairs, admit, to some extent, of arbitrary transpositions and combinations, as is evident from considerations similar to those employed in arts. 13 and 35. Hence we should expect *à priori* that the problem would require the solution of partial differential equations.

It appears, indeed, at first sight, that having any n of the elements *given* functions of the variables, the relations established in art. 9, with the others included in the formula (21.), art. 9, would furnish more than a sufficient number of equations to determine explicitly all the partial differential coefficients of the remaining elements in terms of the variables*, at least in the case in which the principle of *vis viva* subsists, and the given integrals do not contain t. But it is certain from the above considerations that this cannot be the case, and therefore that the equations furnished by those conditions cannot be all independent. I have not at present attempted to show this directly, though it would probably be easy to do so.

Note on art. 2, *Section* I.

The theorem established in this article may be more shortly demonstrated as follows:—

Since $$d\Sigma_i(x_iy_i) = \Sigma_i(x_idy_i) + \Sigma_i(y_idx_i)$$

* The conditions $[a_i, b_i]=1$, $[a_i, b_j]=0$, $[b, b_j]=0$, $[a_i, a_j]=0$ will give, as is easily seen, $\frac{n(n-1)}{2}+n^2$ equations; and the analogous conditions (21.), art. 9, in which the summation refers to the *numerators* of the differential coefficients, will give the same number, so that upon the whole we shall apparently have $3n^2-n$ equations, to determine the $2n^2$ partial coefficients required.

It is not difficult to make mistakes in this subject. I was for some time under the impression that the problem could be solved when *any* n independent integrals were given. Even the illustrious JACOBI himself appears to have been misled, at first sight, as to the consequences of POISSON's theorem (art. 22.). See the beginning of M. BERTRAND's Memoir mentioned above; I do not know the fact from any other source.

become simply $\quad a_i'=[\Omega, a_i],\ b_i'=[\Omega, b_i].$

In these expressions Ω, a_i, b_i are supposed to be expressed in terms of the variables. Now

$$[\Omega, a_i]=\Sigma_j \frac{d(\Omega, a_i)}{d(y_j, x_j)},$$

but, by equation (26.), art. 10, this is equivalent to

$$-\Sigma_j \frac{d(\Omega, a_i)}{d(b_j, a_j)},$$

in which Ω is expressed as a function of the elements and t; and this last expression obviously reduces itself to the single term $-\frac{d\Omega}{db_i}$. In like manner the expression for $[\Omega, b_i]$ reduces itself to $+\frac{d\Omega}{da_i}$; thus the equations for determining the variation of the elements are

$$a_i'=-\frac{d\Omega}{db_i}, \quad b_i'=\frac{d\Omega}{da_i}, \quad \text{(E.)}$$

in which Ω is to be expressed as a function of the elements and t*. This will be a sufficient account of the method for our immediate purpose.

39. The following propositions in spherical trigonometry will be required. If a, b, c be the sides, and α, β, γ the opposite angles of any spherical triangle, then

$$\cos(a+b)=\frac{\frac{(\cos\alpha+\cos\beta)^2}{1-\cos\gamma}-1-\cos\alpha\cos\beta}{\sin\alpha\sin\beta} \quad (40.)$$

$$\cos(a-b)=\frac{-\frac{(\cos\alpha-\cos\beta)^2}{1+\cos\gamma}+1-\cos\alpha\cos\beta}{\sin\alpha\sin\beta}; \quad (41.)$$

and if the sides be considered as functions of the angles, then

$$\frac{da}{d\beta}=\cos\gamma\frac{db}{d\beta}+\cos\beta\frac{dc}{d\beta} \quad (42.)$$

$$\frac{da}{d\gamma}=\cos\gamma\frac{db}{d\gamma}+\cos\beta\frac{dc}{d\gamma}. \quad (43.)$$

The two last are easily verified; but as the others are not so obvious, I shall give the demonstration. Putting x for the expression on the right of the equation (40.), we

* The history of these remarkable formulæ may, I believe, be stated as follows. They were first discovered by LAGRANGE in the case in which a_i, b_i were the *initial values* of x_i, y_i, and Ω contained x_1, &c. but not y_1, &c. They were extended by Sir W. R. HAMILTON to the case in which Ω contains *both* sets of variables; and finally, by JACOBI, to the case in which a_1, &c., b_1, &c. are *any* system of conjugate elements. JACOBI however does not appear to have published a demonstration of them, and the only one which I have seen is by M. DESBOVES, LIOUVILLE's Journal, vol. xiii. p. 397, and differs essentially from that given in the text. Sir W. R. HAMILTON has pointed out the circumstance, that when Ω contains both sets of variables, the varying elements determined by the formula (E.) are not *osculating*.

easily obtain

$$\frac{1-x}{1+x}=\frac{\cos^2\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha+\beta}{2}}\cdot\frac{\sin^2\frac{\gamma}{2}-\cos^2\frac{\alpha+\beta}{2}}{\cos^2\frac{\alpha-\beta}{2}-\sin^2\frac{\gamma}{2}}$$

$$=-\frac{\cos^2\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha+\beta}{2}}\cdot\frac{\cos\frac{\alpha+\beta+\gamma}{2}\cos\frac{\alpha+\beta-\gamma}{2}}{\cos\frac{\alpha-\beta+\gamma}{2}\cos\frac{\beta+\gamma-\alpha}{2}}$$

$$=\frac{\cos^2\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha+\beta}{2}}\tan^2\frac{c}{2}=\tan^2\frac{a+b}{2};$$

whence it is plain that $x=\cos(a+b)$, and in like manner may the equation (41.) be established.

40. Returning now to the problem of rotation, and supposing, for convenience, that the question refers to the motion of the earth about its centre of gravity, the following will be the signification of the symbols employed.

A, B, C are the moments of inertia about the principal axes of the earth, viz. the axes of x, y, z; the last being the polar axis, and the arrangement being such that the positive direction of z is to the *north* pole, and that the positive axis of x *follows* that of y in the actual rotation about the polar axis: p, q, r being the angular velocities about the three principal axes, the usual convention will be adopted as to their signs; so that in the actual case r is positive. The arrangement of the fixed axes of ξ, η, ζ is supposed similar to that of x, y, z, the plane of ξ, η being a fixed ecliptic, and the axis of ξ the origin of longitudes unless another origin be expressly indicated.

Then θ is the oliquity, ψ the longitude of the vernal equinox, and φ the right ascension of the axis of x; all referring to the fixed ecliptic.

Let the "principal plane" signify that which, in the undisturbed problem, is the "invariable plane." Then i is the inclination of the principal plane to the fixed ecliptic, and j is the inclination of the equator to the principal plane.

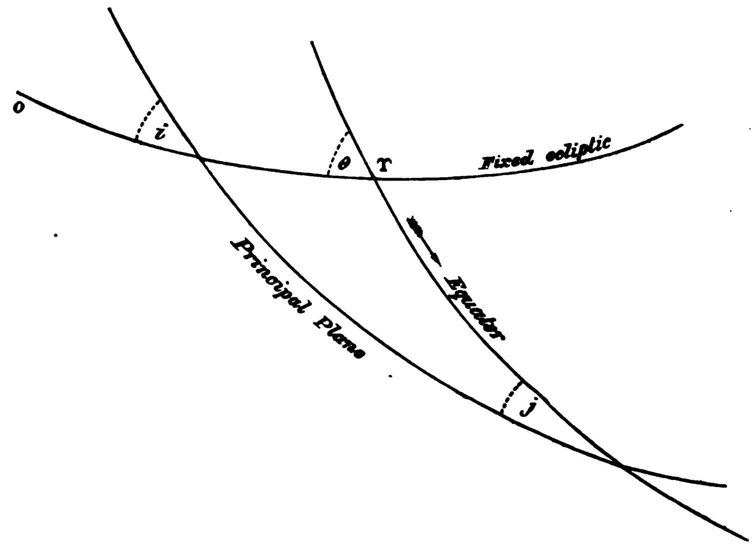
In the case of the earth, A is nearly equal to B, θ never differs sensibly from i, and j is therefore always small. But these conditions are not supposed in what follows. It is assumed however that C is the greatest of the three moments of inertia. These conventions, in which it is very desirable to avoid any ambiguity, may be illustrated by the annexed figure, in which O represents the origin of longitudes.

The angles of the spherical triangle formed by the intersection of the three planes with a spherical surface are $i, j, \pi-\theta$; and the sides opposite to them will be denoted by I, J, Θ. Thus we shall have

$$\cos \mathrm{I}=\frac{\cos i-\cos j\cos\theta}{\sin j\sin\theta},\quad \cos \mathrm{J}=\frac{\cos j-\cos i\cos\theta}{\sin i\sin\theta}$$

$$\cos\theta=\cos i\cos j-\sin i\sin j\cos\Theta.$$

And, in the figure, $O\Upsilon=\psi$, and φ is measured from Υ in the direction indicated by



the arrow, which is also the direction of the rotation about the polar axis. Moreover, if the direction-cosines of the axes of x, y, z referred to the fixed axes, be respectively a, b, c; a', b', c'; a'', b'', c'', we shall have

$$a = \cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta$$
$$a' = -\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta$$
$$a'' = -\sin\psi\sin\theta$$
$$b = \sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta$$
$$b' = -\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta$$
$$b'' = \cos\psi\sin\theta$$
$$c = -\sin\varphi\sin\theta$$
$$c' = -\cos\varphi\sin\theta$$
$$c'' = \cos\theta$$
$$p = -\theta'\cos\varphi - \psi'\sin\varphi\sin\theta$$
$$q = \theta'\sin\varphi - \psi'\cos\varphi\sin\theta$$
$$r = \varphi' + \psi'\cos\theta,$$

hence we obtain the expressions for u, v, w employed in art. 31, viz.

$$u = \frac{dh}{d\theta'} = -\mathrm{A}p\cos\varphi + \mathrm{B}q\sin\varphi$$

$$v = \frac{dh}{d\varphi'} = \mathrm{C}r$$

$$w = \frac{dh}{d\psi'} = -\mathrm{A}p\sin\varphi\sin\theta - \mathrm{B}q\cos\varphi\sin\theta + \mathrm{C}r\cos\theta,$$

from which the following also are easily deduced:

$$u=\frac{A+B}{2}\theta'+\frac{A-B}{2}(\theta'\cos 2\phi+\psi'\sin\theta\sin 2\phi)$$

$$v=C(\phi'+\psi'\cos\theta)$$

$$w=\frac{A+B}{2}\psi'\sin^2\theta+C\cos\theta\,(\phi'+\psi'\cos\theta)+\frac{A-B}{2}\sin\theta\,(\theta'\sin 2\phi-\psi'\sin\theta\cos 2\phi).$$

41. Resuming the three integrals (i.), (ii.), (iii.) of art. 31, we may put the first in the following form:—

$$Z+\Omega=h, \quad \ldots \quad \text{(i.)}$$

in which $2Z=\frac{1}{2}\left(\frac{1}{A}+\frac{1}{B}\right)\left(u^2+\frac{(v\cos\theta-w)^2}{\sin^2\theta}\right)+\frac{v^2}{C}$

$$2\Omega=\frac{1}{2}\left(\frac{1}{A}-\frac{1}{B}\right)\left\{\left(u^2-\frac{(v\cos\theta-w)^2}{\sin^2\theta}\right)\cos 2\phi-\frac{2u(v\cos\theta-w)}{\sin\theta}\sin 2\phi\right\};$$

and the other two are, as before,

$$w^2+u^2+\frac{(v-w\cos\theta)^2}{\sin^2\theta}=k^2 \quad \ldots \quad \text{(ii.)}$$

$$w=g, \quad \ldots \quad \text{(iii.)}$$

in which k is the sum of areas on the invariable plane, and g the sum on fixed ecliptic; moreover $v=Cr$ is the sum of areas projected on the plane of the equator; hence we have

$$g=k\cos i, \quad v=k\cos j.$$

It has been seen that the complete solution of the problem is impracticable in the general case, on account of an algebraical difficulty. If however we suppose $B=A$, this difficulty disappears; and after completing the solution on this supposition we may take account of the terms arising from the inequality of A and B, by treating the function denoted above by Ω (equation (i.)) as a *disturbing function*, and applying the method explained in art. 38. Thus when the action of disturbing forces is considered, the whole disturbing function will consist of two parts; one depending upon the forces, and the other the function which has just been assigned, and of which the effect, as will be seen, is extremely simple.

42. We proceed then first to complete the solution on the supposition $A=B$. The three integrals (i.), (ii.), (iii.) give in this case

$$v^2=\frac{C}{C-A}(k^2-2Ah), \quad w=g \quad \ldots \quad (44.)$$

$$u=\frac{1}{\sin\theta}\left\{k^2-v^2-w^2+2vw\cos\theta-k^2\cos^2\theta\right\}^{\frac{1}{2}},$$

in which latter expression the above constant values of v and w are to be introduced. We may put, as before, $g=k\cos i$, $v=k\cos j$, j being now constant; and the expression for u becomes

$$u=\frac{k}{\sin\theta}\left\{1-\cos^2 i-\cos^2 j+2\cos i\cos j\cos\theta-\cos^2\theta\right\}^{\frac{1}{2}} \quad \ldots \quad (45.)$$

and we shall have (art. 19.)

$$V=k(\psi\cos i+\varphi\cos j)+\int u d\theta\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46.)$$

and we will take

$$h, \quad \cos i, \quad \cos j$$

for *normal elements**, so that k is to be considered as a function of these elements, given by the equation (see (44.))

$$k^2=\frac{2ACh}{C-(C-A)\cos^2 j}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47.)$$

It is to be observed, that, according to the hypotheses admitted above, k is positive; also, the expression for u at the end of art. 40 becomes, in the case now considered, $u=A\theta'$. Thus u has the same sign as θ'; and since θ is evidently comprised between $i-j$ and $i+j$, if we suppose i and j both acute (as in the figure), $\sin\theta$ is always positive; hence in the expression (45.) for u, we have to attribute the sign $+$ or $-$ to the radical, *according as θ is increasing or diminishing*, or according as Θ is between o and π, or not; thus, in the position represented in the figure, the negative sign must be taken.

43. If we put $\pm Q$ for the radical in question, the expression for $ud\theta$ is easily transformed into the following, namely,

$$ud\theta=\pm\frac{k\sin\theta d\theta}{Q}\left\{1-\frac{1}{2}\,\frac{(\cos j-\cos i)^2}{1-\cos\theta}-\frac{1}{2}\,\frac{(\cos j+\cos i)^2}{1+\cos\theta}\right\},$$

in which it is evident that the part within brackets is *positive* upon the whole, but each of the two last terms is essentially *negative*. The integration is now easily performed, and the result is

$$\int u d\theta=\pm kP,$$

where the *sign* is that which belongs to the radical Q, and P is given by the equation

$$\begin{aligned} P= &\cos^{-1}\frac{\cos\theta-\cos i\cos j}{\sin i\sin j} \\ &-\frac{1}{2}(\cos j-\cos i)\cos^{-1}\frac{\dfrac{(\cos j-\cos i)^2}{1-\cos\theta}-1+\cos i\cos j}{\sin i\sin j} \\ &+\frac{1}{2}(\cos j+\cos i)\cos^{-1}\frac{\dfrac{(\cos j+\cos i)^2}{1+\cos\theta}-1-\cos i\cos j}{\sin i\sin j}, \end{aligned}$$

$+$ an arbitrary function of i, j, h.

This apparently complicated expression has a very simple geometrical signification;

* Since g, h, k are normal elements (i. e. satisfy the conditions $[g, h]=0$, $[h, k]=0$, $[k, g]=0$, art. 31.), any three independent combinations of them are obviously normal also.

for, referring to the figure, and using the theorems (40.), we see that it is equivalent to

$$\mathrm{P}=\cos^{-1}(-\cos\Theta)+\frac{\cos j+\cos i}{2}\cos^{-1}(\cos(\mathrm{I}+\mathrm{J}))$$

$$-\frac{\cos j-\cos i}{2}\cos^{-1}(-\cos(\mathrm{I}-\mathrm{J}))+\mathrm{K},$$

where K is put for the arbitrary function. Now the expression for $ud\theta$ (from which this is derived) shows that the three terms in the above value of P must be so interpreted that the differential coefficient of the first (with respect to θ) shall be positive, and those of the two others negative. These conditions will be satisfied by taking*

$$\pm\mathrm{P}=\pi-\Theta+\frac{\cos j+\cos i}{2}(\mathrm{I}+\mathrm{J})$$

$$-\frac{\cos j-\cos i}{2}(\pi-(\mathrm{I}-\mathrm{J}))+\mathrm{K}$$

(in which the upper sign is to be taken when Θ is between o and π, and the under sign when Θ is $>\pi$).

Hence, assuming the arbitrary K so as to destroy the constant part of the expression, we have, without ambiguity, for all values of the variables,

$$\int ud\theta=k(\Theta-\mathrm{I}\cos j-\mathrm{J}\cos i),$$

so that, finally,

$$\mathrm{V}=k\{(\psi-\mathrm{I})\cos j+(\phi-\mathrm{J})\cos i+\Theta\}.\quad.\quad.\quad.\quad.\quad.\quad.\quad(48.)$$

It will be observed that without attention to the proper interpretation of ambiguous symbols, a completely erroneous expression for V might have been obtained.

44. The final equations will be (art. 19.)

$$\frac{d\mathrm{V}}{dh}=t+\tau,\ \frac{d\mathrm{V}}{d\cos i}=\alpha,\ \frac{d\mathrm{V}}{d\cos j}=\beta,$$

τ, α, β being three new arbitrary constants, namely, the elements conjugate respectively to h, $\cos i$, $\cos j$.

In performing the differentiations, it is to be remembered that I, J, Θ do not contain h; and that, by the equations (42.), (43.), art. 39, the terms arising from the differentiation of I, J, Θ with respect to i and j, disappear identically, so that these functions may be considered as exempt from differentiation. Also we have

$$\frac{dk}{dh}=\frac{k}{2h},\ \frac{dk}{d\cos i}=0,$$

$$\frac{dk}{d\cos j}=\frac{(\mathrm{C}-\mathrm{A})k^3\cos j}{2\mathrm{AC}h}$$

* In the figure, as θ diminishes (i and j remaining constant) Θ increases, I+J increases, and I−J increases or diminishes according as $j \lessgtr i$, since $\tan\frac{\mathrm{I}-\mathrm{J}}{2}=\tan\frac{\Theta}{2}\cdot\frac{\sin\frac{i-j}{2}}{\sin\frac{j+i}{2}}$.

(see equation (47.)), and the final equations become, after simple reductions,

$$\left.\begin{aligned}\Theta&=-\frac{\alpha\cos i+\beta\cos j}{k}+\frac{k}{\mathrm{A}}(t+\tau)\\ \phi-\mathrm{I}&=\frac{\beta}{k}-\left(\frac{1}{\mathrm{A}}-\frac{1}{\mathrm{C}}\right)k\cos j.(t+\tau)\\ \psi-\mathrm{J}&=\frac{\alpha}{k}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(\mathrm{R.})$$

These equations comprise a *normal solution* of the problem. The first gives immediately

$$\cos\theta=\cos i\cos j-\sin i\sin j\cos\left(\frac{k}{\mathrm{A}}(t+\tau)-\frac{\alpha\cos i+\beta\cos j}{k}\right)$$

(see art. 40.); and since I, J are given explicit functions of θ, the three variables θ, ϕ, ψ are determined explicitly as functions of t. The third equation (R.) simply expresses that the invariable plane intersects the ecliptic in a fixed line, whose longitude is $\frac{\alpha}{k}$.

45. Let us now introduce the supposition that A and B are unequal, and that the body is acted on by disturbing forces.

We must (see art. 41.) put $\frac{1}{2}\left(\frac{1}{\mathrm{A}}+\frac{1}{\mathrm{B}}\right)$ instead of $\frac{1}{\mathrm{A}}$ in the equations (R.) of the last article; these equations will express the solution of the problem, the elements being now variable, and determined as functions of t by the system of equations

$$k'=-\frac{d\Phi}{d\tau},\ (\cos i)'=-\frac{d\Phi}{d\alpha},\ (\cos j)'=-\frac{d\Phi}{d\beta}$$

$$\tau'=\frac{d\Phi}{dk},\quad \alpha'=\frac{d\Phi}{d\cos i},\quad \beta'=\frac{d\Phi}{d\cos j},$$

where Φ is the disturbing function, expressed in terms of the elements and t.

46. If there are no disturbing forces, Φ reduces itself simply to Ω (art. 41.), whic is now to be transformed by means of the equations (R.), art. 44, as follows.

Since $v=k\cos j$, and $w=k\cos i$, we have

$$\frac{v\cos\theta-w}{\sin\theta}=-k\frac{\cos i-\cos j\cos\theta}{\sin\theta}=-k\sin j\cos\mathrm{I}.$$

Also the expression for u, art. 42, is easily put in the following form:

$$u=\frac{k}{\sin\theta}\left\{\sin^2 i\sin^2 j-(\cos\theta-\cos i\cos j)^2\right\}^{\frac{1}{2}}$$

$$=-\frac{k}{\sin\theta}\sin i\sin j\sin\Theta$$

(with respect to the sign, see art. 42.). And since

$$\frac{\sin\Theta}{\sin\theta}=\frac{\sin\mathrm{I}}{\sin i},$$

this becomes $$u=-k\sin j\sin\mathrm{I}.$$

Introducing these expressions in the value of Ω (art. 41.), we find

$$\Omega=-\frac{k^2}{4}\left(\frac{1}{\mathrm{A}}-\frac{1}{\mathrm{B}}\right)\sin^2 j\cos 2(\phi-\mathrm{I})\,; \quad \ldots \ldots \ldots \quad (49.)$$

and when $\phi-\mathrm{I}$ is expressed in terms of the elements and t $\Big($see equations (R.), art. 44, in which $\frac{1}{2}\left(\frac{1}{\mathrm{A}}+\frac{1}{\mathrm{B}}\right)$ is now to be written for $\frac{1}{\mathrm{A}}\Big)$, this becomes, finally,

$$\Omega=-\frac{k^2}{4}\left(\frac{1}{\mathrm{A}}-\frac{1}{\mathrm{B}}\right)\sin^2 j.\cos 2\left[\frac{\beta}{k}-\left(\frac{1}{2}\left(\frac{1}{\mathrm{A}}+\frac{1}{\mathrm{B}}\right)-\frac{1}{\mathrm{C}}\right)k\cos j.(t+\tau)\right]. \quad \ldots \quad (\Omega.)$$

47. The above expression for Ω does not contain the elements i, α; hence, when there are no disturbing forces, we shall have $(\cos i)'=0$, $\alpha'=0$, or i and α are constant; also

$$k'=-\frac{dk}{dh}\frac{d\Omega}{d\tau}-\frac{dk}{d\cos j}\frac{d\Omega}{d\beta},$$

an expression which is easily found to vanish identically $\Big($see the values of $\frac{dk}{dh}, \frac{dk}{d\cos j}$ in art. 44, observing to put $\frac{1}{2}\left(\frac{1}{\mathrm{A}}+\frac{1}{\mathrm{B}}\right)$ for $\frac{1}{\mathrm{A}}\Big)$. Thus k is also constant; and the "principal plane" is still the "invariable plane," as we know *à priori.*

48. If we now suppose the attraction of another body to be introduced as a disturbing force, we shall have to take for the disturbing function

$$\Phi=\Omega-\mathrm{P},$$

where Ω is the same as above, and P is the potential of one body upon the other, expressed as a function of the elements and the time*. And it follows from the remarks of the last article, that the variation in the *position of the principal plane* depends wholly upon P, and not upon Ω.

I shall here conclude this part of the subject, as it would be beyond the scope of this essay to enter into the details of any of the various problems which might be taken in illustration of the theory, such as those which relate to precession and nutation, or to the motion of the moon about its centre of gravity. The investigations of this section have been introduced, because the results, so far as they go, appeared interesting in themselves, and afforded a remarkable example of the application of the general method.

P.S. Since the last sheets of this essay were in type, I have seen for the first time two papers by Professor Brioschi, in Tortolini's Annali for August and October 1853, of which the titles are "Sulla variazione delle costanti arbitrarie nei problemi della Dinamica," and "Intorno ad un teorema de Meccanica." I have not had an opportunity of examining them sufficiently to judge how far any of the preceding investigations may have been anticipated in them.

June 7.

* The variables which determine the position of the disturbing body are supposed to be given explicit functions of t.

equations of the planetary theory. This investigation, if interesting at all, will probably be so to the mathematician rather than to the astronomer. I think, however, that if the theories of physical astronomy were more frequently treated rigorously and symmetrically, apart from any approximate integrations; and if, when the latter are introduced, more care were taken to give a clear and exact view of the nature of the reasoning employed, it might be possible to draw the attention and secure the cooperation of a class of mathematicians who now may well be excused, if, after a slight trial, they turn from the subject in disgust, and prefer to expatiate in those beautiful fields of speculation which are offered to them by other branches of modern geometry and analysis.

The contents of the two last Sections are more or less closely connected with the subjects of various memoirs by other writers, especially Professor Hansen and the Rev. B. Bronwin. I cannot pretend to that degree of acquaintance with them which would enable me to give an exact statement of the amount of novelty to be found in my own researches. I believe it is enough to justify me in offering them to the Society; beyond this I make no claim.

Oxford, Feb. 15, 1855.

Section IV.

49. The following theorems were demonstrated in the former part of this essay, and are recapitulated here for convenience of reference. (As before, total differentiation with respect to the independent variable t will, in general, be denoted by accents, which will be used *for no other purpose*.)

Theorem I.—If X be a function of n variables $x_1, x_2, \ldots x_n$, and if $y_1, y_2, \ldots y_n$ be n other variables connected with the former by the n equations

$$\frac{d\mathrm{X}}{dx_1}=y_1, \frac{d\mathrm{X}}{dx_2}=y_2, \ldots \frac{d\mathrm{X}}{dx_n}=y_n, \quad \ldots \ldots \quad (50.)$$

then will the values of $x_1, x_2, \ldots . x_n$, expressed by means of these equations in terms of $y_1, \ldots . y_n$, be of the form

$$x_1=\frac{d\mathrm{Y}}{dy_1}, x_2=\frac{d\mathrm{Y}}{dy_2}, \ldots x_n=\frac{d\mathrm{Y}}{dy_n}; \quad \ldots \ldots \quad (51.)$$

and if p be any other quantity explicitly contained in X, then also

$$\frac{d\mathrm{X}}{dp}+\frac{d\mathrm{Y}}{dp}=0 \quad \ldots \ldots \quad (52.)$$

(the differentiation with respect to p being in each case performed only so far as p appears *explicitly* in the function).

The value of Y is given by the equation

$$\mathrm{Y}=-(\mathrm{X})+(x_1)y_1+(x_2)y_2+\ldots+(x_n)y_n, \quad \ldots \ldots \quad (53.)$$

where the brackets indicate that $x_1 \ldots x_n$ are supposed to be expressed in terms of $y_1 \ldots y_n$ (arts. 2, 3.).

Theorem II.—Suppose the function X to contain explicitly, besides the n variables $x_1 \dots x_n$, another variable t, and also n constants $a_1, a_2, \dots a_n$; and in addition to the equations (50.), let the following be assumed:

$$\frac{dX}{da_1}=b_1, \dots \frac{dX}{da_n}=b_n, \quad \dots \dots \dots \quad (54.)$$

where $b_1, \dots b_n$ are n other constants; so that, by virtue of the $2n$ equations (50.), (54.), the $2n$ variables $x_1 \dots x_n, y_1 \dots y_n$, may be considered as functions of the $2n$ constants $a_1, \dots a_n, b_1, \dots b_n$, and t. Then if from the equations (50.), (54.), and their total differential coefficients with respect to t, the $2n$ constants be eliminated, there will result the following $2n$ simultaneous differential equations of the first order; viz.—

$$x'_i=\frac{dZ}{dy_i}, \; y'_i=-\frac{dZ}{dx_i}, \quad \dots \dots \dots \quad (55.)$$

where Z is a function of $x_1 \dots x_n, y_1, \dots y_n$ (which will in general also contain t explicitly), and is given by the equation

$$Z=-\left(\frac{dX}{dt}\right). \quad \dots \dots \dots \quad (56.)$$

In this equation $\frac{dX}{dt}$ represents the partial differential coefficient of X taken with respect to t so far as t appears explicitly in the original expression for X in terms of $x_1 \dots x_n, a_1 \dots a_n$ and t; and the brackets indicate that $a_1, \dots a_n$ are afterwards to be expressed in terms of the variables by means of the equations (50.), (arts. 5, 6.)

Theorem III.—Let the supposition that the $2n$ variables $x_1 \dots x_n, y_1 \dots y_n$ are expressed in terms of the $2n$ constants and t, be called *Hypothesis* I.; and the converse supposition that $a_1 \dots a_n, b_1 \dots b_n$ are expressed in terms of the $2n$ variables and t, *Hypothesis* II.; then will the following relations subsist:

$$\left.\begin{array}{l} \frac{dx_i}{da_j}=-\frac{db_j}{dy_i}, \; \frac{dx_i}{db_j}=\frac{da_j}{dy_i} \\ \frac{dy_i}{da_j}=\frac{db_j}{dx_i}, \; \frac{dy_i}{db_j}=-\frac{da_j}{dx_i} \end{array}\right\} \quad \dots \dots \dots \quad (57.)$$

(In each of these equations the first member refers to *Hyp.* I., and the second to *Hyp.* II.; and since there is no connexion between the indices of the variables and those of the constants, the case of $i=j$ has no peculiarity.)

Theorem IV.—Let the symbol $[p, q]$ be an abbreviation for the expression

$$\Sigma_i\left(\frac{dp}{dy_i}\frac{dq}{dx_i}-\frac{dp}{dx_i}\frac{dq}{dy_i}\right)$$

(where p, q are any functions of the $2n$ variables, which may also contain any other quantities explicitly; and the differentiations are performed only so far as x_1,&c.,y_1,&c. appear explicitly in p, q); then if $a_1, \dots a_n, b_1, \dots b_n$ be expressed (*Hyp.* II.) in terms of the $2n$ variables and t, the following equations subsist identically:

$$[a_i, b_i]=-[b_i, a_i]=1, \quad [a_i, b_j]=[a_i, a_j]=[b_i, b_j]=0 \quad \dots \quad (58.)$$

(i being different from j); and obviously in all cases

$$[p, q] = -[q, p], \text{ and } [p, p] = 0 \text{ (art. 9.).}$$

Theorem V.—If u, v be either (1) any two functions whatever of the $2n$ constants a_1, &c., b_1, &c., or (2) any two functions whatever of the $2n$ variables x_1, &c., y_1, &c. (which may in either case also contain t explicitly)*, then

$$\Sigma_i\left\{\frac{du}{dy_i}\frac{dv}{dx_i} - \frac{du}{dx_i}\frac{dv}{dy_i}\right\} = \Sigma_i\left\{\frac{du}{da_i}\frac{dv}{db_i} - \frac{du}{db_i}\frac{dv}{da_i}\right\}. \quad \text{(59.)}$$

(When u, v represent functions of the constants, the differential coefficients in the first member of this equation refer to *Hyp.* II.; and, when functions of the variables, those in the second member refer to *Hyp.* I.) (art. 10.).

Theorem VI.—Let $x_1, \ldots x_n, y_1, \ldots y_n$ be $2n$ variables concerning which no supposition is made except that they are connected by n equations of the form

$$a_i = \varphi_i(x_1, x_2, \ldots. x_n, y_1, y_2, \ldots y_n) \quad \text{(a.)}$$

(where the functions on the right are only subject to the condition that the n equations (a.) shall be algebraically sufficient to determine $y_1, \ldots y_n$ in terms of $x_1, \ldots x_n$, a_1, &c., and may contain explicitly any other quantities besides x_1, &c., y_1, &c.).

Then, if by means of the equations (a.) the n variables $y_1, y_2, \ldots y_n$ be expressed as functions of x_1, x_2, &c., a_1, &c.; in order that the $\frac{n(n-1)}{2}$ conditions

$$\frac{dy_i}{dx_j} = \frac{dy_j}{dx_i}$$

may subsist identically, it is necessary and sufficient that each of the $\frac{n(n-1)}{2}$ expressions $[a_i, a_j]$ vanish identically.

Theorem VII.—Let Z be any function whatever of $2n$ variables $x_1 \ldots x_n$, $y_1 \ldots y_n$, and t. If of the system of $2n$ simultaneous differential equations of the first order

$$x'_i = \frac{dZ}{dy_i}, \; y'_i = -\frac{dZ}{dx_i} \quad \text{(I.)}$$

there be given n integrals involving n arbitrary constants $a_1, a_2, \ldots a_n$, so that each of these constants may be expressed as a function of the variables x_1, &c., y_1, &c. (with or without t); then *if the* $\frac{n(n-1)}{2}$ *conditions* $[a_i, a_j] = 0$ *subsist identically, the remaining* n *integrals may be found*, as follows. By means of the n given integrals let the n variables $y_1 \ldots y_n$ be expressed in terms of x_1, &c., a_1, &c.; and let (Z) be what Z becomes when $y_1 \ldots y_n$ are thus expressed. These values of $y_1, y_2 \ldots y_n$ and $-(Z)$, will be the partial differential coefficients with respect to $x_1, x_2, \ldots x_n$ and t, of one and the same function; call this function X, then, since its partial differential coefficients are

* It was inadvertently stated in art. 10, that u, v must *not* contain t explicitly. But it is evident that no such limitation is implied in the demonstration of the theorem. The preceding theorem is obviously a particular case of this; namely, the case in which $u = a_j$, $v = b_j$.

(*Hyp.* II.), c becomes also a function of the same; and we have

$$\frac{dc}{dt}=\Sigma_i\left(\frac{dc}{da_i}\frac{da_i}{dt}+\frac{dc}{db_i}\frac{db_i}{dt}\right)=\Sigma_i\left(\frac{dc}{da_i}\frac{dZ}{db_i}-\frac{dc}{db_i}\frac{dZ}{da_i}\right) \quad \text{(61.)}$$

(see the last article). But, by Theorem V., this becomes

$$\frac{dc}{dt}=[c, Z]. \quad \text{(62.)}$$

It is worth observing that both this equation and (60.) might have been obtained indirectly as follows. Since c is constant, we have $c'=0$; that is, $\frac{dc}{dt}+[Z, c]=0$ (see (32.), art. 22.); this gives (62.), since $[Z, c]=-[c, Z]$, and again, by Theorem V., is changed into (61.); and if, in the latter, we put successively $c=a_j$, $c=b_j$, we obtain the system (60.).

SECTION V.—*On the Variation of Elements.*

52. The following general problem includes, I believe, all the cases which occur in practice. Let $P_1, \dots P_n$, $Q_1, \dots Q_n$ be any functions whatever of the $2n$ variables $x_1, \dots x_n$, $y_1, \dots y_n$ and t. It is required to express the $2n$ integrals of the system of $2n$ simultaneous differential equations of the first order

$$x'_i=P_i,\ y'_i=Q_i \quad \text{(63.)}$$

in the same form as the integrals (supposed given) of the canonical system

$$x'_i=\frac{dZ}{dy_i},\quad y'_i=-\frac{dZ}{dx_i}, \quad \text{(I.)}$$

by substituting functions of t for the constant elements of the latter system.

Suppose a *normal solution* (see end of art. 50.) of the system (I.) to be employed. The elements a_i, b_i represent the same functions of x_1, &c., y_1, &c. and t as before, but are now *variable*; consequently we have

$$a'_i=\frac{da_i}{dt}+\Sigma_j\left\{\frac{da_i}{dx_j}x'_j+\frac{da_i}{dy_j}y'_j\right\}=\frac{da_i}{dt}+\Sigma_j\left\{P_j\frac{da_i}{dx_j}+Q_j\frac{da_i}{dy_j}\right\},$$

with a similar expression for b'_i. But, by equations (57.) and (60.), these are immediately transformed into the following:

$$\left.\begin{aligned} a'_i&=\frac{dZ}{db_i}+\Sigma_j\left\{Q_j\frac{dx_j}{db_i}-P_j\frac{dy_j}{db_i}\right\}\\ b'_i&=-\frac{dZ}{da_i}-\Sigma_j\left\{Q_j\frac{dx_j}{da_i}-P_j\frac{dy_j}{da_i}\right\}\end{aligned}\right\} \quad \text{(E.)}$$

where Z, Q_j, P_j, x_j, y_j in the second members are supposed to be expressed (*Hyp.* I.) in terms of the elements and t. Thus the system (63.) is transformed into a system involving the new variables a_i, b_i, instead of the original variables x_i, y_i.

53. If, instead of employing a set of *normal integrals* of the pattern system (I.), we take *any* complete set of integrals c_1, c_2, ... c_{2n}, then c_1, &c. may be considered as

functions of a_1, &c., and again, through them, of the variables. We have then

$$c_i'=\frac{dc_i}{da_1}a_1'+\ldots+\frac{dc_i}{db_1}b_1'+\ldots;$$

and if in this equation the values of a_1', &c. be introduced from the formula (E.) of the last article, the following expression results:

$$c_i'=\{Z, c_i\}+\Sigma_j(Q_j\{x_j, c_i\}-P_j\{y_j, c_i\})$$

(in which the symbol $\{p, q\}$ is used to denote

$$\Sigma_k\left\{\frac{dp}{db_k}\frac{dq}{da_k}-\frac{dp}{da_k}\frac{dq}{db_k}\right\},$$

so that by (59.) (Theorem V.) we have $\{p, q\}=-[p, q]$; but in $\{p, q\}$ p and q are considered as functions of a_1, &c., b_1, &c., whilst in $[p, q]$ they are considered as functions of x_1, &c., y_1, &c.). Now, considering p, q as functions of c_1, &c., and through these, of a_1, &c., we have (by reasoning exactly similar to that employed in deducing equation (24.), art. 9.)

$$\{p, q\}=\Sigma\left(\{c_\alpha, c_\beta\}\left(\frac{dp}{dc_\alpha}\frac{dq}{dc_\beta}-\frac{dp}{dc_\beta}\frac{dq}{dc_\alpha}\right)\right)$$

(the summation referring to all binary combinations of the indices α, β). Hence we have, putting $q=c_i$,

$$\{p, c_i\}=\Sigma_\alpha\left(\{c_\alpha, c_i\}\frac{dp}{dc_\alpha}\right), \quad \ldots \ldots \quad (64.)$$

and consequently the above expression for c_i' becomes

$$c_i'=\{Z, c_i\}+\Sigma_\alpha\Sigma_j\left(\{c_\alpha, c_i\}\left(Q_j\frac{dx_j}{dc_\alpha}-P_j\frac{dy_j}{dc_\alpha}\right)\right), \quad \ldots \ldots \quad (F.)$$

an equation which is easily seen to become identical with (E.), art. 52, when $c_1\ldots c_{2n}$ represent $a_1\ldots a_n$, $b_1\ldots b_n$.

54. The simplest case is that in which the system of equations (63.), whose integrals are sought, are of the *canonical form*; that is, where

$$P_i=\frac{dW}{dy_i}, \quad Q_i=-\frac{dW}{dx_i},$$

W being a given function of the variables (with or without t). In this case the formula (E.) becomes

$$\left.\begin{aligned} a_i'&=\frac{dZ}{db_i}-\frac{dW}{db_i} \\ b_i'&=-\frac{dZ}{da_i}+\frac{dW}{da_i}\end{aligned}\right\} \quad \ldots \ldots \quad (65.)$$

whilst (F.) is easily found to be reducible, by the help of (64.), to either of the following forms:

$$c_i'=\{Z, c_i\}-\{W, c_i\} \quad \ldots \ldots \quad (66.)$$

$$c_i'=\Sigma_\alpha\left(\{c_\alpha, c_i\}\left(\frac{dZ}{dc_\alpha}-\frac{dW}{dc_\alpha}\right)\right). \quad \ldots \ldots \quad (67.)$$

If we put $W=Z+\Omega$, so that Ω may be called the "disturbing function," the above formulæ become

$$a_i'=-\frac{d\Omega}{db_i}, \quad b_i'=\frac{d\Omega}{da_i} \quad \ldots \ldots \ldots \ldots \quad (68.)$$

$$c_i'=\Sigma_\alpha\left(\{c_i, c_\alpha\}\frac{d\Omega}{dc_\alpha}\right). \quad \ldots \ldots \ldots \ldots \quad (69.)$$

On the first of these forms see the note to art. 38. With respect to the form (69.), if we put for $\{c_i, c_\alpha\}$ its equivalent $-[c_i, c_\alpha]$, or $[c_\alpha, c_i]$ (see Theorem V. art. 49.), we obtain the well-known expression

$$c_i'=\Sigma_\alpha\left([c_\alpha, c_i]\frac{d\Omega}{dc_\alpha}\right).$$

The difference between this last form and (69.) consists in this; that in the latter the coefficients $[c_\alpha, c_i]$ are obtained from the expressions for c_1, c_2, &c. *in terms of the variables*; whereas in (69.) the coefficients $\{c_i, c_\alpha\}$ are similarly obtained from the expressions for c_1, &c. *in terms of the normal elements* a_1, &c., b_1, &c.*; and when a *normal solution* of the undisturbed problem has been obtained, the latter process will generally be found much more convenient than the former, since the elements c_1, &c. will usually be much simpler functions of the *normal elements* than of the *variables.*

55. In illustration of this, it will be worth while to deduce the expressions for the variations of the ordinary elliptic elements of a planet's orbit from those of the normal elements given in art. 30.

Let a and e be the semiaxis major and excentricity, ι the inclination of the orbit to a fixed ecliptic, ν the longitude of the node, ϖ the longitude of the perihelion, $nt+(\epsilon)$ the mean longitude of the planet; *longitudes* being reckoned in the plane of the ecliptic (from a fixed origin) *as far as the node*, and then *on the plane of the orbit.* As usual, n stands for $\frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}$. Also let $nt+(\epsilon)=\int_0^t ndt+\epsilon$, so that $\epsilon'=(\epsilon)'+tn'$.

If, then, we call the six normal elements α_1, α_2, α_3, β_1, β_2, β_3, we have (see art. 30.)

$$\alpha_1=\frac{m\mu}{2a}, \qquad \beta_1=\frac{\varpi-(\epsilon)}{n},$$

$$\alpha_2=m\sqrt{\mu a(1-e^2)}, \qquad \beta_2=\varpi-\nu,$$

$$\alpha_3=m\sqrt{\mu a(1-e^2)}.\cos\iota, \qquad \beta_3=\nu;$$

from which, conversely,

$$a=\frac{m\mu}{2\alpha_1}, \qquad \varpi=\beta_2+\beta_3,$$

$$1-e^2=\frac{2\alpha_1\alpha_2^2}{m^3\mu^2}, \qquad \nu=\beta_3,$$

$$\cos\iota=\frac{\alpha_3}{\alpha_2}, \qquad (\epsilon)=\beta_2+\beta_3-\frac{(2\alpha_1)^{\frac{3}{2}}}{m^{\frac{3}{2}}\mu}\beta_1.$$

* $\{c_i, c_\alpha\}=\Sigma_j\left(\frac{dc_i}{db_j}\frac{dc_\alpha}{da_j}-\frac{dc_i}{da_j}\frac{dc_\alpha}{db_j}\right)$.

From these expressions the values of $\{a, e\}$, $\{a, \iota\}$, &c. are found with the greatest simplicity, and the results are

$$m\mu\{a, (\epsilon)\}=2na^2, \quad m\mu\{(\epsilon), e\}=\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2}),$$

$$m\mu\{\varpi, e\} =\frac{na\sqrt{1-e^2}}{e}, \quad m\mu\{(\epsilon), \iota\}=\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2},$$

$$m\mu\{\varpi, \iota\} =\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2}, \quad m\mu\{\nu, \iota\}=\frac{na}{\sin\iota\sqrt{1-e^2}},$$

the rest all vanishing. Hence, observing that if R be taken in its usual signification we have $\Omega=-\mathrm{R}$, we obtain*

$$\mu a' =2na^2\frac{d\mathrm{R}}{d(\epsilon)},$$

$$\mu e' =\frac{-na\sqrt{1-e^2}}{e}\left\{\frac{d\mathrm{R}}{d\varpi}+(1-\sqrt{1-e^2})\frac{d\mathrm{R}}{d(\epsilon)}\right\},$$

$$\mu(\epsilon)'=-2na^2\frac{d\mathrm{R}}{da}+\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\frac{d\mathrm{R}}{de}+\frac{na}{\sqrt{1-e^2}}\tan\frac{\iota}{2}\frac{d\mathrm{R}}{d\iota},$$

$$\mu\varpi' =na\left\{\frac{\sqrt{1-e^2}}{e}\frac{d\mathrm{R}}{de}+\frac{1}{\sqrt{1-e^2}}\tan\frac{\iota}{2}\frac{d\mathrm{R}}{d\iota}\right\},$$

$$\mu\iota' =\frac{-na}{\sqrt{1-e^2}}\left\{\frac{1}{\sin\iota}\frac{d\mathrm{R}}{d\nu}+\tan\frac{\iota}{2}\left(\frac{d\mathrm{R}}{d(\epsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\},$$

$$\mu\nu' =\frac{na}{\sin\iota\sqrt{1-e^2}}\frac{d\mathrm{R}}{d\iota},$$

in which we may, as usual, put ϵ for (ϵ), provided that in forming the term $\frac{d\mathrm{R}}{da}$, nt be exempt from differentiation with respect to a.

56. A comparison of the above process with that by which the corresponding

* If we consider R as a function of p, q instead of ι, ν, where $p=\tan\iota\cos\nu$, $q=\tan\iota\sin\nu$, we find

$$\frac{d\mathrm{R}}{d\iota}=\sec^2\iota\left(\cos\nu\frac{d\mathrm{R}}{dp}+\sin\nu\frac{d\mathrm{R}}{dq}\right)$$

$$\frac{d\mathrm{R}}{d\nu}=\tan\iota\left(\cos\nu\frac{d\mathrm{R}}{dq}-\sin\nu\frac{d\mathrm{R}}{dp}\right),$$

and consequently

$$\mu p'=\frac{-na(\sec\iota)^2}{\sqrt{1-e^2}}\left\{\sec\iota\frac{d\mathrm{R}}{dq}+\tan\frac{\iota}{2}\cos\nu\left(\frac{d\mathrm{R}}{d(\epsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\}$$

$$\mu q'=\frac{na(\sec\iota)^2}{\sqrt{1-e^2}}\left\{\sec\iota\frac{d\mathrm{R}}{dp}-\tan\frac{\iota}{2}\sin\nu\left(\frac{d\mathrm{R}}{d(\epsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)\right\}.$$

The formulæ will then agree with those of the Mécanique Céleste (Supplement to vol. iii. p. 360, ed. 1844), if we allow for the different mode of measuring longitudes, and neglect, as LAPLACE does, terms of the second order with respect to ι and $\frac{d\mathrm{R}}{d\iota}$. (LAPLACE uses R with the opposite sign.) Those in the text agree (allowing for notation) with the expressions given by Professor HANSEN, Astr. Nachr. No. 166, art. 3, equations (2).

expressions are obtained by PONTÉCOULANT*, will show the convenience of using the coefficients $\{c_i, c_j\}$ instead of $[c_i, c_j]$ (in PONTÉCOULANT's notation (c_i, c_j)).

[It will be observed that the formulæ for $(\varepsilon)'$, ϖ', ι' at the end of the last article, do not agree with those of PONTÉCOULANT (p. 330) for the variations of the corresponding quantities ε, ω, φ. The reason of this is as follows:—In PONTÉCOULANT's notation φ expresses the same as ι in this paper, and α the same as ν. But ω (the longitude of the perihelion) is not the same as ϖ; the former being measured *entirely in the plane of the orbit* from a radius vector, *fixed in that plane*†, and assumed as the origin of longitudes. Consequently ε, in PONTÉCOULANT (which we will call $\varepsilon_{,}$ for distinction), is not the same as (ε) in the present paper. In fact, if we equate the expressions for the mean anomaly in the two notations, we have

$$\varepsilon_{,}-\omega=(\varepsilon)-\varpi;$$

also it is evident that if we put β for the angle between the node and the origin from which ω is measured, we have $d\beta=-\cos\iota d\nu$, and $\varpi=\nu+\beta+\omega$, so that

$$d\varpi=d\omega+(1-\cos\iota)d\nu.$$

If then it were allowable to consider R as capable of being expressed as a function of ω and $\varepsilon_{,}$ instead of ϖ and (ε), and if we represented by (R) the expression for R so transformed, we should have

$$\frac{d\mathrm{R}}{d\varpi}d\varpi+\frac{d\mathrm{R}}{d(\varepsilon)}d(\varepsilon)+\&\mathrm{c.}=\frac{d(\mathrm{R})}{d\omega}d\omega+\frac{d(\mathrm{R})}{d\varepsilon_{,}}d\varepsilon_{,}+\&\mathrm{c.};$$

and if, in the two first terms, we put for $d\varpi$ and $d(\varepsilon)$ the values $d\varpi=d\omega+(1-\cos\iota)d\nu$, $d(\varepsilon)=d\varepsilon_{,}+(1-\cos\iota)d\nu$, and compare the two expressions, we find

$$\frac{d\mathrm{R}}{d\varpi}=\frac{d(\mathrm{R})}{d\omega},\quad \frac{d\mathrm{R}}{d(\varepsilon)}=\frac{d(\mathrm{R})}{d\varepsilon_{,}},$$

$$\frac{d\mathrm{R}}{d\nu}+(1-\cos\iota)\left(\frac{d\mathrm{R}}{d(\varepsilon)}+\frac{d\mathrm{R}}{d\varpi}\right)=\frac{d(\mathrm{R})}{d\nu}.$$

These relations, together with the equation

$$\varpi'=\omega'+(1-\cos\iota)\nu',$$

are easily seen to render the expressions at the end of art. 55 identical with those of PONTÉCOULANT; in fact, it is by an equivalent transformation that the latter are finally obtained by that author from the correct expressions in p. 328. But it is to be observed that this proceeding is founded upon a *false assumption*; for it is impossible to express R as a function of a, e, ι, ν, $\varepsilon_{,}$, ω, as is obvious from the consideration that R, in its original form, is not a function of $(\varepsilon)-\varpi$ merely, but also of (ε); whilst (ε) is *not expressible as a function of the new elements*, as is shown by the equation $d(\varepsilon)=d\varepsilon_{,}+(1-\cos\iota)d\nu$ ‡. It would be out of place to enter further into this sub-

* Théorie Anal. du Système du Monde, tome i. pp. 316–330.

† On the meaning of this expression, see below, art. 73.

‡ It would be a work of some trouble to trace *accurately* the process by which LAPLACE arrives at the for-

ject here, especially as some of the most important principles involved in it have been discussed elsewhere *. [See also Appendix B.]

57. Returning to the expression (69.), art. 54, it may be observed that the coefficients $\{c_i, c_j\}$ are to be expressed in terms of c_1, c_2, &c., and this involves no difficulty when *each* of the two sets of elements c_1, &c., a_1, &c. can be expressed *in terms of the other explicitly*, as was the case in the example just discussed. Suppose, however, that the normal set a_1, &c., b_1, &c. are given in terms of the set c_1, &c., but that it is impracticable or inconvenient to obtain the converse equations expressing the latter in terms of the former. In this case we may proceed as follows.

Adopting the notation of art. 1 †, and putting f, g for any two of the set c_1, c_2, &c., we have

$$\{f, g\} = \Sigma_i \frac{d(f,g)}{d(b_i, a_i)};$$

suppose this equation written at length, and then, after multiplying by $\frac{d(b_j, a_j)}{d(f,g)}$, let each side be summed with respect to all binary combinations f, g. The result is (see art. 1, equation (4.)),

$$\Sigma\left(\frac{d(b_j, a_j)}{d(f,g)}\cdot\{f, g\}\right) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70.)$$

(the summation referring to the combinations f, g). Again, if the former equation be multiplied by $\frac{d(p, q)}{d(f, g)}$, where p, q represent any two of the normal elements, a_1, &c. b_1, &c., *except a conjugate pair*, and the sum be taken as before, we have

$$\Sigma\left(\frac{d(p, q)}{d(f, g)}\cdot\{f, g\}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71.)$$

The two formulæ (70.), (71.) give $n(2n-1)$ linear equations for determining the $n(2n-1)$ unknown quantities $\{f, g\}$; the coefficients of the latter being all given functions of c_1, &c. But such cases will hardly occur in practice. (With respect to the form of the above system of linear equations, it is easy to show that the complete determinant of the coefficients is $=1$.)

58. The integration of the formulæ (65.), art. 53, would give the means of expressing the solution of the system

$$x_i' = \frac{dW}{dy_i}, \quad y_i' = -\frac{dW}{dx_i}$$

mulæ alluded to in a preceding note, as the various steps of it are to be found in different places, the notation is somewhat inconsistent, and the results *do not profess to be rigorous*. My impression is, however, that LAPLACE nowhere commits the fallacy of assuming (for example) that R is a function of r, v, z, or r, v, s (see vol. i. p. 295), where v is the angle described by the radius vector on the varying plane of the orbit.

* See JACOBI's two letters to Professor HANSEN in CRELLE's Journal, vol. xlii.

† *i. e.* using $\frac{d(u, v)}{d(x, y)}$ as an abbreviation for $\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$.

in the *form* of a normal solution of any other similar system

$$x_i' = \frac{dZ}{dy_i}, \quad y_i' = -\frac{dZ}{dx_i},$$

which may be chosen as the pattern.

In the most usual examples the function to be chosen for Z is naturally suggested by the circumstance, that W presents itself under the form of the sum of two functions $Z+\Omega$, of which the former, taken alone, gives an integrable system. But this is not necessarily the case; and it is worth while to observe that the formulæ (65.) take a simple and remarkable form whatever Z may be, provided that it be a function *not containing* t *explicitly*. For then, assuming the "integral of vis viva," $Z=h$, as one of the normal integrals of the pattern system*, the element conjugate to h is τ (the constant added to t); and observing that Z, in (65.), being *expressed in terms of the elements*, reduces itself simply to h, we shall have $\frac{dZ}{dh}=1$, whilst the differential coefficients of Z with respect to all the other elements vanish; so that, if we put $a_1, \dots a_{n-1}, b_1, \dots b_{n-1}$ for the remaining elements, the system (65.) takes the following form:—

$$\left.\begin{aligned} h' &= -\frac{dW}{d\tau}, \quad \tau' = -1 + \frac{dW}{dh} \\ a_i' &= -\frac{dW}{db_i}, \quad b_i' = \frac{dW}{da_i} \end{aligned}\right\} \quad \dots\dots\dots \quad (72.)$$

This, in dynamics, gives the process to be used in the following problem: "*To express the solution of any dynamical problem in the form of the solution of any other* (*involving the same number of variables*) *in which the principle of vis viva subsists.*"

59. As an example of the above process we may apply it to determine the motion of a simple free pendulum (not taking into account the earth's rotation).

Let l be the length of the pendulum, and let the mass of the material point m placed at its extremity be represented by unity. Also let x, y, z be the rectangular coordinates of m, the origin being at the position of rest of m, and the axis of z directed vertically upwards. The equation to the sphere described by m is

$$x^2+y^2+z^2-2lz=0,$$

and the *force-function* U is $-gz$.

Hence if we take, as the two independent coordinates, the radius vector ϱ of the projection of m on the plane of xy, and the angle θ between ϱ and the axis of x, we shall have for the differential equations of motion,

$$\left.\begin{aligned} \varrho' &= \frac{dW}{du}, \quad \theta' = \frac{dW}{dv} \\ u' &= -\frac{dW}{d\varrho}, \quad v' = -\frac{dW}{d\theta} \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (A.)$$

* See art. 19 (where h_i in equation (29.) is a misprint for b_i).

where u, v, are the variables conjugate respectively to ϱ, θ, and defined by the equations

$$u=\frac{dT}{d\varrho'},\quad v=\frac{dT}{d\theta'};$$

and W is T—U expressed in terms of ϱ, θ, u, v.

Now $x=\varrho\cos\theta$, $y=\varrho\sin\theta$, $z=l-\sqrt{l^2-\varrho^2}$; hence

$$T\left(=\frac{1}{2}(x'^2+y'^2+z'^2)\right)=\frac{1}{2}\left\{\frac{l^2}{l^2-\varrho^2}\varrho'^2+\varrho^2\theta'^2\right\},$$

from which the following expression for W is easily obtained:

$$W=\frac{1}{2}\left(\frac{l^2-\varrho^2}{l^2}u^2+\frac{v^2}{\varrho^2}\right)+g(l-\sqrt{l^2-\varrho^2}). \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(W.)}$$

Now let us take as a model for the solution of the above system, a set of normal integrals (in polar coordinates) of the system

$$x''+n^2x=0,\quad y''+n^2y=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B.)}$$

where $n^2=\frac{g}{l}$. In this system we have $U=-\frac{1}{2}n^2\varrho^2$; and proceeding exactly as in art. 27, we obtain the following results: the two integrals of *vis viva* and of areas are

$$\left.\begin{aligned} h&=\frac{1}{2}\left(u^2+\frac{v^2}{\varrho^2}+n^2\varrho^2\right) \\ c&=v \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)}$$

these are to be solved for u, v; and then V is to be obtained from the equation $dV=ud\varrho+vd\theta$. This gives

$$V=c\theta+\int d\varrho\left\{2h-n^2\varrho^2-\frac{c^2}{\varrho^2}\right\}^{\frac{1}{2}};$$

and the remaining integrals are given by the equations

$$\frac{dV}{dh}=t+\tau,\quad \frac{dV}{dc}=\varpi,$$

τ and ϖ being the elements conjugate respectively to h and c. Performing the differentiations *first*, and taking the integrals in the second term so as to vanish with the expression $\left\{2h-n^2\varrho^2-\frac{c^2}{\varrho^2}\right\}^{\frac{1}{2}}$ (see Appendix A.), we find easily the final equations

$$\left.\begin{aligned} n^2\varrho^2&=h+\sqrt{h^2-n^2c^2}.\cos 2n(t+\tau) \\ c^2\varrho^{-2}&=h-\sqrt{h^2-n^2c^2}.\cos 2(\theta-\varpi) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii.)}$$

in which ϖ is the angle between the axis of x and a (distant) apse, and $-\tau$ is the time of passage through that apse. The four equations (i.), (ii.) comprise a complete normal solution of the equations (B.). The last is the polar equation to the elliptic orbit; and if we call a, b the semiaxes of the ellipse, we have

$$c=nab,\quad h=n^2\frac{a^2+b^2}{2}.$$

60. The solution of the system (A.) of the last article will now be expressed by the same equations (i.), (ii.), if the elements h, c, τ, ϖ be variables defined by the system (see art. 58.)

$$h'=-\frac{dW}{d\tau},\quad \tau'=-1+\frac{dW}{dh}$$

$$c'=-\frac{dW}{d\varpi},\quad \varpi'=\frac{dW}{dc},$$

where W is to be obtained by substituting in the expression (W.), art. 59, the values of ϱ, θ, u, v in terms of the elements and t, derived from equations (i.), (ii.). The result of this substitution is

$$W=\frac{h}{2}+\frac{c^2}{4l^2}-\frac{h^2}{4n^2l^2}-\frac{1}{2}\sqrt{h^2-n^2c^2}.\cos 2n(t+\tau)$$

$$+\frac{h^2-n^2c^2}{4n^2l^2}\cos 4n(t+\tau)+n^2l(l-\sqrt{l^2-\varrho^2}),$$

in which the value of ϱ^2 in the last term must be understood to be substituted from the first of equations (ii.). If we call φ the angle between the pendulum and the vertical, we shall have evidently

$$n^2l(l-\sqrt{l^2-\varrho^2})=n^2l^2(1-\cos\varphi),$$

and the differential coefficient of this term with respect to any constant k involved in the value of ϱ will be $\frac{n^2}{2\cos\varphi}\cdot\frac{d(\varrho^2)}{dk}$. Observing this, we obtain the following expressions for the variations of the elements:

$$h'=-n(\sec\varphi-1)\sqrt{h^2-n^2c^2}.\sin 2n(t+\tau)+\frac{h^2-n^2c^2}{nl^2}\sin 4n(t+\tau)$$

$$\tau'=-\frac{h}{2n^2l^2}+\frac{1}{2}(\sec\varphi-1)\left(1+\frac{h}{\sqrt{h^2-n^2c^2}}\cos 2n(t+\tau)\right)+\frac{h}{2n^2l^2}\cos 4n(t+\tau)$$

$$c'=0$$

$$\varpi'=\frac{c}{2l^2}-\frac{1}{2}(\sec\varphi-1)\frac{n^2c}{\sqrt{h^2-n^2c^2}}\cos 2n(t+\tau)-\frac{c}{2l^2}\cos 4n(t+\tau).$$

The third of these equations gives $ab=constant$; hence, by means of the equations at the end of art. 59, the following expressions are easily deduced:

$$\frac{a'}{a}=-\frac{b'}{b}=\frac{n}{2}\left\{-(\sec\varphi-1)\sin 2n(t+\tau)+\frac{a^2-b^2}{l^2}\sin 4n(t+\tau)\right\}$$

$$\tau'=\frac{a^2+b^2}{4l^2}\left(-1+\cos 4n(t+\tau)\right)+\frac{1}{2}(\sec\varphi-1)\left(1+\frac{a^2+b^2}{a^2-b^2}\cos 2n(t+\tau)\right)$$

$$\varpi'=nab\left\{\frac{1-\cos 4n(t+\tau)}{2l^2}-(\sec\varphi-1)\frac{\cos 2n(t+\tau)}{a^2-b^2}\right\}.$$

These equations are rigorous, and in general not easier to integrate than the original system of which they are a transformation; but they may be integrated approxi-

[It follows from Theorem I. art. 49, that the system (73.) is equivalent to each of the following:

$$\frac{dQ}{dx_i}=y_i, \quad \frac{dQ}{d\xi_i}=-\eta_i,$$

in which $Q=-P+\Sigma_i(x_iy_i)$, and is expressed in terms of $x_1, \dots x_n, \xi_1, \dots \xi_n$; or

$$\frac{dR}{d\eta_i}=\xi_i, \quad \frac{dR}{dy_i}=-x_i,$$

in which $R=-P+\Sigma_i(\xi_i\eta_i)$, and is expressed in terms of $y_1, \dots y_n, \eta_1, \dots \eta_n$; or lastly,

$$\frac{dS}{d\eta_i}=\xi_i, \quad \frac{dS}{dx_i}=y_i,$$

in which $S=-P+\Sigma_i(x_iy_i+\xi_i\eta_i)$, and is expressed in terms of $x_1, \dots x_n, \eta_1, \dots \eta_n$.

Any one of these forms might be used; but I shall employ the form (73.) for reasons of convenience.]

63. Inasmuch as the equations (73.) of the last article are of the same general form as the system (54.), art. 49, all the conclusions deduced from that form will subsist, *mutatis mutandis*; so that we may apply the Theorems (II.), (III.), (IV.), (V.), art. 49, by merely changing X into P, and

$x_1, \dots x_n, y_1, \dots y_n, a_1, \dots a_n, b_1, \dots b_n,$ respectively into
$\xi_1, \dots \xi_n, \eta_1, \dots \eta_n, y_1, \dots y_n, x_1, \dots x_n$;

observing that instead of x'_i, y'_i we must now write $\frac{d\xi_i}{dt}, \frac{d\eta_i}{dt}$*. We thus obtain the following relations:

$$\frac{d\xi_i}{dt}=\frac{d\Psi}{d\eta_i}, \quad \frac{d\eta_i}{dt}=-\frac{d\Psi}{d\xi_i}, \quad \dots \dots \dots \quad (74.)$$

where Ψ is a function of ξ_1, &c., η_1, &c. and t, defined by the equation

$$\Psi=-\left(\frac{dP}{dt}\right), \quad \dots \dots \dots \quad (75.)$$

the brackets indicating that the expressions for $y_1, \dots y_n$ in terms of the new variables ξ_1, &c., η_1, &c., are to be substituted in $\frac{dP}{dt}$ after the differentiation with respect to t; which is performed so far as t appears explicitly in the original expression for P as a function of $\xi_1 \dots \xi_n, y_1 \dots y_n$ and t. (See Theorem II.)

We have also the system

$$\left.\begin{array}{l} \frac{d\xi_i}{dy_j}=-\frac{dx_j}{d\eta_i}, \quad \frac{d\xi_i}{dx_j}=\frac{dy_j}{d\eta_i} \\ \frac{d\eta_i}{dy_j}=\frac{dx_j}{d\xi_i}, \quad \frac{d\eta_i}{dx_j}=-\frac{dy_j}{d\xi_i} \end{array}\right\} \quad \dots \dots \dots \quad (76.)$$

* For in the original theorems x'_i is the same thing as the differential coefficient of x_i taken with respect to t, as t appears explicitly in the expression for x_i in terms of a_1, &c., b_1, &c. and t; the analogous quantity in the present case is therefore the differential coefficient of ξ_i taken with respect to t, as t appears explicitly in the expression for ξ_i in terms of x_1, &c., y_1, &c. and t. But this must not now be denoted by ξ'_i, inasmuch as x_1, &c., y_1, &c. are themselves afterwards to be considered as functions of t.

(see Theorem III.). And if p, q be any two functions of the variables (with or without t), then

$$\Sigma_i\left(\frac{dp}{dy_i}\frac{dq}{dx_i}-\frac{dp}{dx_i}\frac{dq}{dy_i}\right)=\Sigma_i\left(\frac{dp}{d\eta_i}\frac{dq}{d\xi_i}-\frac{dp}{d\xi_i}\frac{dq}{d\eta_i}\right). \quad \ldots \quad (77.)$$

where p and q in the first member are supposed to be expressed in terms of x_1, &c., y_1, &c., and, in the second, in terms of ξ_1, &c., η_1, &c. In other words, the value of $[p, q]$ is the same, whether it be obtained from the expressions for p, q in terms of the original variables, or by an analogous process from their expressions in terms of the new.

Particular cases of (77.) are the relations

$$[\xi_i, \eta_i]=-1, \quad [\xi_i, \xi_j]=[\eta_i, \eta_j]=[\xi_i, \xi_j]=0. \quad \ldots \quad (78.)$$

(See Theorems IV., V.)

64. The relations (74.), (76.), (77.), (78.) of the last article depend solely upon the form of the equations (73.), art. 62, which connect the new variables with the old; and are independent of any supposition as to the equations which may determine either set of variables as functions of t. Let us now, however, introduce the supposition that the original variables $x_1, \ldots x_n, y_1, \ldots y_n$ are determined as functions of t by the system of differential equations,

$$x_i'=\frac{dZ}{dy_i}, \quad y_i'=-\frac{dZ}{dx_i}. \quad \ldots \quad (I.)$$

The relations just established enable us immediately to transform this system into another involving the new variables instead of the old; for we have

$$\xi_i'=\frac{d\xi_i}{dt}+\Sigma_j\left(\frac{d\xi_i}{dx_j}x_j'+\frac{d\xi_i}{dy_j}y_j'\right);$$

now $$\frac{d\xi_i}{dt}=\frac{d\Psi}{d\eta_i} \text{ (see (74.), art. 63)};$$

and if in the remaining term we substitute for x_j', y_j' their values from (I.), and for $\frac{d\xi_i}{dx_j}$, $\frac{d\xi_i}{dy_j}$ their values $\frac{dy_j}{d\eta_i}$, $-\frac{dx_j}{d\eta_i}$, it becomes

$$\Sigma_j\left(\frac{dZ}{dx_j}\frac{dx_j}{d\eta_i}+\frac{dZ}{dy_j}\frac{dy_j}{d\eta_i}\right),$$

which is equivalent to $\frac{dZ}{d\eta_i}$, if Z be supposed expressed in terms of the new variables.

We have then $$\xi_i'=\frac{d\Psi}{d\eta_i}+\frac{dZ}{d\eta_i},$$

and, exactly in the same way,

$$\eta_i'=-\frac{d\Psi}{d\xi_i}-\frac{dZ}{d\xi_i}.$$

This result may be stated in the form of the following *Theorem* VIII*. If the system

* This theorem, in its general form, is, to the best of my knowledge, new. But that case of it in which P does not contain t explicitly has already been proved in a different way by M. Dessoves, who has, by means

of differential equations (I.) be transformed by the introduction of new variables $\xi_1, \dots \xi_n, \eta_1, \dots \eta_n$, connected with the original variables $x_1, \dots x_n, y_1, \dots y_n$, by the equations $\frac{dP}{dy_i}=x_i$, $\frac{dP}{d\xi_i}=\eta_i$, where P is any function of $\xi_1, \dots \xi_n, y_1, \dots y_n$, which may also contain t explicitly, then the transformed equations are

$$\xi_i'=\frac{d\Phi}{d\eta_i},\quad \eta_i'=-\frac{d\Phi}{d\xi_i}, \quad \dots\dots\dots\dots \quad (79.)$$

in which Φ is defined by the equation

$$\Phi=Z-\frac{dP}{dt},$$

and is to be expressed in terms of the new variables. (The substitution of the new variables in $\frac{dP}{dt}$ is to be made *after* the differentiation. See art. 63.).

Corollary.—If P do not contain t explicitly, $\frac{dP}{dt}=0$ and $\Phi=Z$; so that in this case the transformation is effected merely by expressing Z in terms of the new variables.

65. It follows from (77.), art 63, that if f, g be any two integrals of the system (I.), the value of $[f, g]$ is the same whether it be derived from these integrals in their original form, or similarly obtained from the same integrals after transformation by the introduction of the new variables. And consequently *if* n *integrals* $a_1, a_2, \dots a_n$ *of the original system be given, which satisfy the* $\frac{n(n-1)}{2}$ *conditions* $[a_i, a_j]=0$, *they will continue, after a normal transformation, to satisfy the analogous conditions*, so that the method of finding the remaining integrals given in Theorem VII. art. 49, will also continue to be applicable. We had an instance of this in the case of the problem of central forces (art. 27.), where the above conditions were found to subsist after the transformation from rectangular to polar coordinates. (It will be shown presently that every transformation of coordinates is a *normal transformation.*)

66. It was shown in Part I. (art. 18.), that if W be any function of $x_1, x_2, \dots x_n$, $x_1', x_2', \dots x_n'$ (which may also contain t explicitly), the system of n differential equations of the second order

$$\left(\frac{dW}{dx_i'}\right)'=\frac{dW}{dx_i} \quad \dots\dots\dots\dots \quad (80.)$$

may be changed into a system of $2n$ equations of the first order of the form (I.),

of it, deduced JACOBI's form of the method of the Variation of Elements (namely, the equations (68.), art. 54), from the similar form of LAGRANGE, in which the elements are the initial values of the variables. It will appear in the sequel that the extension to the case in which P may contain t, is of importance. If the expression were not already appropriated, I should have proposed definitively to call P the "modulus of transformation;" and I shall use this term provisionally in the present paper, not being able to suggest a tolerable substitute. After all, as the word "modulus" itself is used without confusion in very different senses according to the subject matter, there is, perhaps, no reason why a similar liberty should not be allowed in the use of the proposed expression.

art. 64, by putting $y_i=\frac{dW}{dx'_i}$, and taking Z a function of x_1, &c., y_1, &c. and t, defined by the equation

$$Z=-W+\Sigma_i(x'_i y_i),$$

in which $x'_1, \ldots x'_n$ are to be expressed in terms of y_1, &c., x_1, &c.*.

Conversely, a system of the form (I.) being given, it may be changed into a system of the form (80.) as follows: by means of the equations $x'_i=\frac{dZ}{dy_i}$, let $y_1 \ldots y_n$ be expressed in terms of x'_1, &c., x_1, &c.; it follows from Theorem I. (art. 49.) that we shall have

$$y_i=\frac{dW}{dx'_i} \quad \ldots \ldots \ldots \ldots \quad \text{(a.)}$$

and

$$\frac{dZ}{dx_i}=-\frac{dW}{dx_i}, \quad \ldots \ldots \ldots \ldots \quad \text{(b.)}$$

where W is a function of x_1, &c., x'_1, &c. defined by equation

$$W=-Z+\Sigma_i(x'_i y_i),$$

in which $y_1, \ldots y_n$ are to be expressed in terms of x'_1, &c., x_1, &c. The n equations $y'_i=-\frac{dZ}{dx_i}$ are then changed by (a.) and (b.) into the form (80.).

On the Transformation of Coordinates.

67. It has been seen in the preceding article, that we can always change the system of $2n$ equations of the first order of the form (I.), art. 64, into a system of n equations of the second order of the form (80.). In this latter form the equations of dynamics naturally present themselves.

Now in the case of the dynamical equations, $x_1, x_2, \ldots x_n$ are the independent *coordinates* of the system (the word *coordinates* being taken in its most general sense), and when the equations are to be changed into the form (I.), the additional variables $y_1, \ldots y_n$ are defined by the equations $\frac{dW}{dx'_i}=y_i$. In this case a *transformation of coordinates*, in the most general sense, consists in taking n new variables $\xi_1, \xi_2, \ldots \xi_n$, connected with the original coordinates $x_1, \ldots x_n$ by n equations, which may also involve t explicitly. It is a well-known theorem, that the transformation of the equations (80.) is effected merely by expressing W in terms of the new coordinates $\xi_1, \ldots \xi_n$, and their differential coefficients $\xi'_1, \ldots \xi'_n$, instead of the old; so that the new equations are

$$\left(\frac{dW}{d\xi'_i}\right)'=\frac{dW}{d\xi_i}: \quad \ldots \ldots \ldots \ldots \quad \text{(81.)}$$

the proof of this theorem does not depend upon the form of the function W; and we know also (see arts. 18 & 66.), that whatever be the form of W, these new equations may be again transformed to a system of the form (I.), by taking n additional

* This theorem is a generalization of Sir W. R. HAMILTON's transformation of the Dynamical Equations. See Part I. art. 18.

variables $\eta_1, \dots \eta_n$, defined by the equations

$$\eta_i = \frac{dW}{d\xi_i'}.$$

[It is to be observed that this last transformation is not, in general, equivalent to expressing the original Z in terms of the new variables; for the original Z is $-W + \Sigma(x_i' y_i)$ (art. 66.), and the analogous expression derived from (81.) is $-W + \Sigma(\xi_i' \eta_i)$, which is not, in general, equivalent to the former. It will be seen presently that the two expressions *are* equivalent when the equations connecting $x_1, \dots x_n$ with $\xi_1, \dots \xi_n$ do not involve t explicitly.]

68. Following the analogy of the dynamical equations, I shall adopt the following as the

Definition of a transformation of coordinates.

The original equations (I.), art. 64, having been changed into the form (80.), art. 66, let $\xi_1, \xi_2, \dots \xi_n$ be n new variables connected with the n variables $x_1, x_2, \dots x_n$ by n equations, which may also involve t explicitly; and let $\eta_1, \eta_2, \dots \eta_n$ be n other new variables defined by the equations $\frac{dW}{d\xi_i'} = \eta_i$ (where W has been expressed in terms of ξ_1, &c., ξ_1', &c.). By means of the $2n$ assumed relations, the $2n$ original variables $x_1, \dots x_n, y_1, \dots y_n$ can be expressed as functions of the $2n$ new variables $\xi_1, \dots \xi_n, \eta_1, \dots \eta_n$. Let this substitution be called a "*transformation of coordinates.*"

It has been seen in the preceding article that the original equations (I.) are changed by a transformation of coordinates into a system of the *same form*, which however cannot in general be obtained by merely expressing Z in terms of the new variables. But we are not at liberty to assume (and it is not generally true) that a change of the system (I.) into another of the same form is a *normal transformation* (art. 62.). It has already been stated, however (end of art. 65.), that this *is true in the present case*; a proposition which I proceed to establish.

69. *Every transformation of coordinates is a normal transformation.*

To prove this theorem, we have to show that every transformation of the kind described in the last article is also of the kind defined in art. 62; in other words, that it is possible to assign a function P, of $\xi_1, \dots \xi_n, y_1, \dots y_n$ (with or without t), such that the given relations between x_1, &c., ξ_1, &c., which define the transformation of coordinates, shall be equivalent to the system of equations

$$\frac{dP}{dy_i} = x_i, \quad \frac{dP}{d\xi_i} = \eta_i.$$

Take $$P = (x_1)y_1 + (x_2)y_2 + \dots + (x_n)y_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82.)$$

(the brackets indicating that $x_1, \dots x_n$ are to be expressed in terms of $\xi_1, \dots \xi_n$; so that P is a function of $\xi_1, \dots \xi_n, y_1, \dots y_n$, with or without t according as the equations connecting $x_1, \dots x_n$ with $\xi_1, \dots \xi_n$ do or do not contain t explicitly). Then P is the func-

and u, v, w the variables conjugate to them; so that, putting $\mathrm{T}=\frac{1}{2}m(\mathrm{x}'^2+\mathrm{y}'^2+\mathrm{z}'^2)$, we have $\mathrm{u}=\frac{d\mathrm{T}}{d\mathrm{x}'}$, &c.; whence $\mathrm{T}=\frac{1}{2m}(\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2)$, and the equations of motion are $\mathrm{x}'=\frac{d\mathrm{Z}}{d\mathrm{u}}$, $\mathrm{u}'=-\frac{d\mathrm{Z}}{d\mathrm{x}}$, &c., where $\mathrm{Z}=\frac{1}{2m}(\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2)-\mathrm{U}$, and U is a given function of x, y, z, with or without t.

Now let r, θ, ϕ be polar coordinates of P, so that

$$\mathrm{x}=r\sin\theta\cos\phi,\quad \mathrm{y}=r\sin\theta\sin\phi,\quad \mathrm{z}=r\cos\theta.$$

Let u, v, w be the variables conjugate to r, θ, ϕ. Then the ordinary process of transformation would be as follows:—

(1) to express x', y', z' in terms of r, θ, ϕ, r', θ', ϕ', and thus transform T into a function of the latter quantities;

(2) to define u, v, w by the equations

$$u=\frac{d\mathrm{T}}{dr'},\quad v=\frac{d\mathrm{T}}{d\theta'},\quad w=\frac{d\mathrm{T}}{d\phi'},$$

and by means of these relations to express r', θ', ϕ' in terms of u, v, w, r, θ, ϕ, so that x', y', z', and therefore, finally, T and Z, might be expressed as functions of the six new variables.

Instead of this, let us adopt the method indicated by the theorem at the beginning of this article.

We have then, for the modulus of transformation,

$$\mathrm{P}=(\mathrm{x})\mathrm{u}+(\mathrm{y})\mathrm{v}+(\mathrm{z})\mathrm{w},$$

in which x, y, z are to be expressed in terms of r, θ, ϕ; so that the proper form of P is

$$\mathrm{P}=\mathrm{u}r\sin\theta\cos\phi+\mathrm{v}r\sin\theta\sin\phi+\mathrm{w}r\cos\theta,$$

and the equations (corresponding to $\eta_i=\frac{d\mathrm{P}}{d\xi_i}$ (art. 69.)) which define the new variables u, v, w, are

$$u=\frac{d\mathrm{P}}{dr},\quad v=\frac{d\mathrm{P}}{d\theta},\quad w=\frac{d\mathrm{P}}{d\phi}.$$

These give

$$u=\sin\theta(\mathrm{u}\cos\phi+\mathrm{v}\sin\phi)+\mathrm{w}\cos\theta,$$

$$v=r\cos\theta(\mathrm{u}\cos\phi+\mathrm{v}\sin\phi)-r\mathrm{w}\sin\theta,$$

$$w=-r\sin\theta(\mathrm{u}\sin\phi-\mathrm{v}\cos\phi),$$

from which the values of u, v, w are easily obtained in terms of the six new variables. But in order to effect the transformation of T, we have only to square each side of these equations, and add them, after dividing the second by r^2 and the third by $(r\sin\theta)^2$; we thus obtain

$$\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2=u^2+\frac{v^2}{r^2}+\frac{w^2}{(r\sin\theta)^2},$$

and the transformed equations of motion are therefore

$$mr'=u,\ m\theta'=vr^{-2},\ m\varphi'=w(r\sin\theta)^{-2},$$

$$mu'=r^{-3}(v^2+w^2(\sin\theta)^{-2})+m\frac{dU}{dr},$$

$$mv'=(r\sin\theta)^{-2}w^2\cos\theta+m\frac{dU}{d\theta},\ \ w'=\frac{dU}{d\varphi}.$$

The preceding process is not, of course, given for the sake of the result, which may easily be verified directly, but in order to illustrate the meaning of the theorem on which it depends. It is hardly necessary to add, that if the problem involved the independent rectangular coordinates of any number of material points, the transformation to polar coordinates would be effected in the same way, by merely adding to P analogous terms for each point.

71. Before proceeding further there is an important remark to be made.

It has been hitherto assumed that the modulus of transformation P was a function of no other quantities than the $2n$ variables $\xi_1,\dots\xi_n,\ y_1,\dots y_n$ and t. But if every step of the demonstrations of the theorems of transformation which have been given (art. 62, &c.) be examined, it will be seen that they continue to hold good in the following more general form.

Take $$P=f(\xi_1,\ \xi_2,\dots\xi_n,\ y_1,\ y_2,\dots y_n,\ p,\ q,\ r,\dots,\ t),$$

where $p,\ q,\ r,\dots$ are any functions of *any or all the variables*, old and new, with or without t.

Let the equations connecting the old and new variables be, as before,

$$\frac{dP}{dy_i}=x_i,\ \frac{dP}{d\xi_i}=\eta_i,\ \ \dots\dots\dots\dots \quad (83.)$$

with the condition that p, q, r, ... *are exempt from differentiation in forming these equations.*

Then take $\Psi=-\left(\frac{d(P)}{dt}\right)$, with the following signification; (1) $\frac{d(P)}{dt}$ denotes the differential coefficient of P with respect to t, so far as t is contained explicitly, and also through the variables in $p,\ q,\ r,\dots$; that is to say,

$$\frac{d(P)}{dt}=\frac{dP}{dt}+\frac{dP}{dp}\cdot p'+\frac{dP}{dq}\cdot q'+\dots$$

(where $p'=\frac{dp}{dt}+\frac{dp}{dx_1}x_1'+$&c. &c., but this substitution *is not to be made* at this stage); (2) $\left(\frac{d(P)}{dt}\right)$ denotes the result of substituting in the above expression the values of $y_1,\dots y_n$ in terms of $\xi_1,\dots\xi_n,\ \eta_1,\dots\eta_n,\ p,\ q,\ r,\dots$ from (83.), so far as $\frac{d(P)}{dt}$ contains $y_1,\dots y_n$ explicitly (*i. e.* not involved in $p,\ q,\ r,\dots$).

Lastly, take $$\Phi=(Z)+\Psi,$$

where (Z) denotes the result of substituting in Z the values of the old variables as

given in terms of $\xi_1, \ldots \eta_1, \ldots p, q, r, \ldots$ from (82.). We shall then have

$$\xi'_i = \frac{d\Phi}{d\eta_i}, \quad \eta'_i = -\frac{d\Phi}{d\xi_i}, \quad \ldots \ldots \ldots \ldots \quad (84.)$$

where Φ is in general a function of

$$\xi_1, \ldots \xi_n, \eta_1, \ldots \eta_n, p, q, r, \ldots p', q', r', \ldots \text{ and } t;$$

and the differentiations with respect to ξ_i, η_i are performed only so far as those variables appear *explicitly* in Φ. But *after these differentiations*, we may introduce the actual values of $p, \ldots p', \ldots$ in terms of the variables and their differential coefficients. It is obvious that the original variables will not, *in general*, have been eliminated from the system (84.); but of course the elimination may be afterwards completed*. Similar considerations apply to that particular case of transformation which we have called a transformation of coordinates (art. 68.). We have then

$$P = (x_1)y_1 + \ldots + (x_n)y_n,$$

and the relations connecting $x_1 \ldots x_n$ with $\xi_1, \ldots \xi_n$, may contain $p, p, r, \ldots$; so that (x_1), &c. are functions of $p, q, r, \ldots$ as well as of $\xi_1, \ldots \xi_n$.

We might have deduced the preceding conclusions from the following simple consideration. Since $p, q, r, \ldots$ *are actually* functions of t, though *unknown* functions, we may imagine them to be *known*, and to be expressed explicitly in terms of t; and then the case resolves itself into that of art. 62, &c., so far as the demonstration is concerned. But as a doubt might possibly have arisen whether any fallacy was involved in the circumstance that $p, q, r, \ldots$ involve (when supposed to be expressed in terms of t) the *arbitrary constants of the problem itself*, it seemed best to refer to the original reasoning; the most important part of which is that contained (*mutatis mutandis*) in art. 6. (For the "mutanda" see the beginning of art. 63.) It is then apparent that this circumstance is perfectly immaterial with reference to the conclusions in question, though it may be important in other points of view.

72. This being premised, we will proceed to an example of transformation more interesting than the former, namely, the

Transformation from fixed to moving axes of coordinates.

Let x, y, z, u, v, w have the same signification as in art. 70, and let x, y, z, u, v, w be the new variables, where x, y, z are rectangular coordinates, referring to a system of moving axes of which the origin always coincides with that of the original fixed axes of x, y, z.

Let the direction cosines of the new (moving) axes with respect to the old be λ_0, μ_0, ν_0; λ_1, μ_1, ν_1; λ_2, μ_2, ν_2, thus†:

* The final equations in this case will not in general have the canonical form.

† I do not know who first used this convenient way of indicating the nine direction cosines by a diagram, but I first saw it in one of M. LAMÉ's works.

	x	y	z
x	λ_0	λ_1	λ_2
y	μ_0	μ_1	μ_2
z	ν_0	ν_1	ν_2

where λ_0, &c. are functions of t, which may be either given explicitly, or implicitly through the variables (see the last article).

The modulus of transformation P is found (art. 69.) by substituting for the variables x, y, z in the expression xu+yv+zw, their values in terms of x, y, z; we have therefore

$$\mathrm{P}=(\lambda_0 x+\lambda_1 y+\lambda_2 z)\mathrm{u}+(\mu_0 x+\mu_1 y+\mu_2 z)\mathrm{v}+(\nu_0 x+\nu_2 y+\nu_2 z)\mathrm{w};$$

and then the three equations

$$\frac{d\mathrm{P}}{dx}=u,\quad \frac{d\mathrm{P}}{dy}=v,\quad \frac{d\mathrm{P}}{dz}=w, \text{ give}$$

$$\left.\begin{aligned} u&=\lambda_0\mathrm{u}+\mu_0\mathrm{v}+\nu_0\mathrm{w}\\ v&=\lambda_1\mathrm{u}+\mu_1\mathrm{v}+\nu_1\mathrm{w}\\ w&=\lambda_2\mathrm{u}+\mu_2\mathrm{v}+\nu_2\mathrm{w}\end{aligned}\right\},\ \text{whence}\left\{\begin{aligned}\mathrm{u}&=\lambda_0 u+\lambda_1 v+\lambda_2 w\\ \mathrm{v}&=\mu_0 u+\mu_1 v+\mu_2 w\\ \mathrm{w}&=\nu_0 u+\nu_1 v+\nu_2 w\end{aligned}\right\}\quad\dots\dots\quad (85.)$$

Also we have (see Theorem VIII. art. 64.), since P is to be considered to contain t explicitly through λ_0, &c., only,

$$\frac{d\mathrm{P}}{dt}=(\lambda_0' x+\lambda_1' y+\lambda_2' z)\mathrm{u}+(\mu_0' x+\mu_1' y+\mu_2' z)\mathrm{v}+(\nu_0' x+\nu_1' y+\nu_2' z)\mathrm{w},$$

in which expression the values of u, v, w in terms of the new variables are to be substituted from (85.). Now if we put ω_0, ω_1, ω_2 for the angular velocities of the moving system of axes about the axes of x, y, z, respectively, so that

$$\omega_0=\lambda_2\lambda_1'+\mu_2\mu_1'+\nu_2\nu_1'=-(\lambda_1\lambda_2'+\mu_1\mu_2'+\nu_1\nu_2'),\ \&\text{c.},$$

it will be immediately seen that the usual relations between the nine direction cosines enable us to put the result of the substitution in the following form:

$$\left(\frac{d\mathrm{P}}{dt}\right)=\omega_0(yw-zv)+\omega_1(zu-xw)+\omega_2(xv-yu).$$

The original differential equations

$$\mathrm{x}'=\frac{d\mathrm{Z}}{d\mathrm{u}},\quad \mathrm{u}'=-\frac{d\mathrm{Z}}{d\mathrm{x}},\ \&\text{c}$$

are then transformed into

$$x'=\frac{d\Phi}{du},\quad u'=-\frac{d\Phi}{dx},\ \&\text{c. (art. 64.)},$$

where $$\Phi=(\mathrm{Z})-\left(\frac{d\mathrm{P}}{dt}\right).$$

Introducing the above value of $\left(\frac{dP}{dt}\right)$, and omitting the brackets, we obtain for the system of transformed equations,

$$\left.\begin{array}{ll} x'=\frac{dZ}{du}+\omega_2 y-\omega_1 z, & u'=-\frac{dZ}{dx}+\omega_2 v-\omega_1 w \\ y'=\frac{dZ}{dv}+\omega_0 z-\omega_2 x, & v'=-\frac{dZ}{dy}+\omega_0 w-\omega_2 u \\ z'=\frac{dZ}{dw}+\omega_1 x-\omega_0 y, & w'=-\frac{dZ}{dz}+\omega_1 u-\omega_0 v \end{array}\right\} \quad \text{(86.)}$$

in which Z is supposed to be expressed in terms of the new variables.

73. On the principles of the integration of this, and of transformed systems in general, I shall make some remarks hereafter. For the present, the following may be observed. If, in the transformation of the last article, we suppose the motion of the new axes *given*, then λ_0, &c., and therefore also $\omega_0, \omega_1, \omega_2$, are given explicit functions of t. But if the motion of the new axes is only given by connecting it with the motion of the point m itself, then the above quantities are given functions of the variables and their differential coefficients.

The most interesting case of the latter kind is that in which the motion of the new axes is assumed to satisfy the equations

$$\frac{\omega_0}{x}=\frac{\omega_1}{y}=\frac{\omega_2}{z}, \quad (\omega.)$$

which express the condition *that the instantaneous axis of rotation* (of the moving axes) *always coincides with the radius vector of the moving point* m*.

The radius vector traces, in fixed space, a certain conical surface. It also traces, with reference to the moving axes, another conical surface; and we might always assume as *one* of the conditions defining their motion, that this latter should be *any proposed surface*; that is, we might assume that the new coordinates x, y, z should always satisfy the equation $\varphi(x, y, z)=0$, φ representing any given homogeneous function. If to this last assumption we add the two conditions expressed by the formula (ω.), we further assume that *the conical surface traced by the radius vector with reference to the moving axes, rolls upon that traced in fixed space.*

Suppose, for example, we assume for the equation $\varphi(x, y, x)=0$, simply $z=0$. This, with the conditions (ω.), will express that the radius vector is always in the plane of xy, and that this plane rolls upon the conical surface traced by the radius vector in fixed space. We may then say that the plane of xy is the "plane of the orbit," and that the axes of xy, or any lines fixed with reference to them in their plane, are "fixed in the plane of the orbit†."

* See JACOBI's first letter to Professor HANSEN (CRELLE's Journal, vol. xlii. p. 21). This letter appears to refer to some unpublished (?) results of Professor HANSEN, which may possibly be similar to those of this article.

† The student of elementary treatises is, I believe, *always* left to find out for himself what this expression means, or ought to mean.

be performed before or after the substitution for $\mathrm{x}_{,}$ $\mathrm{y}_{,}$, $\mathrm{z}_{,}$, in terms of $x_{,}, y_{,}, z_{,}$. If then we put $z=0$ and

$$\mathrm{R}=m_{,}\left(\frac{1}{\delta}-\frac{xx_{,}+yy_{,}}{r_{,}^{3}}\right),$$

we shall have, from the fourth and fifth of the transformed differential equations*,

$$\left.\begin{aligned} x''&=-\frac{\mu x}{r^{3}}+\frac{d\mathrm{R}}{dx}\\ y''&=-\frac{\mu y}{r^{3}}+\frac{d\mathrm{R}}{dy}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87.)$$

from which it is evident, that, *assuming the motion of* $\mathrm{m}_{,}$ *to be known relatively to the new axes*, the variations of the four elements of the orbit of m which determine the dimensions of the orbit, its position relatively to lines *fixed in its own plane*, and the time of perihelion passage, will be expressed in terms of the differential coefficients of R in the same way as if the plane of the orbit were fixed. But the motion of the node of the orbit upon the fixed plane of xy, and its inclination to that plane, must be determined by means of the last of the differential equations, as follows: that equation gives

$$\omega_{1}u-\omega_{0}v=\left(\frac{d\mathrm{Z}}{dz}\right)(z=0);$$

or if we put $-\Omega$ for the term multiplied by $mm_{,}$ in the value of U given above,

$$\omega_{1}u-\omega_{0}v=\frac{d}{dz}\left(-\frac{m\mu}{r}+\Omega\right),$$

and z is to be put $=0$ after the differentiation, which reduces the above to

$$\omega_{1}u-\omega_{0}v=\left(\frac{d\Omega}{dz}\right).$$

Let $\omega_{0}^{2}+\omega_{1}^{2}=\alpha^{2}$, so that α is the angular velocity of the plane of the orbit about the radius vector; then (observing that $u=mx'$, &c.) we have

$$\frac{\omega_{0}}{x}=\frac{\omega_{1}}{y}=\frac{\alpha}{r}=\frac{v\omega_{0}-u\omega_{1}}{m(xy'-x'y)},$$

whence
$$v\omega_{0}-u\omega_{1}=\frac{m\alpha}{r}(xy'-x'y)$$

$$=\frac{m\alpha}{r}\sqrt{\mu a(1-e^{2})}\,\dagger,$$

and therefore
$$m\alpha=-\frac{r}{\sqrt{\mu a(1-e^{2})}}\left(\frac{d\Omega}{dz}\right),$$

which gives α in terms of the four elements referred to above, and of t. And if we put ι for the inclination of the orbit to the plane of xy, ν for the longitude of the node

* These equations (87.) have been obtained in a different way by Mr. BRONWIN. Camb. Math. Journ. vol. iv. p. 245.

† See below, art. 80.

referred to the axis of x, and β for the angle between the axis of x and the node, the usual formulæ of rotation give

$$\left.\begin{aligned} \iota' &= \omega_0 \cos\beta - \omega_1 \sin\beta \\ \nu' \sin\iota &= \omega_0 \sin\beta + \omega_1 \cos\beta \\ \beta' &= \omega_2 - \nu' \cos\iota \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (88.)$$

If in these expressions we put $\omega_0 = \frac{x}{r}\alpha$, $\omega_1 = \frac{y}{r}\alpha$, $\omega_2 = 0$, and call ϑ the angle between the radius vector and the node, so that $\omega_0 \cos\beta - \omega_1 \sin\beta = r\cos\vartheta$, $\omega_0 \sin\beta + \omega_1 \cos\beta = r\sin\vartheta$, we obtain finally

$$\iota' = \alpha\cos\vartheta, \quad \nu' = \alpha\frac{\sin\vartheta}{\sin\iota}, \quad \beta' = -\alpha\sin\vartheta\cot\iota,$$

in which the expression above given for α is to be substituted.

The actual value of $\left(\frac{d\Omega}{dz}\right)$ is $mm_{\prime}z_{\prime}\left(\frac{1}{r_{\prime}^3} - \frac{1}{\delta^3}\right)$, which (since $z_{\prime}$, $r_{\prime}$, &c. are supposed given in terms of t) may be expressed in terms of the four elements first mentioned, and t.

I propose to consider the transformation of the differential equations of the planetary theory in a more general manner in the following section. At present I shall add some remarks on normal transformations in general.

74. *Theorem.* If

$$x_i' = \frac{dZ}{dy_i}, \quad y_i' = -\frac{dZ}{dx_i} \quad \ldots \ldots \ldots \ldots \quad (89.)$$

be a system of $2n$ simultaneous differential equations, where

$$Z = f(x_1, y_1, x_2, y_2, \ldots p, q, r, \ldots, t),$$

and $p, q, r, \ldots$ are also explicit functions of x_1, &c., y_1, &c. and t, but are *exempt from differentiation* in taking the differential coefficients $\frac{dZ}{dx_i}$, $\frac{dZ}{dy_i}$; and if these equations be transformed by a normal substitution of new variables ξ_1, &c., η_1, &c. (art. 62, equation (73.)), then the transformed equations are, as in art. 64,

$$\xi_i' = \frac{d\Psi}{d\eta_i} + \frac{dZ}{d\eta_i}, \quad \eta_i' = -\frac{d\Psi}{d\xi_i} - \frac{dZ}{d\xi_i},$$

in which Z is expressed in terms of ξ_1, &c., η_1, &c., but the differentiations with respect to ξ_i, η_i are performed *before* the substitution of these variables in p, q, r, &c.; in other words, $p, q, r, \ldots$ are still to be exempt from differentiation in forming the differential equations.

This may be proved simply by repeating the reasoning of art. 64. The only difference is, that in the term

$$\Sigma_j\left\{\frac{dZ}{dx_j}\frac{dx_j}{d\eta_i} + \frac{dZ}{dy_j}\frac{dy_j}{d\eta_i}\right\}$$

the differential coefficients $\frac{dZ}{dx_j}$, $\frac{dZ}{dy_j}$ are now taken only so far as Z contains x_j, y_j inde-

pendently of p, q, r, &c.; and therefore the term represents the differential coefficient $\frac{dZ}{d\eta_i}$ taken so far as Z contains η_i independently of p, q, r, &c. The same reasoning applies to the corresponding term in the value of η'_i. The theorem is thus established.

It is evident that it may be combined with that given in art. 71, where other functions analogous to $p, q, r, \ldots$ are introduced by the modulus of transformation P.

If we call the form of the system of differential equations (89.) *canonical* when the differentiations of Z with respect to x_1, &c., y_1, &c. are *total*, we might call it *pseudo-canonical* when Z contains functions of x_1, &c., y_1, which are exempt from differentiation in forming the differential equations.

In like manner, if we call a transformation of variables *normal*, when the differentiations of the modulus P (equations (73.), art. 62.) with respect to ξ_1, &c., y_1, &c. are total (as in art. 62.), we might call the transformation *pseudo-normal* when P contains functions of the variables which are exempt from differentiation in forming the equations of transformation (as in art. 71.).

Adopting these designations, we may enunciate the following general theorem of transformation:—

Theorem X.—If a *pseudo-canonical* system be transformed by a *normal* or *pseudo-normal* substitution, the transformed equations are also *pseudo-canonical*, and may be formed by the rules applying to normal transformations of canonical systems, provided that the functions which are originally exempt from differentiation with respect to the variables, be continued exempt to the end of the process; but if such functions occur in the modulus of transformation P, they are subject to total differentiation *with respect to* t in forming the term $\frac{dP}{dt}$. (See art. 71.)

[With respect to this theorem there is one important remark to be made. If u, v be any two functions of x_1, &c., y_1, &c. (with or without $p, q, r, \ldots$ and t), the equation

$$\Sigma_i\left(\frac{du}{dy_i}\frac{dv}{dx_i}-\frac{du}{dx_i}\frac{dv}{dy_i}\right)=\Sigma_i\left(\frac{du}{d\eta_i}\frac{dv}{d\xi_i}-\frac{du}{d\xi_i}\frac{dv}{d\eta_i}\right) \text{ (art. 63.)}$$

is now only true on condition that the substitution of the actual values of $p, q, r, \ldots$ in terms of the variables be not performed till after all the differentiations.]

75. The theory of the variation of elements affords an interesting example of th theorem given in the last article. Consider the following system of differenti equations,

$$x'_i=\frac{dZ}{dy_i}+\frac{d\Omega}{dy_i},\quad y'_i=-\frac{dZ}{dx_i}-\frac{d\Omega}{dx_i}, \quad \ldots\ldots\ldots \quad (9($$

where in $\frac{dZ}{dx_i}, \frac{dZ}{dy_i}$ the differentiations are *total*, but Ω is supposed to contain functic of x_1, &c., y_1, &c., which are exempt from differentiation in forming the above eq tions. The system

$$x'_i=\frac{dZ}{dy_i},\quad y'_i=-\frac{dZ}{dx_i}$$

is *canonical.* Let us assume then that a complete set of normal integrals $a_1, \dots a_n$, $b_1, \dots b_n$ of this latter system is known, so that we have

$$a_i = \phi_i(x_1, \&c., y_1, \&c., t), \quad b_i = \chi_i(x_1, \&c., y_1, \&c., t).$$

The assumption of these last equations to represent the solution of the complete system (90.) is simply a *transformation of variables,* (a_1, &c., b_1, &c. being the new variables); it is also a *normal transformation,* since the equations connecting the new and old variables may be put in the form (see Theorem VII. art. 49, and art. 62.)

$$\frac{dX}{dx_i} = y_i, \quad \frac{dX}{da_i} = b_i,$$

where X (the modulus of transformation) is a function of $x_1, \dots x_n, a_1, \dots a_n, t$. The function Ψ of art. 62 is now obtained by expressing $-\frac{dX}{dt}$ in terms of $a_1, \dots b_1, \dots$; but since Z is $-\frac{dX}{dt}$ expressed in terms of x_1, &c., y_1, &c., it follows that when Z is expressed in terms of the new variables a_1, &c., b_1, &c., it becomes identical with Ψ. Now if the process of art. 63 be followed, *mutatis mutandis,* it will be seen that in the present case we obtain

$$a_i' = \frac{d\Psi}{db_i} - \frac{dZ}{db_i} - \Sigma_j\left(\frac{d\Omega}{dy_j}\frac{dy_j}{db_i}\right),$$

in which expression the first two terms destroy one another, and the remaining term is evidently the differential coefficient of Ω with respect to b_i, taken so far as Ω contains b_i *independently of those functions which were exempt from differentiation* in forming the original differential equations (90.). Similar reasoning applies to the expression for b_i'.

As this result will be useful, I shall enunciate it separately as

Theorem XI.—If the original system of differential equations be formed by treating certain functions, $p, q, r, \dots$, contained in the disturbing function Ω, as exempt from differentiation with respect to x_1, &c., y_1, &c., the equations which determine the variations of any set of normal elements a_1, &c., b_1, &c.

$$a_i' = -\frac{d\Omega}{db_i}, \quad b_i' = \frac{d\Omega}{da_i}$$

on condition that $p, q, r, \dots$ be treated, in forming these equations, as exempt from differentiation with respect to a_1, &c., b_1, &c.

[It is important to recollect, that *after these equations are formed,* p, q, r, &c. are to be expressed in terms of a_1, &c., b_1, &c., and in the integration of the system a_1, &c., b_1, &c. are to be treated indiscriminately as variables, whether they originally entered through $p, q, r, \dots$ or not].

The Theorem XI. may also be immediately obtained from the general equations (E.) of art. 52 (in which it is to be remembered that Z *includes* the disturbing

(where Q is expressed in terms of the new variables, and ω_0, ω_1, ω_2 (the angular velocities of the moving system of axes about the three moving axes themselves) are marked with the horizontal line to show that these quantities are exempt from differentiation in forming the following system of differential equations, though they may be functions of the variables); then the system (91.) is transformed into

$$x'_{(i)}=\frac{dZ}{du_{(i)}}+\frac{d\Omega}{du_{(i)}},\quad u'_{(i)}=-\frac{dZ}{dx_{(i)}}-\frac{d\Omega}{dx_{(i)}}, \quad \ldots \ldots \quad (92.)$$

with similar equations for y'_i, &c.

In these equations Z is to be expressed in terms of the new variables; and it is evident from the original form of Z and Q, that when so expressed, these two quantities are the same functions of the new variables that they were of the old, and involve (see art. 74.) the quantities exempt from differentiation in the same way*. Thus the transformed system (92.) contains no terms *explicitly depending upon the motion of the axes*, except those introduced by the three terms multiplied by ω_0, ω_1, ω_2 in the value of Ω given above; and the addition of these terms constitutes the only difference between the form of the old and of the new system.

79. We may now apply the method of the variation of elements to the system (92.) as follows:—

The system obtained by omitting the disturbing function Ω, namely,

$$\left.\begin{array}{l} x'_{(i)}=\dfrac{dZ}{du_{(i)}},\quad y'_{(i)}=\dfrac{dZ}{dv_{(i)}},\quad z'_{(i)}=\dfrac{dZ}{dw_{(i)}} \\ u'_{(i)}+\dfrac{dZ}{dx_{(i)}}=0,\quad v'_{(i)}+\dfrac{dZ}{dy_{(i)}}=0,\quad w'_{(i)}+\dfrac{dZ}{dz_{(i)}}=0 \end{array}\right\} \quad \ldots \ldots \quad (93.)$$

is *canonical*, and consists simply of the aggregate of the equations representing, for each planet, undisturbed elliptic motion about the sun† (relatively to the new axes of coordinates).

The integrals of these equations may therefore be expressed in any of the usual forms. We will suppose that the elements chosen are

$$a,\ e,\ \varpi,\ (\epsilon),\ \iota,\ \nu,$$

with significations corresponding to those given to the same symbols in art. 55. These letters unaccented will apply to the planet m, and $a_{,}$, $e_{,}$, &c., $a_{,,}$, $e_{,,}$, &c., $a_{(i)}$, $e_{(i)}$, &c., to the planets $m_{,}$, $m_{,,}$, $m_{(i)}$, &c.

The *definitions* of the elements a, e, ϖ, &c. are their expressions in terms of the six

* Since the direction cosines λ_0, &c. are exempt from differentiation in forming the equations connecting the old and new variables from the modulus P, they continue exempt throughout. (Theorem X. art. 74.) Hence we have
$$\mathrm{x}\bar{\mathrm{x}}_{,}+\mathrm{y}\bar{\mathrm{y}}_{,}+\mathrm{z}\bar{\mathrm{z}}_{,}=(\bar\lambda_0 x+\bar\lambda_1 y+\bar\lambda_2 z)(\overline{\lambda_0 x_{,}+\lambda_1 y_{,}+\lambda_2 z_{,}})+\&c.=x\bar{x}_{,}+y\bar{y}_{,}+z\bar{z}_{,},$$
and similarly for the rest.

† $Z=\Sigma\left(\frac{u^2+v^2+w^2}{2m}-\frac{\mu m}{r}\right)$, the summation extending to all the planets.

The complete variations of the elements are then found by adding to the terms just written the expressions given in art. 25.

It is easily seen that the expressions (95.) might have been deduced from geometrical considerations alone, if we had been at liberty to assume beforehand that the *mechanical* and *geometrical* parts of the variations might be calculated separately; the former as if the axes were at rest, and the latter as if there were no disturbing forces. It would not, I believe, be difficult to establish by *à priori* and simple reasoning the validity of such an assumption, and then the above results would only serve as a verification of the method which has been employed to obtain them.

80. In order however that no obscurity may rest upon the interpretation of the formulæ obtained in the last articles, it is necessary to consider the physical (or rather geometrical) meaning of the elements a, e, &c., which we have so far only defined by means of their expressions in terms of the variables x, y, z, u, &c., and to ascertain what relation they bear to the elements similarly defined by means of the original variables x, y, &c., which refer to axes whose directions are invariable.

The relations between the variables ((85.), art. 72.) give immediately

$$x^2+y^2+z^2=\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2$$

$$u^2+v^2+w^2=\mathrm{u}^2+\mathrm{v}^2+\mathrm{w}^2$$

$$ux+vy+zw=\mathrm{ux}+\mathrm{vy}+\mathrm{zw};$$

and if we put
$$yw-zv=A,\quad zu-xw=B,\quad xv-yu=C$$

$$\mathrm{yw}-\mathrm{zv}=\mathrm{A},\quad \mathrm{zu}-\mathrm{xw}=\mathrm{B},\quad \mathrm{xv}-\mathrm{yu}=\mathrm{C},$$

we find, by virtue of the relations $\mu_1\nu_2-\nu_1\mu_2=\lambda_0$, &c., the following equations:

$$A=\lambda_0\mathrm{A}+\mu_0\mathrm{B}+\nu_0\mathrm{C}$$

$$B=\lambda_1\mathrm{A}+\mu_1\mathrm{B}+\nu_1\mathrm{C}$$

$$C=\lambda_2\mathrm{A}+\mu_2\mathrm{B}+\nu_2\mathrm{C}$$

and
$$A^2+B^2+C^2=\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2.$$

Now $\mathrm{A}(=\mathrm{yw}-\mathrm{zv}=m(\mathrm{yz}'-\mathrm{zy}'))$ is the projection on the plane of yz of the areal velocity of m (relative to fixed space) multiplied by the mass, and B, C have analogous meanings; hence it is evident from the above equations that A, B, C are the projections on the three moving coordinate planes of yz, zx, xy of the absolute areal velocity of m relative to fixed space, multiplied by the mass. (The projections of the areal velocity *relative to the moving axes* would be $yz'-zy'$, &c., which are not proportional to $yw-zv$, &c., since u, v, w are not the same as mx', my', mz', except on a particular hypothesis as to the motion of the axes. See art. 73.).

Inasmuch as the definitions of the elements a, e, ι involve the variables only in the forms $x^2+y^2+z^2$, $u^2+v^2+w^2$, A, B, C, it follows that these three elements are respectively the *semiaxes, excentricity, and inclination to the plane of xy, of the absolute osculating ellipse of the orbit in fixed space.* Thus the instantaneous ellipse, relatively to the moving axes, is of the same dimensions and in the same plane as the true

The definitions of all the elements (relative to the moving axes) in terms of the six new variables x, y, z, u, v, w, have the same form as those of the corresponding elements (relative to the fixed axes) in terms of x, y, z, u, v, w. The two relative elements a, e are the same as the corresponding *absolute* elements; ι is the inclination of the plane of the ellipse to the moving plane of xy, and ν the longitude of the node reckoned from the axis of x; and since the place of the body in the ellipse is evidently the same, the relations between the remaining elements ϖ and (ϵ) (or ϵ) and the corresponding *absolute elements* are purely geometrical.

Comparing these results with those of art. 79, we see that the independence of the formulæ for the mechanical and geometrical variations of the elements of the true osculating ellipse is completely established.

83. In all that precedes, the three variables ω_0, ω_1, ω_2 (the angular velocities of the system of moving axes about the axes themselves) are entirely arbitrary; they may be either explicit functions of t, involving only determinate constants, or they may depend in any way upon the relative or absolute elements of the orbits of any or all the planets, and their differential coefficients with respect to t. In the case in which the expressions for $\omega_0, \omega_1, \omega_2$ involve only the *relative* elements, when these expressions are introduced in the formulæ (95.), art. 79, and these formulæ completed by the addition of the terms in art. 55, and when the corresponding sets of equations are formed for each planet, we obtain a set of simultaneous differential equations involving all the elements of all the orbits, and their differential coefficients with respect to t. The integration of these equations would determine all the elements as functions of t, and thus the motion of all the planets, relatively to the axes of coordinates, would be known. Lastly, the motion of the whole system, relatively to fixed space, would be found by integrating the system of equations

$$\left.\begin{aligned} \mathrm{L}' &= \omega_0 \cos \mathrm{X} - \omega_1 \sin \mathrm{X} \\ \mathrm{N}' \sin \mathrm{L} &= \omega_0 \sin \mathrm{X} + \omega_1 \cos \mathrm{X} \\ \mathrm{X}' &= \omega_2 - \mathrm{N}' \cos \mathrm{L} \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (96.)$$

where ω_0, ω_1, ω_2 are now given functions of t, and L is the inclination of the plane of xy to that of xy, N is the longitude of the ascending* node of the plane of xy, reckoned from the axis of x, and X is the longitude of the axis of x, reckoned upon the plane of xy, *from the node*, in the direction of positive rotation.

In the case in which $\omega_0, \omega_1, \omega_2$ cannot be expressed in terms of the *relative* elements, the integrations which determine the relative motion of the system cannot be separated from those which determine the position of the axes in fixed space; but the equations (96.) must be considered simultaneously with the other differential equations of the problem.

* *Ascending* relatively to a positive rotation, *i. e.* from x to y.

with similar equations for $\Omega_{,}$. In what follows, I shall, as an abridgment, employ the symbol E to denote the operation $\frac{d}{d\epsilon}+\frac{d}{d\varpi}$, and in like manner I shall put $E_{,}$ for $\frac{d}{d\epsilon_{,}}+\frac{d}{d\varpi_{,}}$, so that

$$E\Omega=\frac{d\Omega}{d\epsilon}+\frac{d\Omega}{d\varpi}, \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99.)$$

Since r and θ, when expressed in terms of the elements and t, do not contain $\iota, \iota_{,}, \nu, \nu_{,}$, we have

$$\frac{d\Omega}{d\nu}=\frac{d\Omega}{d\cos\chi}\cdot\frac{d\cos\chi}{d\nu}, \quad \frac{d\Omega}{d\iota}=\frac{d\Omega}{d\cos\chi}\cdot\frac{d\cos\chi}{d\iota}$$

and

$$\frac{d\cos\chi}{d\nu}=-\left(1+\frac{dv}{d\nu}\right)\frac{d\cos\chi}{d\theta}-\frac{dv_{,}}{d\nu}\,\frac{d\cos\chi}{d\theta_{,}}$$
$$+\sin\iota\sin\iota_{,}\sin(\theta-v-\nu)\sin(\theta_{,}-v_{,}-\nu_{,})\sin(\nu_{,}-\nu),$$

in which expression the values of $\frac{dv}{d\nu}$, $\frac{dv_{,}}{d\nu_{,}}$ are to be obtained by differentiating the equations (98.). In like manner we find

$$\frac{d\cos\chi}{d\iota}=-\frac{dv}{d\iota}\,\frac{d\cos\chi}{d\theta}-\frac{dv_{,}}{d\iota}\,\frac{d\cos\chi}{d\theta_{,}}$$
$$+\sin(\theta-v-\nu)\sin(\theta_{,}-v_{,}-\nu_{,})\big(-\sin\iota\cos\iota_{,}+\cos\iota\sin\iota_{,}\cos(\nu_{,}-\nu)\big).$$

Analogous expressions may be found for $\frac{d\Omega_{,}}{d\nu_{,}}$, $\frac{d\Omega_{,}}{d\iota_{,}}$.

85. Hitherto we have assumed nothing concerning the motion of the axes of coordinates. Let us now however take as a first assumption that *the plane of xy shall always pass through the line of nodes.* This implies the conditions*

$$\nu_{,}=\nu, \quad v=0, \quad v_{,}=0,$$

and consequently $I=\iota_{,}-\iota$.

Introducing these conditions in the expressions at the end of the last article, and

* The legitimacy of the following processes will be apparent to the reader who shall have followed the general reasoning of former articles, though probably not to others. In either case it may be useful here briefly to recapitulate the principles now to be applied. The results of art. 79, and in particular the expressions (94.), (95.), were established independently of any assumption as to the values of ω_0, ω_1, ω_2, which are perfectly arbitrary. We are therefore at liberty to assume that ω_0, ω_1, ω_2 are such functions of t that any three functions of the variables shall constantly $=0$. Thus the first assumption made in the text is that $\nu_{,}-\nu$ shall constantly $=0$. But since such assumptions may cause (as in this case) certain elements to disappear from th expression of Ω, it is necessary to perform the differentiations of Ω with respect to such elements *first*; the di ferential coefficients may or may not vanish, on afterwards introducing the assumptions; if not, we find e pressions for them in terms of differential coefficients with respect to *other elements which do not disappear*; that instead of differentiations which ought to be performed *before* the assumed conditions are introduced, have finally only such as may be performed *afterwards*. The expression "disappear" does not, it mus observed, necessarily mean the same as "vanish." Thus the condition $\nu_{,}-\nu=0$, causes ν and $\nu_{,}$ both to *appear* from Ω, but does not of course imply that they both *vanish*.

The variations of the elements will now be found by introducing the above values of

$$\frac{d\Omega}{d\nu},\quad \frac{d\Omega_{\prime}}{d\nu_{\prime}},\quad \frac{d\Omega}{d\iota},\quad \frac{d\Omega_{\prime}}{d\iota_{\prime}}$$

in the expressions given in art. 55, and completing these expressions by the addition of the terms (95.), art. 79. But the angular velocities ω_0, ω_1, ω_2 are no longer wholly arbitrary, since we have made *one* assumption concerning the motion of the axes, which implies *one* relation between these quantities and the elements. In order to determine ω_0, ω_1, ω_2 completely, it will be necessary to make two more assumptions; but first we will investigate the relation already implied.

86. The complete expressions for ν', $\nu'_{\prime}$, obtained in the way mentioned in the last article, from arts. 55 and 79, may be written in the following form: put for brevity

$$m\sqrt{\mu a(1-e^2)}=p,\quad m_{\prime}\sqrt{\mu_{\prime}a_{\prime}(1-e_{\prime}^2)}=p_{\prime},$$

and put $\mu^{\frac{1}{2}}a^{-\frac{3}{2}}$ for n, and $\mu_{\prime}^{\frac{1}{2}}a_{\prime}^{-\frac{3}{2}}$ for $n_{\prime}$; then

$$\left.\begin{aligned}\nu'&=\frac{1}{p\sin\iota}\frac{d\Omega}{d\iota}+\cot\iota.(\omega_0\sin\nu-\omega_1\cos\nu)-\omega_2\\ \nu'_{\prime}&=\frac{1}{p_{\prime}\sin\iota_{\prime}}\frac{d\Omega_{\prime}}{d\iota_{\prime}}+\cot\iota_{\prime}.(\omega_0\sin\nu_{\prime}-\omega_1\cos\nu_{\prime})-\omega_2\end{aligned}\right\}\quad\ldots\ldots\quad(102.)$$

and if the latter equation be subtracted from the former, and the conditions

$$\nu_{\prime}=\nu,\quad \frac{d\Omega}{d\iota}=-\frac{d\Omega}{d\mathrm{I}},\quad \frac{d\Omega_{\prime}}{d\iota_{\prime}}=\frac{d\Omega_{\prime}}{d\mathrm{I}},\quad \iota_{\prime}-\iota=\mathrm{I}$$

be introduced, the result is easily found to be

$$(\omega_0\sin\nu-\omega_1\cos\nu)\sin\mathrm{I}=\frac{\sin\iota}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}+\frac{\sin\iota_{\prime}}{p}\frac{d\Omega}{d\mathrm{I}}.\quad\ldots\ldots\quad(103.)$$

This is the relation between ω_0, ω_1 and the elements and t, implied by the one assumed condition that the plane of xy passes through the line of nodes. The angular velocity ω_2 of the system of axes about the axis of z, is so far left, as it evidently ought to be, perfectly arbitrary.

Ω and $\Omega_{\prime}$ are now functions of the following elements* *only*:—

$$a,\ e,\ (\epsilon),\ \varpi,\ a_{\prime},\ e_{\prime},\ (\epsilon_{\prime}),\ \varpi_{\prime},\ \mathrm{I},\ \nu.$$

And we now have

$$\cos\chi=\cos(\theta-\nu)\cos(\theta_{\prime}-\nu)+\sin(\theta-\nu)\sin(\theta_{\prime}-\nu)\cos\mathrm{I}.$$

87. The complete expression for ι', derived from arts. 55 and 79, is easily put in the form

$$\iota'=-\frac{1}{p\sin\iota}\left\{\frac{d\Omega}{d\nu}+(1-\cos\iota)\mathrm{E}\Omega\right\}-(\omega_0\cos\nu+\omega_1\sin\nu);$$

and, on introducing the assumptions of the preceding articles, this will be found to become, after simple reductions,

$$p\sin\mathrm{I}.\iota'=-(\cos\mathrm{I}.\mathrm{E}\Omega+\mathrm{E}_{\prime}\Omega)-(\omega_0\cos\nu+\omega_1\sin\nu)\,p\sin\mathrm{I};$$

* On the difference between (ϵ) and ϵ see above, art. 55. We cannot strictly call Ω a function of a and ϵ.

and the required condition of a maximum will evidently be

$$p \sin \iota + p_{\prime} \sin \iota_{\prime} = 0\,* ; \qquad (107.)$$

adding the squares of these expressions, we obtain

$$\sigma^2 = p^2 + p_{\prime}^2 + 2pp_{\prime} \cos \mathrm{I}, \qquad (108.)$$

which determines the actual value of σ; moreover we have

$$-\frac{\sin \iota}{p_{\prime}} = \frac{\sin \iota_{\prime}}{p} = \frac{\sin \mathrm{I}}{\sigma} ; \qquad (109.)$$

and it is easy to find

$$\left. \begin{aligned} \sigma \cos \iota &= p + p_{\prime} \cos \mathrm{I} \\ \sigma \cos \iota_{\prime} &= p_{\prime} + p \cos \mathrm{I} \end{aligned} \right\} . \qquad (110.)$$

so that σ, $\sin \iota$, $\sin \iota_{\prime}$, $\cos \iota$, $\cos \iota_{\prime}$ are all simply expressible in terms of p, $p_{\prime}$ and I. The variation of σ is easily found by means of the equation (105.), which gives

$$\sigma \sigma' = (p_{\prime} \mathrm{E}_{\prime} - p \mathrm{E})(\Omega_{\prime} - \Omega). \qquad (111.)$$

The equation (103.) now gives (see (109.))

$$\sigma(\omega_1 \cos \nu - \omega_0 \sin \nu) = \frac{d}{d\mathrm{I}}(\Omega_{\prime} - \Omega) ; \qquad (112.)$$

and from (106.) we obtain

$$\sigma^2 \sin \mathrm{I}\, (\omega_0 \cos \nu + \omega_1 \sin \nu) = \{(p + p_{\prime} \cos \mathrm{I}) \mathrm{E}_{\prime} + (p_{\prime} + p \cos \mathrm{I}) \mathrm{E}\} (\Omega_{\prime} - \Omega). \quad (113.)$$

The two last equations determine the motion of the principal plane in space, irrespectively of any arbitrary *sliding* motion which we may attribute to it in its own plane. For they give the angular velocities with which it is at any instant moving about two lines at right angles to one another in its own plane (see the end of art. 87.). They may be put in another form as follows:—the actual value of $\Omega_{\prime} - \Omega$ is

$$mm_{\prime}\left(\frac{r}{r_{\prime}^2} - \frac{r_{\prime}}{r^2}\right) \cos \chi,$$

where $\cos \chi$ has the value given above (end of art. 86.); and if the operations indicated be actually performed, observing that $\mathrm{E}r = 0$, $\mathrm{E}_{\prime}r_{\prime} = 0$, &c., and that

$$\mathrm{E} \cos \chi = \frac{d \cos \chi}{d\theta}, \text{ \&c.},$$

the results will be found to be

$$\sigma(\omega_1 \cos \nu - \omega_0 \sin \nu) = mm_{\prime} \sin \mathrm{I} . \left(\frac{r_{\prime}}{r^2} - \frac{r}{r_{\prime}^2}\right) \sin(\theta - \nu) \sin(\theta_{\prime} - \nu)$$

$$\sigma^2(\omega_0 \cos \nu + \omega_1 \sin \nu) = mm_{\prime} \sin \mathrm{I} . \left(\frac{r_{\prime}}{r^2} - \frac{r}{r_{\prime}^2}\right) \times (p \cos(\theta - \nu) \sin(\theta_{\prime} - \nu) + p_{\prime} \sin(\theta - \nu) \cos(\theta_{\prime} - \nu)).$$

Here $\theta - \nu$, $\theta_{\prime} - \nu$ represent, it will be remembered, the angular distances of the planets from the line of nodes.

We will assume for the present the condition $\omega_2 = 0$, so that the plane of xy may have no sliding motion, but *roll* upon the conical surface to which it is always a tan-

* Referring to the arrangement supposed in the diagram, it will be seen that ι becomes *negative* in the case now considered.

(where δ is the distance between m and $m_{,}$); and in like manner

$$\mathrm{A}'_{,}=-mm_{,}(\mathrm{yz}_{,}-\mathrm{zy}_{,})(\delta^{-3}-\mathrm{r}_{,}^{-3});$$

we have therefore $\qquad \mathrm{A}'+\mathrm{A}'_{,}=mm_{,}(\mathrm{yz}_{,}-\mathrm{zy}_{,})(\mathrm{r}_{,}^{-3}-\mathrm{r}^{-3}),$

with similar expressions for $\mathrm{B}'+\mathrm{B}'_{,}$, $\mathrm{C}'+\mathrm{C}'_{,}$; so that, when the common factor $mm_{,}(\mathrm{r}_{,}^{-3}-\mathrm{r}^{-3})$ is omitted, the equation (115.) becomes

$$(\mathrm{yz}_{,}-\mathrm{zy}_{,})\xi+(\mathrm{zx}_{,}-\mathrm{xz}_{,})\eta+(\mathrm{xy}_{,}-\mathrm{yx}_{,})\zeta=0,$$

which is evidently the equation to the plane containing the two radii vectores. Thus the theorem in question is verified.

90. To return from this digression: the motion of the line of nodes *in the principal plane* will be given by putting $\omega_{2}=0$ in either of the equations (102.), and introducing the value of $\omega_{0}\sin\nu-\omega_{1}\cos\nu$ from (103.). In this way we find, after slight reductions,

$$pp_{,}\sin\mathrm{I}.\nu'=p_{,}\cos\iota_{,}\frac{d\Omega}{d\mathrm{I}}+p\cos\iota\frac{d\Omega_{,}}{d\mathrm{I}},$$

in which we may substitute for $\cos\iota$, $\cos\iota_{,}$, the values given by equations (110.); this gives

$$\sigma pp_{,}\sin\mathrm{I}.\nu'=(p_{,}^{2}+pp_{,}\cos\mathrm{I})\frac{d\Omega}{d\mathrm{I}}+(p^{2}+pp_{,}\cos\mathrm{I})\frac{d\Omega_{,}}{d\mathrm{I}};$$

or, if we introduce the actual values of $\frac{d\Omega}{d\mathrm{I}}$, $\frac{d\Omega_{,}}{d\mathrm{I}}$, we find

$$pp_{,}\nu'=-mm_{,}rr_{,}\sin(\theta-\nu)\sin(\theta_{,}-\nu)\times\{p\cos\iota.(\delta^{-3}-r^{-3})+p_{,}\cos\iota_{,}(\delta^{-3}-r_{,}^{-3})\}.$$

It is not my purpose however to enter further into details; and I shall conclude this subject by briefly examining the consequences of a slightly different assumption as to the motion of the axes of coordinates. I shall suppose, namely, that the plane of xy still always coincides with the principal plane, but has a *sliding* motion such that the axis of x *always coincides with the line of nodes.*

91. The assumption made at the end of the last article implies the condition $\nu=0$; and ω_{2} will no longer be 0, but must be determined by equations (102.); either of these gives (putting $\nu'=\nu'_{,}=0$, and reducing by means of (103.), (109.), &c.)

$$pp_{,}\sin\mathrm{I}.\omega_{2}=p_{,}\cos\iota_{,}\frac{d\Omega}{d\mathrm{I}}+p\cos\iota\frac{d\Omega_{,}}{d\mathrm{I}}, \qquad (116.)$$

which coincides, as of course it ought, with the expression given for ν' on the former hypothesis (art. 90.). The difference is that Ω, $\Omega_{,}$ now no longer contain ν.

The values of ω_{0}, ω_{1} are obtained at once by putting $\nu=0$ in the equations (112.), (113.); and all the conclusions which were derived independently of any supposition as to the value of ω_{2}, subsist as before, when modified by putting $\nu=0$.

We may add one more equation, which is required in forming some of the expressions for the variations of the elements; namely,

$$\tan\frac{\iota}{2}=\frac{-p_{,}\sin\mathrm{I}}{\sigma+p+p_{,}\cos\mathrm{I}}. \qquad (117.)$$

$p, p_{\prime}, \sigma$ are defined by the equations

$$p=m\sqrt{\mu a(1-e^2)},\quad p_{\prime}=m_{\prime}\sqrt{\mu_{\prime}a_{\prime}(1-e_{\prime}^2)},\quad \sigma^2=p^2+p_{\prime}^2+2pp_{\prime}\cos \mathrm{I}.$$

$\iota, \iota_{\prime}$ are the angles between the *principal plane* and the planes of the two orbits, and are given functions of the elements $a, a_{\prime}, e, e_{\prime}, \mathrm{I}$, by virtue of the equations

$$-p\sigma \sin\iota=p_{\prime}\sigma \sin\iota_{\prime}=pp_{\prime}\sin \mathrm{I},$$

from which we have also

$$\sigma\cos\iota=p+p_{\prime}\cos \mathrm{I},\quad \sigma\cos\iota_{\prime}=p_{\prime}+p\cos \mathrm{I}$$

(for the values of $\tan\frac{\iota}{2}$, $\tan\frac{\iota_{\prime}}{2}$, see art. 91.).

ω_0 is the angular velocity of the principal plane about the line of nodes;

ω_1 the angular velocity of the principal plane about a line in itself perpendicular to the line of nodes;

ω_2 the angular velocity of the line of nodes estimated in the direction of the principal plane.

Differential Equations of the Problem.

93. The nine *intrinsic elements*, as we may perhaps appropriately call them, namely,

$$a,\ a_{\prime},\ e,\ e_{\prime},\ \epsilon,\ \epsilon_{\prime},\ \varpi,\ \varpi_{\prime},\ \mathrm{I},$$

are determined as functions of t by the following system of nine simultaneous differential equations of the first order:

$$m\mu a'=2na^2\frac{d\Omega}{d\epsilon},\quad m_{\prime}\mu_{\prime}a_{\prime}'=2n_{\prime}a_{\prime}^2\frac{d\Omega_{\prime}}{d\epsilon_{\prime}}$$

$$m\mu e'=-\frac{na\sqrt{1-e^2}}{e}\left\{\frac{d\Omega}{d\varpi}+(1-\sqrt{1-e^2})\frac{d\Omega}{d\epsilon}\right\}$$

$$m_{\prime}\mu_{\prime}e_{\prime}'=-\frac{n_{\prime}a_{\prime}\sqrt{1-e_{\prime}^2}}{e_{\prime}}\left\{\frac{d\Omega_{\prime}}{d\varpi_{\prime}}+(1-\sqrt{1-e_{\prime}^2})\frac{d\Omega_{\prime}}{d\epsilon_{\prime}}\right\}$$

$$m\mu\epsilon'=-2na^2\frac{d\Omega}{da}+\frac{na\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\frac{d\Omega}{de}$$

$$-\frac{m\mu}{\sin \mathrm{I}}\left\{\frac{\cos \mathrm{I}}{p}\frac{d\Omega}{d\mathrm{I}}+\frac{1}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}\right\}$$

$$m_{\prime}\mu_{\prime}\epsilon_{\prime}'=-2n_{\prime}a_{\prime}^2\frac{d\Omega_{\prime}}{da_{\prime}}+\frac{n_{\prime}a_{\prime}\sqrt{1-e_{\prime}^2}}{e_{\prime}}(1-\sqrt{1-e_{\prime}^2})\frac{d\Omega_{\prime}}{de_{\prime}}$$

$$-\frac{m_{\prime}\mu_{\prime}}{\sin \mathrm{I}}\left\{\frac{\cos \mathrm{I}}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}+\frac{1}{p}\frac{d\Omega}{d\mathrm{I}}\right\}$$

$$m\mu\varpi'=\frac{na\sqrt{1-e^2}}{e}\frac{d\Omega}{de}-\frac{m\mu}{\sin \mathrm{I}}\left\{\frac{\cos \mathrm{I}}{p}\frac{d\Omega}{d\mathrm{I}}+\frac{1}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}\right\}$$

$$m_{\prime}\mu_{\prime}\varpi_{\prime}'=\frac{n_{\prime}a_{\prime}\sqrt{1-e_{\prime}^2}}{e_{\prime}}\frac{d\Omega_{\prime}}{de_{\prime}}-\frac{m_{\prime}\mu_{\prime}}{\sin \mathrm{I}}\left\{\frac{\cos \mathrm{I}}{p_{\prime}}\frac{d\Omega_{\prime}}{d\mathrm{I}}+\frac{1}{p}\frac{d\Omega}{d\mathrm{I}}\right\}$$

$$pp_{\prime} \sin \mathrm{I}.\mathrm{I}' = p_{\prime}\left\{\cos \mathrm{I}\left(\frac{d\Omega}{d\epsilon}+\frac{d\Omega}{d\varpi}\right)+\frac{d\Omega}{d\epsilon_{\prime}}+\frac{d\Omega}{d\varpi_{\prime}}\right\}$$
$$+p\left\{\cos \mathrm{I}\left(\frac{d\Omega_{\prime}}{d\epsilon_{\prime}}+\frac{d\Omega_{\prime}}{d\varpi_{\prime}}\right)+\frac{d\Omega_{\prime}}{d\epsilon}+\frac{d\Omega_{\prime}}{d\varpi}\right\}.$$

94. The only parts of the preceding expressions of which the deduction is not perfectly obvious, are the terms involving I in the values of ϵ', $\epsilon'_{\prime}$, ϖ', $\varpi'_{\prime}$. They are obtained, as has been sufficiently explained, from the expressions in art. 55, to which are to be added the values of $\frac{\partial\epsilon}{dt}$, &c. ((95.), art. 79.); on putting $\nu=0$, ω_0 disappears from the latter; and the values of ω_1, ω_2, $\cos\iota$, $\cos\iota_{\prime}$, $\tan\frac{\iota}{2}$, $\tan\frac{\iota_{\prime}}{2}$ are to be introduced (equations (110.), (112.), (116.), (117.)). After some rather troublesome reductions, the expressions above given will be found.

In these equations it will be recollected that the mean longitude of m is represented by $\int_0^t ndt+\epsilon$, and the differentiation with respect to a in $\frac{d\Omega}{da}$ is only performed so far as a appears explicitly. If we wished that the mean longitude should be expressed by $nt+\epsilon$, the only change in the equations would be that the differentiation with respect to a must be total; i.e. must extend to a as contained in n. A similar remark applies of course to $\epsilon_{\prime}$.

In actual use it would be more convenient to introduce R, $\mathrm{R}_{\prime}$ instead of Ω, $\Omega_{\prime}$; the latter functions give a rather more symmetrical form to the equations, and are more convenient in general investigations. (The relation between them here is merely $\Omega=m\mathrm{R}$, $\Omega_{\prime}=m_{\prime}\mathrm{R}_{\prime}$; in another part of the paper the symbol Ω was used for $-m\mathrm{R}$ (art. 55.)).

95. If the equations of art. 93. were completely integrated, the *intrinsic* motion of the system would be completely determined; that is, we should know at any instant the dimensions of the two orbits, the mutual inclination of their planes, the position of their axes with respect to the line of nodes, the place of each planet in its orbit, and (by (110.)) the inclination of each orbit to the principal plane.

The position of the system relatively to fixed space would then have to be separately determined as follows:—

The three quantities ω_0, ω_1, ω_2 (see end of art. 92.), of which the values are ((112.), (113.), (116.)) given by the equations

$$\sigma^2 \sin \mathrm{I}.\omega_0=\left\{(p+p_{\prime}\cos \mathrm{I})\left(\frac{d}{d\epsilon_{\prime}}+\frac{d}{d\varpi_{\prime}}\right)+(p_{\prime}+p\cos \mathrm{I})\left(\frac{d}{d\epsilon}+\frac{d}{d\varpi}\right)\right\}(\Omega_{\prime}-\Omega)$$

$$\sigma\omega_1=\frac{d}{d\mathrm{I}}(\Omega_{\prime}-\Omega)$$

$$\sigma \sin \mathrm{I}.\omega_2=\left(\frac{p_{\prime}}{p}+\cos \mathrm{I}\right)\frac{d\Omega}{d\mathrm{I}}+\left(\frac{p}{p_{\prime}}+\cos \mathrm{I}\right)\frac{d\Omega_{\prime}}{d\mathrm{I}},$$

would be given functions of t. Then if we call

J the inclination of the principal plane to an arbitrary fixed plane;

Ω the longitude of the line of intersection of these two planes, reckoned in the fixed plane from a fixed line;

N the angle between this line of intersection and the *line of nodes*;

we should have (as in art. 83. with a different notation)

$$\left.\begin{aligned} J' &= \omega_0 \cos N - \omega_1 \sin N \\ \Omega' \sin J &= \omega_0 \sin N + \omega_1 \cos N \\ N' &= \omega_2 - \cot J(\omega_0 \sin N + \omega_1 \cos N) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (118.)$$

and the integration of this system would give J, Ω, N as functions of t, and so determine at any instant the position of the principal plane and of the line of nodes, relatively to fixed space.

With respect to the motion of the principal plane, the following may be added. It has already been shown (art. 88, 89.) that the line about which it is at any instant turning, coincides with that in which it is intersected by the plane of the radii vectores; and the values of ω_0, ω_1 (see art. 88, putting $\nu=0$ in the expressions there given) may be put in the form

$$\sigma^2\omega_0 = mm_, \sin I.\left(\frac{r_,}{r^2}-\frac{r}{r_,^2}\right)(p\cos\theta\sin\theta_, + p_,\sin\theta\cos\theta_,)$$

$$\sigma\omega_1 = mm_, \sin I\left(\frac{r_,}{r^2}-\frac{r}{r_,^2}\right)\sin\theta\sin\theta_,.$$

If the latter of these be multiplied by σ, and then both sides of each squared, and the results added (after putting for σ^2 on the right its value $p^2+p_,^2+2pp_,\cos I$), we find, observing that $\cos\theta\cos\theta_, + \sin\theta\sin\theta_,\cos I = \cos\chi$,

$$\sigma^2\sqrt{\omega_0^2+\omega_1^2} = mm_, \sin I\left(\frac{r_,}{r^2}-\frac{r}{r_,^2}\right)(p^2\sin^2\theta_, + p_,^2\sin^2\theta + 2pp_,\sin\theta\sin\theta_,\cos\chi)^{\frac{1}{2}},$$

an expression which may be further transformed as follows. Let λ, $\lambda_,$ be the *latitudes* of m, $m_,$ (with reference to the principal plane); then $\sin\lambda = \sin\theta\sin\iota$, $\sin\lambda_, = \sin\theta_,\sin\iota_,$; hence, since $p=\frac{\sigma\sin\iota_,}{\sin I}$, $p_,=\frac{-\sigma\sin\iota}{\sin I}$, we obtain

$$\sigma\sqrt{\omega_0^2+\omega_1^2} = mm_,\left(\frac{r_,}{r^2}-\frac{r}{r_,^2}\right)(\sin^2\lambda + \sin^2\lambda_, - 2\sin\lambda\sin\lambda_,\cos\chi)^{\frac{1}{2}}.$$

This gives the absolute angular velocity with which the principal plane is at any instant changing its direction in space; it is evident that (if the supposition $r_,=r$ be excluded) it can never vanish except when both planets are in the line of nodes. The *direction* of the rotation is determined by the signs of ω_0 and ω_1.

96. The system of differential equations given in art. 93. affords an example of the so-called "elimination of the nodes" in the problem of three bodies. Jacobi, by a very remarkable and ingenious transformation, has effected the elimination in a quite

different manner*. The equations of art. 93. are *merely* transformations of the original differential equations of the problem, without any integrations; they are however in a form which might perhaps be used advantageously in certain cases for the purposes of physical astronomy. Those of JACOBI are obtained by employing all the four usual integrals of the problem, and are shown to include an *additional integration.* They have however the disadvantage of substituting the coordinates of two fictitious bodies for those of the actual planets, and would probably be inconvenient for ordinary practical use; though in a theoretical point of view they seem to deserve more attention than they have hitherto received. It would be wrong to take leave of this celebrated problem without referring to another transformation by M. BERTRAND†, which, as has been remarked by a recent writer in the same journal, effects *six* integrations, and therefore represents the furthest advance which has yet been made towards a rigorous solution.

APPENDIX A.

When the method described in Theorem VII. (art. 49.) is applied to the solution of a system of equations of the form (I.), of which n integrals, $a_1 \dots a_n$, satisfying the conditions $[a_i, a_j]=0$, are given, the first step is to express the $n+1$ partial differential coefficients $\frac{dX}{dx_1}$, &c., and $\frac{dX}{dt}$; namely, the values of $y_1, \dots y_n$ and $-Z$ in terms of $x_1, \dots x_n, a_1, \dots a_n$ and t. The *direct* process is then to find X by integrating the expression $y_1dx_1+y_2dx_2+\dots+y_ndx_n-Zdt$, and afterwards to form the remaining integral equations $\frac{dX}{da_1}=b_1$, &c.: when this process is adopted, the inferior limits in the integrations are perfectly arbitrary; in other words, we may add to X an arbitrary function of $a_1, a_2, \dots a_n$, without altering any of the general properties of the final system of integrals.

But it is generally much more convenient to perform the differentiations with respect to $a_1, \dots a_n$ *first*, and integrate afterwards; thus we obtain the remaining equations in the form

$$b_i=\int\left(\frac{dy_1}{da_i}dx_1+\frac{dy_2}{da_i}dx_2+\dots+\frac{dy_n}{da_i}dx_n-\frac{dZ}{da_i}dt\right).$$

When this plan is followed, the limits are still arbitrary if it be only required that the equations thus obtained shall be *true*; but if it be required that they shall form, with the given integrals, a *normal solution*, it is necessary to take the limits in such a manner that the functions equated to $b_1, b_2, \dots b_n$ shall be the partial differential coefficients with respect to $a_1, a_2, \dots a_n$, of *one and the same function*; which will not generally be the case unless care be taken that it should be so.

In practice, the expression for dX usually consists of several terms, of which each

* Comptes Rendus, 1842, part 2. p. 236, &c.

† LIOUVILLE's Journal, 1852.

contains one of the variables *only*. Suppose one of these terms is

$$\phi(x, a_1, a_2, \dots a_n)dx,$$

so that, so far as this term is concerned, we have

$$X=\int_A^x \phi(x, a_1, a_2, \dots a_n)dx,$$

where A is an arbitrary function of $a_1, \dots a_n$. Consequently

$$\frac{dX}{da_i}=\int_A^x \frac{d\phi}{da_i}dx-\phi(A, a_1, \dots a_n)\frac{dA}{da_i},$$

and we see that we should not *in general* obtain the differential coefficients with respect to $a_1, \dots a_n$ of one and the same function X, by merely integrating $\frac{d\phi}{da_1}, \frac{d\phi}{da_2}$, &c., with respect to x, from the same inferior limit A, chosen at hazard.

But it is evident that we shall attain this end if we adopt the following simple rule:—

Integrate $\frac{d\phi}{da_1}$, &c. with respect to x, taking the same inferior limit in each case, namely, either

(1) a value A of x which satisfies the equation $\phi(x, a_1, \dots a_n)=0$, or

(2) any *determinate* constant (i.e. not a function of $a_1, \dots a_n$).

For example, in the problem of central forces (Part I., art. 28, &c.), we had (see art. 19.)

$$dX=-hdt+cd\theta+\left(2m(h+\phi(r))-k^2r^{-2}\right)^{\frac{1}{2}}dr+(k^2-c^2\sec^2\lambda)^{\frac{1}{2}}d\lambda$$

(where r, θ, λ are the three variables).

The very troublesome process of differentiating X with respect to h, k and c *after* first finding X by integrating the above expression, is avoided by the method adopted in art. 29; namely, by differentiating first, and integrating afterwards. In the integrations with respect to r, the inferior limit is one of the roots of the equation

$$2m(h+\phi(r))-k^2r^{-2}=0,$$

namely (in the case of elliptic motion), the perihelion distance; and in those with respect to λ, the inferior limit is 0; so that the rule above given is observed.

At the time of writing the article referred to, neither the rule itself, nor the necessity of attending to the limits, had occurred to me; it was therefore, strictly speaking, accidental that the final integrals were obtained in a *normal form*.

In treating the problem of rotation (Section III.), I perceived the necessity of caution as to the limits, if the former order of proceeding were adopted; but preferred avoiding the risk of error altogether, by performing the integrations first, so as to obtain the actual expression for V. The final equations (R.), art. 44, might however be obtained in a more simple way by differentiating *first*; thus we should have (see equations (45.), (46.)), observing that $\frac{dk}{dh}=\frac{k}{2h}$, &c. (art. 44.), and putting

$$(1-\cos^2 i-\cos^2 j+2\cos i\cos j\cos\theta-\cos^2\theta)^{\frac{1}{2}}=Q,$$

$$\frac{dV}{dh}=\frac{k}{2h}\left\{\psi\cos i+\phi\cos j+\int\frac{Q}{\sin\theta}d\theta\right\}=t+\tau$$

$$\frac{dV}{d\cos i}=k\left\{\psi+\int\frac{\cos j\cos\theta-\cos i}{Q\sin\theta}d\theta\right\}=\alpha \quad \text{(i.)}$$

$$\frac{dV}{d\cos j}=\frac{(C-A)k^3\cos j}{2ACh}\left\{\psi\cos i+\phi\cos j+\int\frac{Q}{\sin\theta}d\theta\right\}$$
$$+k\left\{\phi+\int\frac{\cos i\cos\theta-\cos j}{Q\sin\theta}d\theta\right\}=\beta.$$

In order to get rid of the troublesome integration involved in the term $\int\frac{Q}{\sin\theta}d\theta$, we may (1) eliminate this term between the first and last of these equations, and (2) eliminate ψ between the first and second. We thus find the two following equations,

$$\phi+\int\frac{\cos i\cos\theta-\cos j}{Q\sin\theta}d\theta=\frac{\beta}{k}-\frac{C-A}{AC}k\cos j.(t+\tau) \quad \text{(ii.)}$$

$$\cos j\left\{\phi+\int\frac{\cos i\cos\theta-\cos j}{Q\sin\theta}d\theta\right\}+\int\frac{\sin\theta d\theta}{Q}=\frac{2h}{k}(t+\tau)-\frac{\alpha\cos i}{k},$$

which last, combined with the preceding, gives

$$\int\frac{\sin\theta d\theta}{Q}=\frac{k}{A}(t+\tau)-\frac{\alpha\cos i+\beta\cos j}{k}; \quad \text{(iii.)}$$

and we may take (i.), (ii.) and (iii.) as expressing the solution of the problem.

Now we have $\cos I=\frac{\cos i-\cos j\cos\theta}{\sin j\sin\theta}$, from which it is easy to find (observing the conditions which determine the sign of Q)

$$-dI=\frac{\cos i\cos\theta-\cos j}{Q\sin\theta}d\theta, \text{ and similarly,}$$

$$-dJ=\frac{\cos j\cos\theta-\cos i}{Q\sin\theta}d\theta;$$

we have moreover $\qquad d\Theta=\frac{\sin\theta d\theta}{Q}.$

All the integrations may therefore now be performed immediately; and we may take for the inferior limit of θ any value which satisfies the equation $Q=0$, or

$$(\cos\theta-\cos i\cos j)^2-\sin^2 i\sin^2 j=0;$$

this is satisfied by $\theta=i+j$, which evidently corresponds to $I=0$, $J=0$, $\Theta=0$, and it is manifest that equations (i.), (ii.), (iii.) will thus become identical with equations (R) of art. 44.

I do not regret however having introduced the rather prolix investigation of arts. 39 and 43, because it is interesting to know the actual value of V (equation (48.)), which the method just given leaves undetermined.

Appendix B.

On the subject of the transformation of elements, the following additional remarks will hardly be superfluous. Suppose Ω is originally a function of the elements a, b, c, &c. with t; and let α, β, γ, &c. be other quantities connected with a, b, c, &c., by equations such as

$$da = \mathrm{A}d\alpha + \mathrm{B}d\beta + \mathrm{C}d\gamma + \dots + \mathrm{K}dt, \quad \dots \dots \dots \quad (a.)$$

where A, B, C, ... K are given functions of $\alpha, \beta, \gamma, \dots t$; or by equations such as

$$d\alpha = \mathrm{A}_{,}da + \mathrm{B}_{,}db + \dots + \mathrm{K}_{,}dt, \quad \dots \dots \dots \quad (\alpha.)$$

where $\mathrm{A}_{,}$, &c. are given functions of $a, b, \dots, t$. In either case, if each of the equations be integrable *per se*, we may consider $a, b, c, \dots$ as functions of $\alpha, \beta, \gamma, \dots, t$; and such equations as

$$\frac{d\Omega}{d\alpha} = \mathrm{A}\frac{d\Omega}{da} + \mathrm{B}\frac{d\Omega}{db} + \dots \quad \dots \dots \dots \quad (\Omega.)$$

are both significant and true.

But if the expressions on the right of the equations (a.) be *not* differentials *per se*, the equations (Ω.) are either unmeaning or untrue. For the symbol $\frac{d\Omega}{d\alpha}$ implies one of two things; either, that Ω is expressed in terms of $\alpha, \beta, \dots t$ *without* arbitrary constants (i. e. that the transformation of Ω can be actually effected *without* integrating the differential equations of the problem), which is manifestly impossible, unless (a.), &c. be integrable *per se*; or else, that the differential equations are to be conceived to have been completely integrated, so that a, b, &c., and consequently $\mathrm{A}_{,}, \mathrm{B}_{,}$, &c., are known as *functions of* t *and arbitrary constants*, whereby the right-hand side of (α.) becomes an *explicit function of* t (and arbitrary constants), so that α, β, &c. may by integration be expressed in the same way, and, by means of (a.), a, b, &c. may be similarly expressed, and finally, by algebraical elimination, a, b, &c. become functions of α, β, &c., t, and arbitrary constants. On this supposition, $\frac{d\Omega}{d\alpha}$ has a meaning, but the equation (Ω.) is *untrue*; for we must have

$$\frac{d\Omega}{d\alpha} = \frac{d\Omega}{da}\frac{da}{d\alpha} + \frac{d\Omega}{db}\frac{db}{d\alpha} + \dots;$$

and it is manifestly not true that $\frac{da}{d\alpha} = \mathrm{A}$, &c. in this case, because the equation

$$da = \mathrm{A}d\alpha + \mathrm{B}d\beta + \dots + \mathrm{K}dt,$$

not being integrable *per se*, only subsists for those variations of α, β, &c. which *actually take place* during the instant dt; whereas the equation

$$da = \frac{da}{d\alpha}d\alpha + \frac{da}{d\beta}d\beta + \dots + \frac{da}{dt}dt$$

subsists for *arbitrary variations* of all the variables. This view of the subject entirely

coincides, in substance, with that taken by JACOBI; but the above mode of stating it may tend to make it clearer, and to call attention to a matter which, so far as I know, is not so much as mentioned in any of the elementary works usually in the hands of students of physical astronomy.

[*Addition to* APPENDIX B.]

Received March 1, 1855.

The remark made above, that the symbol $\frac{d\Omega}{d\alpha}$ is *unmeaning* in the case considered, is not of course intended to imply that a meaning *may not be given to it*; but then such meaning is different from the ordinary signification of the symbol, which is a *partial derived function.*

The whole matter may be strikingly illustrated by a simple example.

Consider the movement of a rigid body about a fixed point. Adopting the notation of art. 40 (Part I.), we have

$$pdt=-\cos\phi d\theta-\sin\theta\sin\phi d\psi$$
$$qdt=\sin\phi d\theta-\sin\theta\cos\phi d\psi$$
$$rdt=d\phi+\cos\theta d\psi.$$

Let α, β, γ be three new variables, defined by the equations

$$\alpha=\int_0^t pdt,\quad \beta=\int_0^t qdt,\quad \gamma=\int_0^t rdt;$$

so that $d\alpha=pdt$, &c. Then the above equations give

$$d\theta=-\cos\phi d\alpha+\sin\phi d\beta,\quad d\phi=d\gamma+\cot\theta(\sin\phi d\alpha+\cos\phi d\beta)$$
$$d\psi=-\operatorname{cosec}\theta(\sin\phi d\alpha+\cos\phi d\beta).$$

Here α, β, γ are the sums of the elementary angles described about the axes in the course of the motion; and no one would maintain that θ, ϕ, ψ are *functions* of α, β, γ, for the values of the latter variables at any time *do not determine* the values of the former. If therefore we choose to write such equations as

$$\frac{d\theta}{d\alpha}=-\cos\phi,\ \frac{d\theta}{d\beta}=\sin\phi,\ \text{\&c.},$$

we must admit that $\frac{d\theta}{d\alpha}$, $\frac{d\theta}{d\beta}$, &c. are not partial derived functions in the ordinary sense. At most, $\frac{d\theta}{d\alpha}$ is the derived function of *that function of* α *which* θ *would become* if β and γ were maintained invariable, *i. e.* if the motion were restricted to a rotation about the A-axis. Again, if we admit such symbols as $\frac{d^2\theta}{d\beta d\alpha}$, $\frac{d^2\theta}{d\alpha d\beta}$, we must interpret them as follows:

$$\frac{d^2\theta}{d\beta d\alpha}=\frac{d}{d\beta}\frac{d\theta}{d\alpha}=-\frac{d\cos\phi}{d\beta}=\sin\phi\frac{d\phi}{d\beta},$$

but $\frac{d\phi}{d\beta} = \cot\theta \cos\phi$, and therefore

$$\frac{d^2\theta}{d\beta d\alpha} = \cot\theta \sin\phi \cos\phi\,;$$

and in like manner we find the same value for $\frac{d^2\theta}{d\alpha d\beta}$, so that in *this particular case* the condition $\frac{d^2\theta}{d\alpha d\beta} = \frac{d^2\theta}{d\beta d\alpha}$ is verified.

But if we take $\frac{d^2\theta}{d\gamma d\alpha}$ and $\frac{d^2\theta}{d\alpha d\gamma}$ in the same way, we find *the former* $= \sin\phi$ *and the latter* $=0$, so that in this case the condition *is not verified.* The geometrical meaning of this is obvious; analytically it is merely an instance of a general fact, pointed out by JACOBI; namely, that the effect of two successive *pseudo-differentiations* with respect to two independent variables, is not generally independent of the order of operations.

If V be the potential of another body, given in position, upon the body considered, then V is a function of θ, ϕ, ψ, and

$$dV = \frac{dV}{d\theta} d\theta + \frac{dV}{d\phi} d\phi + \frac{dV}{d\psi} d\psi\,;$$

and if we substitute for $d\theta$, $d\phi$, $d\psi$ their values in terms of $d\alpha$, $d\beta$, $d\gamma$, we obtain an expression which we may call $Ld\alpha + Md\beta + Nd\gamma$, L, M, N being functions of θ, ϕ, ψ, $\frac{dV}{d\theta}$, $\frac{dV}{d\phi}$, $\frac{dV}{d\psi}$, of which, as is well known, the mechanical meanings are the moments of the attraction of the second body about the three axes. Here again no one would maintain that V is a *function* of α, β, γ; and if, as is often done, we say $\frac{dV}{d\alpha} = L$, &c., the above remarks apply in all respects to these equations.

I should have thought it superfluous to dwell so much on these points if it had not appeared that writers on physical astronomy have in some instances either overlooked the distinction between *real* and *pseudo*-differentiation, or at least have failed to point it out to their readers. The only discussion of the subject which I have met with is that given by JACOBI, in the correspondence referred to.

It may be added, that in general investigations, where symbols such as $\frac{dV}{d\alpha}$, &c. may be used without defining the nature of V, or the precise meaning of α, β, &c., serious errors might be committed if it were assumed that the condition $\frac{d^2V}{d\alpha d\beta} = \frac{d^2V}{d\beta d\alpha}$ always subsisted.

APPENDIX C.

The theorems relating to the transformation of coordinates, given in Section VI., may be made more general, and in many cases more useful, as follows:—

If $x_1, x_2, \ldots x_n$ be the coordinates employed in the first statement of any dynamical problem, the differential equations are comprehended in the formula

$$\Sigma_i\left\{\left(\frac{dW}{dx'_i}\right)' - \frac{dW}{dx_i}\right\}\delta x_i = 0. \quad \ldots \ldots \ldots \quad \text{(D.)}$$

[If there be any forces, such as those arising from a resisting medium, which do not satisfy the natural conditions of integrability, then on the right-hand side of the formula (D.), instead of 0 we shall have an expression such as $\Sigma_i(X_i\delta x_i)$; but such terms are easily introduced and allowed for separately, and do not affect the following investigation. I shall therefore here assume that they do not exist.]

In the above formula, W is a given function of $x_1, \ldots x_n, x'_1, \ldots x'_n$, which may also explicitly contain t.

In Section VI. the only case contemplated was that in which $x_1, \ldots x_n$ are *independent* coordinates; in which case the formula (D.) is equivalent to n separate equations, since δx_1, &c. are wholly arbitrary and independent.

In practice, however, it is often more convenient to use, at first, a set of coordinates more in number than the independent variables of the problem, and therefore subject to equations of condition.

Let us assume then that the n coordinates $x_1, \ldots x_n$ are subject to r equations of condition,

$$L_1=0,\ L_2=0,\ \ldots\ L_r=0,$$

where L_1, &c. may explicitly contain t, besides the n variables x_1, &c.

If we introduce the n conjugate variables $y_1, \ldots y_n$ defined by the equations $y_i = \frac{dW}{dx'_i}$, and take Z a function of x_1, &c., y_1, &c. (with or without t), defined by the equation

$$Z = -[W] + [x'_1]y_1 + \ldots + [x'_n]y_n$$

(the brackets indicating that x'_1, &c. are expressed in terms of y_1, &c.), then it follows exactly as in art. 18 (Part I.), that the formula (D.) will be changed into the system

$$\left.\begin{array}{r} x'_i = \dfrac{dZ}{dy_i} \\ \Sigma_i\left(y'_i + \dfrac{dZ}{dx_i}\right)\delta x_i = 0 \end{array}\right\} \quad \ldots \ldots \ldots \ldots \quad \text{(E.)}$$

[In the most usual problems W is of the form T+U, where T is homogeneous and of the second degree in x'_1, &c., and U does not contain x'_1, &c. at all. In this case Z is only T−U expressed in terms of y_1, &c., instead of x'_1, &c. But T is *not necessarily* homogeneous; in fact it is not so in problems relating to motion *relative to the earth*, as affected by the earth's rotation.]

Let us now suppose that the system (E.) is to be transformed by the introduction of the m *independent* coordinates $\xi_1, \xi_2, \ldots \xi_m$, and of the new conjugate variables $\eta_1, \eta_2, \ldots \eta_m$; where $m=n-r$. And let it be required to investigate a theorem by means of which the transformation may be effected *without recurring to the original formula* (D.).

The definitions of the new coordinates ξ_1, &c. will furnish m equations (which may explicitly contain t) by means of which $\xi_1, \ldots \xi_m$ may be expressed as functions of $x_1, \ldots x_n$ (with or without t); and conversely, by means of these m equations, together with the r equations of condition $L_1=0$, &c., the n variables $x_1, x_2, \ldots x_n$ may be expressed as functions of $\xi_1, \ldots \xi_m$, with or without t. When $x_1, \ldots x_n$ are so expressed, let them be represented by $(x_1), \ldots (x_n)$. We shall have then

$$x'_i=\frac{d(x_i)}{dt}+\frac{d(x_i)}{d\xi_1}\xi'_1+\ldots+\frac{d(x_i)}{d\xi_m}\xi'_m, \quad \ldots \ldots \quad (x'.)$$

so that x'_1, &c. are expressible (and in only one way) as functions of ξ_1, &c., ξ'_1, &c.

If then the formula (D.) be transformed by expressing x_1, &c., x'_1, &c. in this manner, it becomes, as is well known,

$$\Sigma_i\left\{\left(\frac{d(\mathrm{W})}{d\xi'_i}\right)'-\frac{d(\mathrm{W})}{d\xi_i}\right\}\delta\xi_i=0,$$

where (W) represents the result of transforming W as above; and since $\delta\xi_1$, &c. are now independent, this formula breaks up into the m separate equations

$$\left(\frac{d(\mathrm{W})}{d\xi'_i}\right)'=\frac{d(\mathrm{W})}{d\xi_i}. \quad \ldots \ldots \quad (\mathrm{F.})$$

Moreover, if we now define η_i by the equation $\frac{d(\mathrm{W})}{d\xi'_i}=\eta_i$, and put

$$\Psi=-(\mathrm{W})+(\xi'_1)\eta_1+\ldots+(\xi'_m)\eta_m,$$

where (ξ'_1), &c. are expressed in terms of η_1, &c., we know already (art. 18.) that the system (F.) becomes

$$\xi'_i=\frac{d\Psi}{d\eta_i}, \quad \eta'_i=-\frac{d\Psi}{d\xi_i}. \quad \ldots \ldots \quad (\mathrm{G.})$$

Now let P be a function of the m new variables $\xi_1, \ldots \xi_m$, and of the n old variables $y_1, \ldots y_n$ (with or without t), defined by the equation

$$\mathrm{P}=(x_1)y_1+(x_2)y_2+\ldots+(x_n)y_n.$$

Since $\eta_i=\frac{d(\mathrm{W}}{d\xi'_i}$, and since (W) contains ξ'_i, &c., only through x'_i, &c., we have, observing that $\frac{d\mathrm{W}}{dx'_i}=y_i$,

$$\eta_i=y_1\frac{dx'_1}{d\xi'_i}+y_2\frac{dx'_2}{d\xi'_i}+\ldots+y_n\frac{dx'_n}{d\xi'_i};$$

but from the equation $(x'.)$ we have $\quad \frac{dx'_j}{d\xi'_i}=\frac{d(x_j)}{d\xi_i}$,

consequently $$\eta_i=y_1\frac{d(x_1)}{d\xi_i}+y_2\frac{d(x_2)}{d\xi_i}+\ldots+y_n\frac{d(x_n)}{d\xi_i},$$

an expression evidently equivalent to $\frac{d\mathrm{P}}{d\xi_i}$. Thus η_i *may be defined by the equation*

$$\frac{d\mathrm{P}}{d\xi_i}=\eta_i. \quad \ldots \ldots \quad (\eta.)$$

(3) Ψ is defined by the equation $\Psi = Z - \frac{dP}{dt}$, in which (after the explicit differentiation of P with respect to t), x_1, &c., y_1, &c. are to be expressed in terms of the new variables. y_1, &c. are thus expressible by the help of the m equations $\frac{dP}{d\xi} = \eta_i$ and the $n-m$ equations $\frac{dL}{dt} + \Sigma_i\left(\frac{dL}{dx_i}\frac{dZ}{dy_i}\right) = 0$.

If (x_1), &c., do not contain t explicitly, then $\frac{dP}{dt} = 0$, and Ψ is obtained merely by expressing Z in terms of the new variables.

It may be observed that the whole of the above reasoning would apply to the case in which the new variables $\xi_1, \ldots \xi_m$ are more in number than the independent variables of the problem (or $m > n - r$), *with this exception*; that the m equations $\frac{dP}{d\xi_i} = \eta_i$, together with the r equations obtained by differentiating the equations of condition totally with respect to t, would be *more than sufficient* to express $y_1, \ldots y_n$ in terms of the new variables; consequently y_1, &c. might be so expressed in *different ways*, and therefore, although the *value* of Ψ obtained by the above rule would certainly be the same as that obtained by recurring to the original formula (D.), the *form* of Ψ might be different, and therefore the resulting formula erroneous.

There must doubtless exist some rule for choosing $n-m$ combinations of the equations of condition in such a way as to lead to the correct *forms* of $y_1, \ldots y_n$ as functions of the new variables; but I have not at present attempted to investigate it, and perhaps it would be hardly worth while. The theorem in the case in which the new coordinates are independent, may, I believe, be practically useful.

ERRATA IN PART I.

Art. 1. equation (4.), for dx read dx_i.

Art. 10. In paragraph preceding equation (26.) *omit* the words "not containing t explicitly."

Art. 18. equation (β), for y_i read y'_i.

Art. 19. equation (29.), for h_i read b_i.

Art. 24. second line after equation (L.), *for* "such as h, k" *read* "such as f, g."

Art. 30. The expressions equated to h, k, c, and the three terms in the left-hand column of the table of elements, should each be multiplied by m.

Art. 42. near the end, *for* "according as Θ is between o and π, or not" *read* "according as Θ is between π and 2π, or between o and π."

- 5 -

From the PHILOSOPHICAL TRANSACTIONS for 1858.—PART III.

XXXVI. *On the Comparison of Transcendents, with certain applications to the Theory of Definite Integrals.* *By* GEORGE BOOLE, *Esq., Professor of Mathematics in the Queen's University.* *Communicated by Professor* W. F. DONKIN, *F.R.S.*

Received March 16,—Read May 7, 1857.

1. THE following objects are contemplated in this paper:—

1st. The demonstration of a fundamental theorem for the summation of integrals whose limits are determined by the roots of an algebraic equation.

2ndly. The application of that theorem to the problem of the comparison of algebraic transcendents.

The immediate object of this application will in each case be the finite expression of the value of the sum of a series of integrals, $\Sigma\int Xdx$, the differential coefficient, X, being an algebraic function, and the values of x at the limits being determined by the roots of an algebraic equation.

3rdly. The application of the same theorem in a new, and, as is conceived, more remarkable line of investigation, to the comparison of functional transcendents.

The terms 'algebraic' and 'functional' are not here used by way of logical division to indicate classes of transcendents wholly distinct, but the term functional transcendent is simply employed to designate an integral $\int Xdx$ in which X involves an arbitrary symbol of functionality.

Under this third head of the comparison of functional transcendents will fall the most important special result of the entire investigation. A case will arise in which, without any limitation of the functional symbol, the several integrals included under the form $\Sigma\int Xdx$ will close up, if the expression may be allowed, into a single integral taken between the limits $-\infty$ and ∞. The result is a very remarkable theorem of definite integration, fruitful in important consequences. In its general form, this theorem is, I believe, entirely new. A particular case of it was discovered by me several years ago, and was published without demonstration in the Cambridge and Dublin Mathematical Journal*, and in LIOUVILLE's Journal de Mathématiques†. A memoir by CAUCHY on integrals taken between the limits 0 and ∞‡, contains also a very limited case of the same theorem. It appears there, however, as an isolated result, quite apart from the doctrine of the comparison of transcendents.

In the concluding sections of this paper I shall apply the results of this part of the investigation to the extension of the theory of multiple definite integrals.

As respects the methods and processes which will be employed in this paper, the only

* Vol. iv. p. 14. † Tom. xiii. ‡ Exercices, vol. i. p. 54.

peculiarity to which it seems necessary to direct attention, is the introduction of a symbol, differing in interpretation only by the addition of one element, from that which CAUCHY has employed in his 'Calculus of Residues.' Of the nature of this connexion I was not aware until my researches were nearly completed; and I should then have abandoned my own symbol and adopted the other, already associated by the labours of its distinguished inventor with so many important discoveries in the higher departments of the integral calculus, if it had not appeared to me that the several elements combined in the interpretation of the former symbol were so allied that no one of them could without a manifest defect of completeness be omitted. It seemed to me also that many of CAUCHY'S own applications of his symbol would gain in simplicity and in generality of expression, by the adoption of the more enlarged interpretation.

2. Beside the above special objects, in the attainment of which whatever claim to originality this paper may possess will consist, I have proposed to myself, as a general object, the simplification of a branch of analysis which possesses some practical and much speculative importance. To this object the introduction of the symbol above referred to, contributes in a very important degree. The necessity of simplification will, I think, be admitted by all who are acquainted with the literature of the subject. As presented in the writings of ABEL and of those who immediately followed in his steps, the doctrine of the comparison of transcendents is repulsive from the complexity of the formulæ in which its general conclusions are embodied. The particular result known as ABEL'S theorem, the only one of its class which has been adopted into English works of education, will at once suggest itself in confirmation of this remark. Perhaps this complexity will not be thought surprising if we consider the nature of the problems involved,—the discovery of finite relations among integrals which derive their very name from the circumstance that individually their finite expression transcends the powers of analysis. On the other hand, and this is a juster ground of inference, the theory upon which such applications rest is far from being difficult or recondite, and, considered *à priori*, should be capable of a simpler analytical development than it has yet received. I hope that I shall be able to show that this anticipation is confirmed by the results of the present inquiry. Simplicity, though it is not to be gained at the expense of that which is the chief object of scientific methods, the discovery of truth, is nevertheless a highly valuable quality. And so far from being inconsistent with generality in the processes and the results of analysis, it is sometimes an indication of the measure of our approach to completeness and unity. I think that this is more especially the case where, as through the labours of ABEL in the present instance, the subject matter of investigation has been clearly defined, and the entire series of methods and results foreshown to be the evolution of some one general principle or idea.

3. It will be proper, before entering upon the special investigation, to give some general account of the doctrine of the comparison of transcendents. In doing this, I cannot but refer to the able report of Mr. LESLIE ELLIS on the Progress of Analysis, published in the Report of the British Association for 1846. It contains a most valuable

$a_1, a_2, \ldots a_r$ being the independent constants in the equation (1.) by which the limits are determined, or the form

$$\Sigma\int^{x_i} \mathrm{X}dx = \psi(x_1, x_2, \ldots x_n) + \mathrm{C}, \qquad (6.)$$

$\psi(x_1, x_2, \ldots x_n)$ being a function, and manifestly a symmetrical function, of the limits $x_1, x_2, \ldots x_n$. Either of these forms may be converted into the other by means of (1.), but the first is the one to which we shall give the preference.

We may remark, that if in the equation (1.) $a_1, a_2, \ldots a_r$ vary, the values of x, determined by the solution of that equation, will vary also. For each set of values of $a_1, a_2, \ldots a_r$, there will exist a simultaneous set of values of x. We may in this way consider the variables x in the several integrals under the sign Σ as always, in the course of their transition from the lower to the upper limits of integration, determined by the roots of the equation

$$\mathrm{E}(x, a_1, a_2, \ldots a_r) = 0.$$

According to this more general view, $a_1, a_2 \ldots a_r$ become a set of variables with which x is connected by the above equation, but the variation of x in each integral represents only the variation of one root of the equation. And the determination of the values of x at the upper or at the lower limits of integration by the solution of that equation, particular values being assigned to $a_1, a_2, \ldots a_r$, is only a special case of the determination of the simultaneous values of the variable x.

4. Thus the problem with which we are concerned may be more briefly expressed in this form. *Required the value of the expression* $\Sigma\int \mathrm{X}dx$, *the simultaneous values of* x *in the several integrals being determined by an equation of the form*

$$\mathrm{E}(x, a_1, a_2, \ldots a_r) = 0, \qquad (7.)$$

in which $a_1, a_2, \ldots a_r$ *are variable quantities, by the assigning of particular values to which in the solution of the equation, the particular values of* x *at the limits of integration are determined.*

The solution of the above problem is to be effected by giving to the expression $\Sigma\int \mathrm{X}dx$ the equivalent form $\int\Sigma \mathrm{X}dx$, transforming $\Sigma \mathrm{X}dx$ into a complete differential with respect to the variables $a_1, a_2, \ldots a_r$, and then effecting the integration. For this reason I shall designate (7.), or, as it may for convenience be written, $\mathrm{E}=0$, as the 'transforming equation,' except when it is employed to determine the limits, in which case the designation of 'equation of the limits' will be preferable.

For the solution of the problem, as thus stated, it is usually necessary that the form of the function E in the transforming equation, and the form of the function X under the sign of integration, should have a certain connexion. The connexion implied is the following. The transforming equation must in general be such, that it may be possible by means of it to reduce the differential expression X under the sign of integration to a form $\mathrm{F}(x, a_1, a_2, \ldots a_r)$, which, considered as an explicit function of x and of $a_1, a_2, \ldots a_n$, shall be rational with respect to x. This is not a necessity *à priori*. It is a necessity founded in the limitations of the powers of analysis.

5. To illustrate these remarks, let us take as an example the integral

$$\int\frac{dx}{\sqrt{1+x^4}},$$

which is known to express the length of the arc of the lemniscate. The following would be a legitimate example of the kind of problem which it is proposed to investigate, viz. Required the value of the expression

$$\Sigma\int\frac{dx}{\sqrt{1+x^4}},$$

the simultaneous values of x in the successive integrals being given by the roots of the equation

$$1+x^4=(a+x+x^2)^2, \quad \ldots \ldots \quad (8.)$$

reducible to the form

$$x^3+(a+\tfrac{1}{2})x^2+ax+\frac{a^2-1}{2}=0. \quad \ldots \ldots \quad (9.)$$

We shall represent these roots by x_1, x_2, x_3.

The solution of the problem would then assume either the form

$$\Sigma\int\frac{dx}{\sqrt{1+x^4}}=\varphi(a)+\mathrm{C}, \quad \ldots \ldots \quad (10.)$$

C being the constant of integration and $\varphi(a)$ the function to be determined, or the form

$$\Sigma\int\frac{dx}{\sqrt{1+x^4}}=\psi(x_1, x_2, x_3)+\mathrm{C}, \quad \ldots \ldots \quad (11.)$$

wherein ψ is to be determined.

It will be observed that the transforming equation (8.) is such as to enable us, in accordance with the requirement of art. 4, to reduce the function under the sign of integration to a form in which, considered as an explicit function of x and a, it is rational with respect to x. For it gives

$$\int\frac{dx}{\sqrt{1+x^4}}=\int\frac{dx}{a+x+x^2}, \quad \ldots \ldots \quad (12.)$$

the second member of which fulfils the condition in question. It is to be remarked, however, that a and x are still connected by the equation (8.) or (9.), and that we are not permitted to integrate as if a were constant.

In the above problem, only one arbitrary element, a, presents itself in the transforming equation. There might, however, have been two or any greater number of arbitrary elements. Thus the equation might have been

$$1+x^4=(a+bx+x^2)^2,$$

or

$$x^3+\frac{b^2+2a}{2b}x^2+ax+\frac{a^2-1}{2b}=0, \quad \ldots \ldots \quad (13.)$$

and the solution of the problem would then have assumed one of the two following forms, viz. either the form

$$\Sigma\int\frac{dx}{\sqrt{1+x^4}}=\varphi(a, b)+\mathrm{C}, \quad \ldots \ldots \quad (14.)$$

or the form

$$\Sigma\int\frac{dx}{\sqrt{1+x^4}}=\psi(x_1, x_2, x_3)+C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15.)$$

From these solutions the corresponding solutions of the previous and less general problem would be obtained, in the one case by making $b=1$, in the other by imposing upon the limits x_1, x_2, x_3 conditions thereto equivalent.

6. And not only the solution, but the original statement of a problem may be exhibited in the two forms above described. Instead of supposing the limits determined by an equation involving arbitrary quantities in its coefficients, we may suppose them directly connected by symmetrical equations, *i. e.* we may suppose those relations *explicitly* given which are only *implied* in the equation determining the limits, and can only thence be deduced by eliminating the arbitrary elements. Thus (9.) furnishes us with the three following equations:

$$x_1+x_2+x_3=-a-\tfrac{1}{2}.$$

$$x_1x_2+x_2x_3+x_3x_1=a.$$

$$x_1x_2x_3=\frac{1-a^2}{2}.$$

From which, eliminating the arbitrary quantity a, we have

$$\left.\begin{array}{l} x_1+x_2+x_3=x_1x_2+x_2x_3+x_3x_1-\frac{1}{2} \\ 2x_1x_2x_3=1-(x_1x_2+x_2x_3+x_3x_1)^2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16.)$$

The problem first considered now assumes the following form. Required the value of the integral expression

$$\Sigma\int\frac{dx}{\sqrt{1+x^4}},$$

when the superior limits x_1, x_2, x_3 are connected by the explicit relations (16.).

The second problem, similarly transformed, would, as there are two arbitrary elements to be eliminated, lead to a single equation between x_1, x_2, x_3, in place of the two equations (16.).

7. We may observe, from the above examples, that when the number of integrals to be added is three, the existence of two arbitrary elements in the equation of the limits involves the existence of one symmetrical equation among the limits, and the existence of one arbitrary constant in the equation of the limits involves the existence of two symmetrical equations among the limits themselves. And thus generally if there be n integrals to be added, the existence of r arbitrary elements in the equation of the limits will involve the existence of $n-r$ symmetrical equations among the limits. The converse of this proposition is obviously true also. If any number r of symmetrical equations among the limits $x_1, x_2, .. x_n$ are given, and if we regard $x_1, x_2, .. x_n$ as roots of the equation of the nth degree,

$$x^n+p_1x^{n-1}+p_2x^{n-2}..+p_n=0,$$

the symmetrical conditions referred to, will, by the theory of equations, establish among the coefficients $p_1, p_2, .. p_n$, a system of relations by means of which we can determine r of

those coefficients as functions of the others deemed arbitrary,—or choosing in some other way $n-r$ arbitrary elements, express all the coefficients by means of those elements.

It is further seen that in the one form of the problem the greater the number of arbitrary quantities in the equation determining the limits,—in the other form the smaller the number of symmetrical equations connecting the limits with each other,—the more general is the statement of the problem itself.

8. The Norwegian and German mathematicians, by whom this branch of analysis has been chiefly cultivated, have almost universally followed ABEL in his mode of stating the general problem, *i. e.* they have regarded the limits as roots of an equation involving a greater or less number of arbitrary elements in its coefficients. Mr. FOX TALBOT, in his interesting papers "On the Comparison of Transcendents," published in the Philosophical Transactions for the years 1836–37, has in the earlier examples of his method expressed by symmetrical equations among the limits the conditions to which the latter are subject: in his later examples he adopts the more succinct notation of ABEL. Indeed, this mode of statement, as it replaces a system of equations by a single equation from which such systems may be considered as derived, is far better suited to general investigations, and will be adopted in this paper.

9. The complete solution of the problem of the comparison of transcendents, as above explained, involves two distinct steps,—1st, regarding the equation

$$E=0$$

of (7.), art. 4, as an equation expressing the dependence of the variable x upon the variables $a_1, a_2, .. a_r$, we must seek to convert the differential expression ΣXdx into a complete differential relative to $a_1, a_2 .. a_r$ as independent variables, and which will therefore virtually be in the form

$$\Sigma Xdx=A_1da_1+A_2da_2..+A_rda_r,$$

each of the differential coefficients $A_1, A_2, .. A_r$ being a function of $a_1, a_2, .. a_r$; 2ndly, we must integrate this expression.

The introduction of the symbol of operation adverted to in art. 1 will enable us to dispense with the explicit determination of the coefficients $A_1, A_2, .. A_r$, and to reduce the corresponding differential expression to one involving only a single variable. I shall now proceed to define the symbol in question, and to investigate its chief properties.

Definition and Properties of the Symbol Θ.

10. It is an evident consequence of TAYLOR's theorem that we can develope any function of x, $f(x)$, in ascending powers of $x-a$, provided that neither $f(x)$ nor any one of its differential coefficients becomes infinite when $x=a$. To effect this development, we have only to assume $x-a=z$; then $x=a+z$, whence

$$\left.\begin{aligned} f(x)&=f(a+z)\\ &=f(a)+f'(a)z+f''(a)\frac{z^2}{1.2}+\&c.\\ &=f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{1.2}+\&c.\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1.)$$

Now $f(x)$ being as above, let $\mathrm{F}(x)=\frac{f(x)}{(x-a)^m}$ where m is an integer. Then

$$\mathrm{F}(x)=\frac{f(a)}{(x-a)^m}+\frac{f'(a)}{(x-a)^{m-1}}\cdot\cdot+\frac{f^{(m-1)}(a)}{\underline{|m-1}}\,\frac{1}{x-a}+\frac{f^{(m)}(a)}{\underline{|m}}+\frac{f^{(m+1)}(a)\times(x-a)}{\underline{|m+1}}+\&\mathrm{c.} \quad . \quad . \quad . \quad (2.)$$

Here we obtain, as before, a development of $\mathrm{F}(x)$ in ascending powers of $x-a$, but the development begins with negative powers of that quantity. To this species of development we shall have frequent occasion to refer, and no doubt after this explanation will arise as to what is meant when we speak of the development of any function $\mathrm{F}(x)$ in ascending powers of $x-a$.

These things being premised, let the symbol Θ be thus defined, *viz. If* $\varphi(\mathrm{x})\mathrm{f}(\mathrm{x})$ *be any function of* x *composed of two factors* $\varphi(\mathrm{x})$ *and* $\mathrm{f}(\mathrm{x})$ *whereof* $\varphi(\mathrm{x})$ *is rational, let*

$$\Theta[\varphi(x)]f(x)$$

denote the result obtained by successively developing the function in ascending powers of each distinct simple factor $\mathrm{x}-\mathrm{a}$ *in the denominator of* $\varphi(\mathrm{x})$, *taking in each development the coefficient of* $\frac{1}{\mathrm{x}-\mathrm{a}}$, *adding together the coefficients thus obtained from the several developments, and subtracting from the result the coefficient of* $\frac{1}{\mathrm{x}}$ *in the development of the same function* $\varphi(\mathrm{x})\mathrm{f}(\mathrm{x})$ *in descending powers of* x.

It is seen from the above that the interpretation of Θ is relative. It directs us to obtain certain developments, but the nature of these developments, if we except the last of them, depends upon the nature of the function within the brackets [ ]. Thus while in the expression $\Theta[\varphi(x)]f(x)$ the operation of the symbol Θ extends over the entire function $\varphi(x)f(x)$, the interpretation of Θ, by which the nature of that operation is defined, is derived solely from the factor $\varphi(x)$.

Thus to take an example of some generality, let it be required to deduce an expression for

$$\Theta\left[\frac{x^2}{(x-a)(x-b)^2}\right]f(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

where $f(x)$ denotes some function of x which does not become infinite when $x=a$ or b.

The distinct simple factors in the denominator of the function within the brackets [ ] are $x-a$ and $x-b$. If we make $x-a=z$, or $x=a+z$, we have to develope

$$\frac{(a+z)^2}{z(a-b+z)^2}f(a+z)$$

in ascending powers of z. The coefficient of $\frac{1}{z}$ in that expansion is

$$\frac{a^2f(a)}{(a-b)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Again, making $x-b=z$ and $x=b+z$, we have to develope the function

$$\frac{(b+z)^2}{z^2(b-a+z)}f(b+z)$$

in ascending powers of z. The development is

$$\frac{1}{z^2}\left\{\frac{b^2 f(b)}{b-a}+\frac{d}{db}\frac{b^2 f(b)}{b-a}z+\frac{1}{1.2}\frac{d^2}{db^2}\frac{b^2 f(b)}{b-a}z^2+\&c.\right\},$$

in which the coefficient of $\frac{1}{z}$ is

$$\frac{d}{db}\cdot\frac{b^2 f(b)}{b-a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

Lastly, the coefficient of $\frac{1}{x}$ in the development of the function

$$\frac{x^2 f(x)}{(x-a)(x-b)^2}$$

in descending powers of x being represented according to a familiar notation by

$$\mathrm{C}_{\frac{1}{x}}\frac{x^2 f(x)}{(x-a)(x-b)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

we have, on adding (4.) and (5.), and subtracting (6.) from the sum,

$$\Theta\left[\frac{x^2}{(x-a)(x-b)^2}\right]f(x)=\frac{a^2 f(a)}{(a-b)^2}+\frac{d}{db}\frac{b^2 f(b)}{b-a}-\mathrm{C}_{\frac{1}{x}}\frac{x^2 f(x)}{(x-a)(x-b)^2}. \quad . \quad . \quad . \quad (7.)$$

As a particular illustration, let $f(x)=\log\left(c+\frac{1}{x}\right)$, and let us seek the value of the last term in the above expression. Now

$$\frac{x^2}{(x-a)(x-b)^2}=\frac{1}{x}+\frac{a+2b}{x^2}+\ \&c.$$

on developing in descending powers of x, and

$$\log\left(c+\frac{1}{x}\right)=\log c+\frac{1}{cx}-\frac{1}{c^2x^2}+\ \&c.$$

Multiplying these together, the coefficient of $\frac{1}{x}$ in the result is $\log c$. Whence

$$\Theta\left[\frac{x^2}{(x-a)(x-b)^2}\right]\log\left(c+\frac{1}{x}\right)=\frac{a^2\log\left(c+\frac{1}{a}\right)}{(a-b)^2}+\frac{d}{db}\frac{b^2\log\left(c+\frac{1}{b}\right)}{b-a}-\log c, \quad . \quad . \quad (8.)$$

in which it only remains to perform the differentiation in the second term.

Formulæ applicable to the determination of the result of the operation Θ in any case, may readily be found by the aid of TAYLOR's theorem. Thus we should have, $f(x)$ not becoming infinite when $x=a$ or $x=b$,

$$\left.\begin{aligned}\Theta\left[\frac{1}{(x-a)^m(x-b)^n}\right]f(x)&=\frac{1}{1.2\ldots(m-1)}\left(\frac{d}{da}\right)^{m-1}\frac{f(a)}{(a-b)^n}\\&+\frac{1}{1.2\ldots(n-1)}\left(\frac{d}{db}\right)^{n-1}\frac{f(b)}{(b-a)^m}-\mathrm{C}_{\frac{1}{x}}\frac{f(x)}{(x-a)^m(x-b)^n}\end{aligned}\right\} \quad . \quad . \quad (9.)$$

an expression in which the general law of such formulæ is manifest.

If there be n distinct simple factors in the denominator of the rational fraction within the brackets, the result of the operation Θ will consist of $n+1$ terms, the first n of which

will be determined as above by TAYLOR's theorem. The $n+1$th term will involve the operation denoted by the symbol $C_{\frac{1}{x}}$, and it is by this term only that the interpretation of Θ differs from that of CAUCHY's symbol, $\mathcal{E}$. We have in fact

$$\Theta = \mathcal{E} - C_{\frac{1}{x}}, \qquad (10.)$$

the complete interpretation of Θ involving two *distinct* elements.

It happens in certain problems that one of those elements alone presents itself, the symbol Θ becoming equivalent either to $\mathcal{E}$ or to $-C_{\frac{1}{x}}$. In other problems they both appear, but I am not aware of any problems in which the vanishing of one of the elements is not due to some special circumstance of restriction or limitation affecting the interpretation of the symbol Θ in the case supposed. In a Note to this paper I have endeavoured to illustrate the above remark by employing the symbol Θ in certain problems, in which CAUCHY has made use of the symbol of residues, and to that Note the reader who is interested in the comparison is referred.

11. The properties of the symbol Θ are now to be considered. The two following are the most important of them:—

1st. *The operation Θ is distributive as respects both the function without and the function within the brackets, provided that those functions do not become together infinite for any finite value of* x.

PROOF.—First, as respects the function without the brackets, we have the theorem

$$\Theta[\phi(x)]\big(f_1(x)+f_2(x)\ldots+f_n(x)\big)=\Theta[\phi(x)]f_1(x)+\Theta[\phi(x)]f_2(x)\ldots+\Theta[\phi(x)]f_n(x)\,; \qquad (1.)$$

for the coefficient of a particular term, as $\frac{1}{x-a}$, $\frac{1}{x}$, &c. in the development of a function is equal to the sum of the coefficients of all the corresponding terms in the development of the several component functions from which the proposed function is formed by addition.

Secondly, as respects the function within the brackets, we have the theorem

$$\Theta[\phi_1(x)+\phi_2(x)\ldots+\phi_n(x)]f(x)=\Theta[\phi_1(x)]f(x)+\Theta[\phi_2(x)]f(x)\ldots+\Theta[\phi_n(x)]f(x). \qquad (2.)$$

Here, it is to be observed that $\phi_1(x)$, $\phi_2(x)$, $\ldots\phi_n(x)$ represent any rational fractions into which the rational fraction $\phi(x)$ within the brackets [ ] is resolved. Thus the theorem might be written in the form

$$\Theta[\phi(x)]f(x)=\Theta[\phi_1(x)]f(x)+\Theta[\phi_2(x)]f(x)\ldots+\Theta[\phi_n(x)]f(x), \qquad (3.)$$

wherein

$$\phi(x)=\phi_1(x)+\phi_2(x)\ldots+\phi_n(x).$$

To prove this theorem I shall show that it is necessarily true for each of the operations into which Θ is resolvable. Let one of the *distinct* factors of the denominator of $\phi(x)$ be $x-a$, then one of the component operations in Θ consists in developing in ascending powers of $x-a$, and taking in the development the coefficient of $\frac{1}{x-a}$. Now

whether this operation be performed at once upon the function $\phi(x)f(x)$ or separately upon the several functions

$$\phi_1(x)f(x),\ \phi_2(x)f(x)\ldots\phi_n(x)f(x)$$

of which that function is composed, and the several results then collected together, is a matter of indifference. If we represent this part of the operation Θ by R, we have therefore

$$\mathrm{R}\phi(x)f(x)=\mathrm{R}\phi_1(x)f(x)+\mathrm{R}\phi_2(x)f(x)\ldots+\mathrm{R}\phi_n(x)f(x)\ldots \qquad (4.)$$

Now any term, $\mathrm{R}\phi_i(x)f(x)$, in the second member of the above will either form a part of the corresponding term $\Theta[\phi_i(x)]f(x)$ in the second member of (3.), or will vanish. The former will obviously be the case if $x-a$ is contained in the denominator of $\phi_i(x)$; the latter will be the case if $x-a$ is not included in the denominator of $\phi_i(x)$, for then the function $\phi_i(x)f(x)$ not becoming infinite when $x=a$, is developable in a series of the form $\mathrm{A}+\mathrm{B}(x-a)+\mathrm{C}(x-a)^2\ldots$ Art. 10, and in this series the coefficient of $\frac{1}{x-a}$ is 0. Hence, all the terms in (4.) are contained in (3.); neither are there any terms resulting from the component part of the operation Θ denoted by R which are not contained in the second member of (4.).

Hence, if we cause R to stand in succession for each of the component operations of Θ, and add the several equations thus obtained and typically represented by (4.) together, we shall obtain the theorem (3.).

As an example, we have

$$\Theta\left[\frac{2a}{x^2-a^2}\right]f(x)=\Theta\left[\frac{1}{x-a}\right]f(x)+\Theta\left[\frac{-1}{x+a}\right]f(x).$$

which is easily verified, since the first member gives

$$f(a)-f(-a)-\mathrm{C}_{\frac{1}{x}}\frac{2af(x)}{x^2-a^2},$$

and the second member

$$f(a)-f(-a)-\mathrm{C}_{\frac{1}{x}}\frac{f(x)}{x-a}+\mathrm{C}_{\frac{1}{x}}\frac{f(x)}{x+a},$$

an equivalent result.

2ndly. *If* f(x) *be a rational and entire function of* x, *we have always*

$$\Theta[\phi(x)]f(x)=0. \qquad (5.)$$

Proof.—As $f(x)$ must be of the form $\Sigma\mathrm{A}x^i$, i being an integer, we have

$$\Theta[\phi(x)]f(x)=\Theta[\phi(x)]\Sigma\mathrm{A}x^i$$
$$=\Sigma\mathrm{A}\Theta[\phi(x)]x^i$$

by the last proposition. Thus we have to consider a series of terms of the form

$$\Theta[\phi(x)]x^i. \qquad (6.)$$

Again, $\phi(x)$ being a rational fraction may be resolved into a series of terms which will be of the form ax^m or of the form $\frac{b}{(x-c)^n}$. Hence, availing ourselves of the distributive

property of Θ with respect to the term within the brackets, we see that (6.) is resolvable ultimately into a series of terms falling under the two typical forms

$$\Theta[x^m]x^i, \qquad \Theta\left[\frac{1}{(x-a)^n}\right]x^i.$$

All terms of the first form obviously vanish, since they can have no negative indices. It only remains, therefore, to consider terms of the second form.

First, let i be equal to or greater than n. Then putting $x-a=z$, and $x=a+z$, the coefficient of $\frac{1}{z}$ in the development of the function $\frac{(a+z)^i}{z^n}$, to which $\frac{x^i}{(x-a)^n}$ is reduced, in ascending powers of z will be

$$\frac{i(i-1)..(i-n+2)}{1.2..(n-1)}a^{i-n+1}; \quad \ldots \ldots \ldots \ldots \quad (7.)$$

and the coefficient of $\frac{1}{x}$ in the development of the function $\frac{x^i}{(x-a)^n}$ in descending powers of x will be

$$\frac{n(n+1)..i}{1.2..(i-n+1)}a^{i-n+1}. \quad \ldots \ldots \ldots \ldots \quad (8.)$$

These expressions are equal, as may be shown by equating them and clearing of fractions. Hence, in this case,

$$\Theta\left[\frac{1}{(x-a)^n}\right]x^i=0. \quad \ldots \ldots \ldots \ldots \quad (9.)$$

Secondly, if $i=n-1$, each of the above expressions (7.) and (8.) reducing to unity, the equation (9.) is still true.

Lastly, if i is less than $n-1$, neither will any term containing $\frac{1}{z}$ present itself in the ascending development, nor any term containing $\frac{1}{x}$ in the descending development, so that the equation (9.) remains true in this case also.

Wherefore the theorem is proved generally. See Note A.

General Theorem of Transformation.

12. The foregoing properties of the symbol Θ have an important bearing upon the general theorem for the transformation of integrals under the sign Σ, to the demonstration of which we shall now proceed.

THEOREM.—*If* $\mathrm{E}=0$ *be an equation connecting the variable* x *with another set of variables* $a_1, a_2, .. a_r$, *the function* E *being rational and entire with respect to* x, *and if* F *be any function of* x *and of* $a_1, a_2, .. a_r$ *which is rational with respect to* x, *then, provided that* F *does not become infinite when* $\mathrm{E}=0$, *we have*

$$\Sigma \mathrm{F}dx=\Theta[\mathrm{F}]\frac{\delta \mathrm{E}}{\mathrm{E}},$$

where δ *indicates complete differentiation with respect to the variables* $a_1, a_2, .. a_r$, *and the*

symbol Θ *directs us, according to previous definition, to develope the function* $F\frac{\delta E}{E}$ *in ascending powers of* $x-h_1$, $x-h_2$.. *the distinct simple factors of the denominator of* F, *to take in those successive developments the coefficients of* $\frac{1}{x-h_1}$, $\frac{1}{x-h_2}$, *&c., and from the sum of these coefficients to subtract the coefficient of* $\frac{1}{x}$ *in the development of the same function in descending powers of* x.

The object of this theorem will become apparent if it be compared with the statement in Art. 4. It will be observed that E stands for the rational and entire function $E(x, a_1, a_2, .. a_r)$, and F for the rational function $F(x, a_1, a_2 .. a_r)$ in that article. Thus F is that rational function of x to which the differential coefficient X in the integral $\int Xdx$ is supposed to be capable of being reduced by means of the equation of transformation $E=0$.

DEMONSTRATION.—13. First, it will be necessary to prove the following subsidiary proposition:—

PROPOSITION.—*If* $\phi(x)$ *be any rational function of* x, *and if* $E=0$ *be any equation rational and entire with respect to* x, *by which* x *is connected with a new set of variables* $a_1, a_2, .. a_r$, *then, provided that* $\phi(x)$ *does not become infinite when* $E=0$, *we have*

$$\Sigma\phi(x)=-\Theta[\phi(x)]\frac{d\log E}{dx}.$$

PROOF.—As $\phi(x)$ is a rational function of x, it is capable of being resolved into a series of terms, each of which is either of the form ax^i, or of the form $\frac{a}{(x-p)^i}$, a being constant, and i an integer.

Consider then, first, the expression

$$\Sigma ax^i, \quad \ldots \ldots \quad (1.)$$

the different values of x in the several terms under the sign Σ being roots of the equation $E=0$. Representing these roots by $x_1, x_2, .. x_n$, any two or more of which may be equal, we have

$$E=A(x-x_1)(x-x_2)..(x-x_n), \quad \ldots \ldots \quad (2.)$$

A being constant. Hence

$$\frac{d}{dx}\log E=\frac{1}{x-x_1}+\frac{1}{x-x_2}..+\frac{1}{x-x_n}. \quad \ldots \ldots \quad (3.)$$

Developing the several terms of the second member in descending powers of x, the aggregate coefficient of $\frac{1}{x^{i+1}}$ will be

$$x_1^i+x_2^i..+x_n^i,$$

or

$$\Sigma x^i.$$

Hence

$\Sigma ax^i=a\times$ coefficient of $\frac{1}{x^{i+1}}$ in the development of $\frac{d}{dx}\log E$ in descending powers of x.

$=$ coefficient of $\frac{1}{x}$ in the development of $ax^i\frac{d}{dx}\log E$ in descending powers of x.

Now $\Theta[ax^i]\frac{d}{dx}\log E = -$coefficient of $\frac{1}{x}$ in the development of $ax^i\frac{d}{dx}\log E$ by the definition of Θ, since, as the function within the brackets is entire, there will be no ascending developments. Hence

$$\Sigma ax^i = -\Theta[ax^i]\frac{d}{dx}\log E. \qquad (4.)$$

Consider, secondly, the expression

$$\Sigma\frac{a}{(x-p)^i}.$$

Now expressing (3.) in the form

$$\frac{d}{dx}\log E = \frac{1}{x-p-(x_1-p)} + \frac{1}{x-p-(x_2-p)} \dots + \frac{1}{x-p-(x_n-p)},$$

and developing the several terms in the second member in ascending powers of $x-p$, the aggregate coefficient of $(x-p)^{i-1}$ will be

$$-\left\{\frac{1}{(x_1-p)^i} + \frac{1}{(x_2-p)^i} \dots + \frac{1}{(x_n-p)^i}\right\} = -\Sigma\frac{1}{(x-p)^i}.$$

Hence $\Sigma\frac{1}{(x-p)^i} = -$coefficient of $(x-p)^{i-1}$ in the development of $\frac{d}{dx}\log E$ in ascending powers of $x-p$,

$= -$coefficient of $\frac{1}{x-p}$ in the development of $\frac{1}{(x-p)^i}\frac{d}{dx}\log E$ in ascending powers of $x-p$;

$\therefore \quad \Sigma\frac{a}{(x-p)^i} = -$coefficient of $\frac{1}{x-p}$ in the development of $\frac{a}{(x-p)^i}\frac{d}{dx}\log E$ in ascending powers of $x-p$.

But $\Theta\left[\frac{a}{(x-p)^i}\right]\frac{d}{dx}\log E =$ coefficient of $\frac{1}{x-p}$ in the development of $\frac{a}{(x-p)^i}\frac{d}{dx}\log E$ in ascending powers of $x-p$.

For 1st, the only distinct simple factor in the denominator of the expression within the brackets is $x-p$; 2ndly, there will be no term of the form $\frac{A}{x}$ in the development of $\frac{a}{(x-p)^i}\frac{d}{dx}\log E$ in descending powers of x, the first term of that development being evidently $\frac{na}{x^{i+1}}$. Hence

$$\Sigma\frac{a}{(x-p)^i} = -\Theta\left[\frac{a}{(x-p)^i}\right]\frac{d}{dx}\log E. \qquad (5.)$$

It appears from the above, that when we decompose the rational fraction $\phi(x)$ into a series of terms $\phi_1(x)$, $+\phi_2(x) \dots + \phi_m(x)$, which are individually either of the form ax^i or of the form $\frac{a}{(x-p)^i}$, we have

$$\Sigma\phi_1(x) + \Sigma\phi_2(x) \dots + \Sigma\phi_m(x) = -\Theta[\phi_1(x)]\frac{d\log E}{dx} - \Theta[\phi_2(x)]\frac{d\log E}{dx} \dots - \Theta[\phi_m(x)]\frac{d\log E}{dx}$$

$$= -\Theta[\phi_1(x) + \phi_2(x) \dots + \phi_m(x)]\frac{d\log E}{dx},$$

by art. 11. Whence

$$\Sigma\phi(x) = -\Theta[\phi(x)]\frac{d\log \mathrm{E}}{dx}, \quad \ldots \ldots \ldots \quad (6.)$$

which is the expression of the subsidiary proposition in question.

By this proposition, $\Sigma\phi(x)$, when the different values of x in the terms under the sign Σ are the roots of an equation $\mathrm{E}=0$, is determined as a function of any independent quantities $a_1, a_2, \ldots a_r$ with which x is connected by that equation. In order that these quantities may be independent, it is, of course, necessary that their number should not exceed the index of the degree of the equation $\mathrm{E}=0$.

14. It may be well before continuing the demonstration to exemplify the theorem just obtained. Suppose it then required to determine the value of the expression $\Sigma\frac{x}{x-a}$, the values of x being the roots of the algebraic equation $px^2+(1-p)x+q=0$. Here we have

$$\Sigma\frac{x}{x-a} = -\Theta\left[\frac{x}{x-a}\right]\frac{2px+1-p}{px^2+(1-p)x+q}.$$

Developing the function

$$-\frac{x}{x-a}\cdot\times\frac{2px+1-p}{px^2+(1-p)x+q}$$

in ascending powers of $x-a$, the coefficient of $\frac{1}{x-a}$ will evidently be

$$-a\frac{2pa+1-p}{pa^2+(1-p)a+q}.$$

Again, developing the same function in descending powers of x, the coefficient of $\frac{1}{x}$ will be -2.

Hence

$$\Sigma\frac{x}{x-a} = -\frac{2pa^2+(1-p)a}{pa^2+(1-p)a+q}+2 = \frac{(1-p)a+2q}{pa^2+(1-p)a+q}, \quad \ldots \ldots \quad (7.)$$

which is easily verified by the known theory of equations.

The quantity a may be itself a function of p and q without affecting the truth of the above result. The reasoning by which (6.) is established remains quite unaffected by the consideration whether the function $\phi(x)$ contains, together with x, any of the independent quantities $a_1, a_2, \ldots a_r$ or not, provided only that if they do enter into its expression they enter determinately, *e. g.* that the same value which is given to any radical, as $\sqrt{1+a^2}$, in one of the functions $\phi(x)$ shall be retained in all.

15. Now let us resume the expression $\Sigma \mathrm{F}dx$, in which the values of x are subject to the condition

$$\mathrm{E}=0.$$

As by this equation the value of x is made to depend upon the quantities $a_1, a_2, \ldots a_r$, we have, on differentiating with respect to all the variables at once,

$$\frac{d\mathrm{E}}{dx}dx+\frac{d\mathrm{E}}{da_1}da_1+\frac{d\mathrm{E}}{da_2}da_2\cdots+\frac{d\mathrm{E}}{da_r}da_r=0. \quad \ldots \ldots \ldots \quad (8.)$$

Or if we appropriate the symbol δ to express complete differentiation with respect to $a_1, a_2, \ldots a_r$,

$$\frac{dE}{dx}dx+\delta E=0.$$

Whence

$$dx=-\frac{\delta E}{\frac{dE}{dx}}. \quad \ldots \ldots \quad (9.)$$

Wherefore

$$\Sigma F dx=-\Sigma\frac{F\delta E}{\frac{dE}{dx}}. \quad \ldots \ldots \quad (10.)$$

Now F being a rational, and E a rational and entire function of x, the expression $\frac{F\delta E}{\frac{dE}{dx}}$ will be rational with respect to x, whence, by the subsidiary proposition just demonstrated,

$$\Sigma\frac{F\delta E}{\frac{dE}{dx}}=-\Theta\left[F\frac{\delta E}{\frac{dE}{dx}}\right]\frac{d\log E}{dx}.$$

Therefore

$$\Sigma F dx=\Theta\left[F\frac{\delta E}{\frac{dE}{dx}}\right]\frac{\frac{dE}{dx}}{E}. \quad \ldots \ldots \quad (11.)$$

Now the distinctive part of the performance of the operation Θ in the second member, consists in developing the entire function

$$F\frac{\delta E}{\frac{dE}{dx}}\times\frac{\frac{dE}{dx}}{E}, \text{ or } F\frac{\delta E}{E} \quad \ldots \ldots \quad (12.)$$

in ascending powers of certain simple factors of the form $x-p$, those simple factors being, 1st, such as are found in the denominator of F; 2ndly, such as are not found in the denominator of F, but are found in the denominator of $\frac{\delta E}{\frac{dE}{d}}$. It may be shown that the result of that portion of the operation Θ which depends upon the latter class of factors is 0. For the only factors of the form $x-p$ which produce terms of the form $\frac{a}{x-p}$ in the ascending development of (12.), and which are not found in the denominator of F, must be found in E. Let $x-p$ be any such factor, then we may write

$$E=H(x-p)^m,$$

where H does not contain $x-p$.

Therefore

$$\frac{\delta E}{\frac{dE}{dx}}=\frac{(x-p)^m\delta H-mH(x-p)^{m-1}\delta p}{(x-p)^m\frac{dH}{dx}+mH(x-p)^{m-1}}=\frac{(x-p)\delta H-mH\delta p}{(x-p)\frac{dH}{dx}+mH}. \quad \ldots \ldots \quad (13.)$$

The denominator of this expression does not contain $x-p$ as a factor. Hence there will be no factor of the above description in the denominator of the fraction within the brackets, and therefore no corresponding development in the performance of Θ. Thus the only factors which produce any effect are those found in the denominator of F. The part of the operation Θ expressed by $-C_{\frac{1}{x}}$ is of course unaffected by the nature of the function within the brackets.

On these accounts then the theorem (11.), seeing that its second member indicates the performance of the operation Θ on the function $F\frac{\delta E}{E}$, the interpretation of that operation being derived solely from the factor F, is reduced to the comparatively simple form

$$\Sigma F dx = \Theta[F]\frac{\delta E}{E}. \qquad (14.)$$

And in this form it constitutes the general theorem of transformation which it was required to demonstrate.

Application of the general Theorem of Transformation to the Comparison of Algebraical Transcendents.

16. In treating of the algebraical transcendents, I shall first exemplify the direct application of the general theorem of transformation to the solution of special problems, and for this application I shall select by preference examples known and familiar. I shall subsequently apply the theorem to the investigation of general formulæ from which the solution of all special problems may be derived.

There is no difficulty in the direct application of the theorem to special problems. The following directions will meet every case.

Let $\Sigma \int X dx$ be the expression whose finite value is to be found, the simultaneous values of x being determined by an equation

$$X = F(x, a_1, a_2 \ldots a_r), \qquad (1.)$$

$a_1, a_2, \ldots a_r$ being the new variables in terms of which the value of the integral expression is to be obtained. The second member, which we shall represent by F, is supposed rational with respect to x. Let also the equation (1.), made rational and entire with respect to x by reduction to the form

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} \ldots + p_n = 0 \qquad (2.)$$

be represented by $E=0$.

Then observing that X does not become infinite when $E=0$, we have

$$\left.\begin{aligned}\Sigma\int X dx &= \Sigma\int F dx\\ &= \int\Theta[F]\frac{\delta E}{E}\end{aligned}\right\}. \qquad (3.)$$

On performing the operation Θ in the second member, the function under the sign of integration will be an exact differential relatively to $a_1, a_2, \ldots a_r$, and, being integrated, will

give the value sought. If the number of integrals under the sign Σ is specified, suppose it n, the function F must be so chosen that the reduced equation $E=0$ may be of the nth degree.

The algebraic sign of each of the integrals in $\Sigma\int X dx$ will be the same as the sign of the corresponding function F, which, as being rational, is not ambiguous.

17. *Example* 1.—The following theorem is given by a writer in the Cambridge Mathematical Journal, vol. i. p. 268, as a generalization of a theorem of Mr. Fox TALBOT'S relating to the arcs of the equilateral hyperbola. The equation of any hyperbola referred to its asymptotes being $xy=\frac{a^2+b^2}{2}$, or for simplicity, $xy=m^2$, we have, supposing θ the angle between the asymptotes,

$$\text{arc} = \int \frac{\sqrt{x^4-2m^2x^2\cos\theta+m^4}}{x^2}\,dx.$$

Supposing then three values of x to be determined by the equation

$$\sqrt{x^4-2m^2x^2\cos\theta+m^4}=vx+m^2, \quad \text{(1.)}$$

the sum of the corresponding arcs will be

$$\frac{3}{2}v+\frac{m^2\cos\theta}{v}+\text{const.} \quad \text{(2.)}$$

To demonstrate this theorem, it must be observed that the equation (1.), reduced to a rational and integral form, becomes

$$x^3-(2m^2\cos^2\theta+v^2)x-2m^2v=0, \quad \text{(3.)}$$

which occupies the place of $E=0$, art. 16. By virtue of the same equation we have

$$\Sigma\int\frac{\sqrt{x^4-2m^2x^2\cos\theta+m^4}}{x^2}\,dx=\Sigma\int\frac{vx+m^2}{x^2}\,dx.$$

Thus we have, 1st, to transform the expression

$$\Sigma\frac{vx+m^2}{x^2}\,dx, \quad \text{(4.)}$$

the simultaneous values of x being determined by (3.); 2ndly, to integrate the result with respect to the new variable v.

Applying the general theorem of transformation, art. 12, we have

$$\Sigma\frac{vx+m^2}{x^2}\,dx=\Theta\left[\frac{vx+m^2}{x^2}\right]\frac{-(2xv+2m^2)\delta v}{x^3-(2m^2\cos\theta+v^2)x-2m^2v}. \quad \text{(5.)}$$

Here we must develope the function

$$\frac{vx+m^2}{x^2}\times\frac{-(2xv+2m^2)\delta v}{x^3-(2m^2\cos\theta+v^2)x-2m^2v}. \quad \text{(6.)}$$

in ascending powers of x, and take therein the coefficient of $\frac{1}{x}$. From this we must subtract the coefficient of $\frac{1}{x}$ in the development of the same function in descending powers of x.

Finally, we must integrate the result relatively to v.

Developing the two factors of (6.) by ordinary division in ascending powers of x, we may express it in the form

$$\left(\frac{m^2}{x^2}+\frac{v}{x}\right)\left(\frac{\delta v}{v}+\frac{v^2-2m^2\cos\theta}{2m^2v^2}x\delta v+\&c.\right).$$

Whence, on multiplication of the factors, we find for the coefficient of $\frac{1}{x}$,

$$\left(\frac{3}{2}-\frac{m^2\cos\theta}{v^2}\right)\delta v. \quad \ldots \quad (7.)$$

The coefficient of $\frac{1}{x}$ in the descending development will evidently be 0. Integrating, we have

$$\left(\frac{3}{2}v+\frac{m^2\cos\theta}{v}\right)+\mathrm{C} \quad \ldots \quad (8.)$$

for the value sought.

It must be observed that, in applying the theorem, the signs of the integrals under the symbol Σ, which would otherwise be ambiguous from the square root, are made determinate by the equivalent rational expression (4.). Each must be of the same sign as the corresponding value of the function $\frac{vx+m^2}{x^2}$.

18. In the example we have just considered, the equation of transformation connects x with but a single new variable v. In the following examples, which are intended to illustrate the doctrine of the comparison of the different orders of elliptic functions, two new variables are introduced.

Example 2.—Required the finite value of the expression

$$\Sigma\int\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}, \quad \ldots \quad (1.)$$

the simultaneous values of x being determined by the equation

$$\sqrt{(1-x^2)(1-c^2x^2)}=1+vx+wx^2. \quad \ldots \quad (2.)$$

By virtue of this equation (1.) assumes the rational form

$$\Sigma\int\frac{dx}{1+vx+wx^2}.$$

Again, the equation (2.) becomes rational and integral when expressed in the form

$$(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)=0, \quad \ldots \quad (3.)$$

and occupies the place of $\mathrm{E}=0$ in the general theorem. Hence we have

$$\Sigma\int\frac{dx}{1+vx+wx^2}=\int\Theta\left[\frac{1}{1+vx+wx^2}\right]\frac{2(1+vx+wx^2)(xdv+x^2dw)}{(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)}.$$

Here the function $$\frac{2(xdv+x^2dw)}{(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)}$$

is to be developed in ascending powers of the simple factors of $1+vx+wx^2$. Let $x-h$ be one of those factors. Then it is evident that there cannot be a term of the form $\frac{\mathrm{A}}{x-h}$ in the development of the function in ascending powers of $x-h$, inasmuch as that

factor does not exist in the term $(1-x^2)(1-c^2x^2)$. Again, there cannot be a term of the form $\frac{A}{x}$ in the development of the function in descending powers of x. Hence the result of the operation Θ is 0, and we have, finally,

$$\Sigma\int\frac{dx}{\sqrt{(1-x^2)(1-c^2x^2)}}=\text{const.} \qquad (4.)$$

The above constitutes in reality what is usually termed the fundamental theorem for the comparison of elliptic functions of the first order. The equation (3.), arranged according to the powers of x, gives

$$(w^2-c^2)x^3+2vwx^2+(v^2+2w+c^2+1)x+2v=0.$$

If we then represent the simultaneous values of x by x_1, x_2, x_3, we have

$$x_1+x_2+x_3=\frac{-2vw}{w^2-c^2}$$

$$x_1x_2+x_2x_3+x_3x_1=\frac{v^2+2w+c^2+1}{w^2-c^2}$$

$$x_1x_2x_3=\frac{2v}{w^2-c^2},$$

whence, eliminating v and w, we find

$$(1-x_1^2)(1-x_2^2)(1-x_3^2)=(2-x_1^2-x_2^2-x_3^2+c^2x_1^2x_2^2x_3^2)^2. \qquad (5.)$$

Now this is the form to which the known relation connecting the amplitudes ϕ, ψ, σ and the modulus c in the equation

$$\pm F_c(\phi)\pm F_c(\psi)\pm F_c(\sigma)=0,$$

viz. the relation, $\cos\sigma=\cos\phi\cos\psi-\sin\phi\sin\psi\sqrt{1-c^2\sin^2\sigma}$, is reduced, when we therein make $\sin\phi=x_1$, $\sin\psi=x_2$, $\sin\sigma=x_3$, and rationalize the resulting equation. The signs of the integrals will of course be determined by the signs of the function $1+vx+wx^2$.

To obtain a formula for the comparison of elliptic functions of the second order, we must deduce a finite expression for

$$\Sigma\int\sqrt{\frac{1-c^2x^2}{1-x^2}}\,dx,$$

subject to the relation (2.). We have then

$$\Sigma\int\sqrt{\frac{1-c^2x^2}{1-x^2}}\,dx=\Sigma\int\frac{1-c^2x^2}{1+vx+wx^2}\,dx$$

$$=\int\Theta\left[\frac{1-c^2x^2}{1+vx+wx^2}\right]\frac{2(1+vx+wx^2)(x\delta v+x^2\delta w)}{(1+vx+wx^2)-(1-x^2)(1-c^2x^2)}.$$

Here, as before, the effect of that part of the operation Θ which depends upon $1+vx+wx^2$ is 0, so that we have simply

$$\Sigma\int\sqrt{\frac{1-c^2x^2}{1-x^2}}\,dx=-\int C_{\frac{1}{x}}\,\frac{(1-c^2x^2)\times 2(x\delta v+x^2\delta w)}{(1+vx+wx^2)-(1-x^2)(1-c^2x^2)},$$

$C_{\frac{1}{x}}$ denoting the coefficient of $\frac{1}{x}$ in the development of the function in descending powers of x. Now developing the numerator and the denominator separately, we have

$$2(c^2x^2-1)(x^2\delta w+x\delta v)=2c^2x^4\delta w+2c^2x^3\delta v\,..$$
$$(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)=(w^2-c^2)x^4+2vwx^3\,..$$

Dividing the first of these by the second, we have a quotient

$$\frac{2c^2\delta w}{w^2-c^2}+\left(\frac{2c^2\delta v}{w^2-c^2}-\frac{4c^2wv\delta w}{(w^2-c^2)^2}\right)\frac{1}{x}+\&c.$$

Integrating the coefficient of $\frac{1}{x}$, which is a complete differential relatively to v and w, we have

$$\Sigma\int\sqrt{\frac{1-c^2x^2}{1-x^2}}\,dx=\frac{2c^2v}{w^2-c^2}+\text{constant}=c^2x_1x_2x_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

The signs of the three integrals under the sign Σ, are of course determined by the signs of the calculated values of $\frac{1-c^2x^2}{1+vx+wx^2}$.

We might in the same way deduce the known formulæ for the comparison of elliptic functions of the third order. Or we might at once investigate a formula for the comparison of elliptic functions of every order. For the latter purpose we should have to evaluate the expression

$$\Sigma\int\frac{(a+bx^2)dx}{(1+nx^2)\sqrt{(1-x^2)(1-c^2x^2)}},$$

under which the three canonical forms of elliptic functions are comprehended, in subjection to the condition (2.). By the general theorem of transformation this value i

$$\int\Theta\left[\frac{a+bx^2}{(1+nx^2)(1+vx+wx^2)}\right]\frac{2(1+vx+wx^2)(x\delta v+x^2\delta w)}{(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)},$$

which may be reduced at once to the form

$$\int\Theta\left[\frac{a+bx^2}{1+nx^2}\right]\frac{2x\delta v+2x^2\delta w}{(1+vx+wx^2)^2-(1-x^2)(1-c^2x^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

The rest of the solution involves no difficulty. We must develope the entire function following Θ in ascending powers of $x+\frac{1}{\sqrt{n}}\sqrt{-1}$ and $x-\frac{1}{\sqrt{n}}\sqrt{-1}$ successively, and take therein the coefficients of $\frac{1}{x+\frac{1}{\sqrt{n}}\sqrt{-1}}$ and $\frac{1}{x-\frac{1}{\sqrt{n}}\sqrt{-1}}$. From their sum we must subtract the coefficient of $\frac{1}{x}$ in the development of the same function in descending powers of x, and integrate the result as a complete differential with respect to v and w.

The above results are entirely founded upon the assumed theorem of transformation,

$$\sqrt{(1-x^2)(1-c^2x^2)}=1+vx+wx^2.$$

But any other transformation which would connect x with two new variables, through

the medium of an equation of the third degree with reference to x, would lead to results possessing the same degree of generality. Thus the equation

$$\sqrt{(1-x^2)(1-c^2x^2)}=v+wx+cx^2,$$

which connects x with v and w, and constitutes when freed from surds an equation of the third degree with reference to x, might have been employed. I am not aware that the above forms have been employed before. LEGENDRE, in deducing the properties of Elliptic Functions from ABEL's theorem, sets out from a different assumption, and as I think a less simple one, leading however to equivalent results.

It is not my intention to enter here into the subject of the *connexion* of the different solutions which may thus be obtained. The theory of that connexion can however present no difficulties to those who are acquainted with the labours of JACOBI, RICHELOT and others, upon the differential equations on the integration of which the doctrine of the comparison of transcendents, as contemplated from another point of view, depends.

19. Before applying the general theorem of transformation to the investigation of general formulæ for the comparison of transcendents, I will say a few words upon ABEL's theorem, as well as upon the class of theorems to which it belongs.

ABEL's theorem is virtually an expression for

$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\phi(x)}},$$

$f(x)$ and $\phi(x)$ being polynomials, the simultaneous values of x in the several integrals being connected by the equation

$$\sqrt{\phi(x)} = \chi(x),$$

or

$$\phi(x) = \{\chi(x)\}^2,$$

where $\chi(x)$ is not restricted to being a polynomial, but is a rational function of x, in terms of the coefficients of which the value of the integral sum is to be determined. ABEL expresses $\phi(x)$ as the product of two polynomials $\phi_1(x)$, $\phi_2(x)$, a form which is obviously given to it in order to meet the case in which a rational fraction occurs under the radical sign, since we have

$$\Sigma\int\frac{f(x)}{x-a}dx\sqrt{\frac{\phi_1(x)}{\phi_2(x)}} = \Sigma\int\frac{f(x)\phi_1(x)dx}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}.$$

BROCH and others, including, I believe, ABEL himself, have considered very fully the more general case in which the polynomial $\phi(x)$ is raised to any fractional power whatever, and to this case may be reduced the still more general one in which a rational fraction takes the place of the polynomial. The reduction is however obviously far more complex than in the simpler case in which the index is $\frac{1}{2}$. I intend here to discuss this problem in a form sufficiently general to render all such reductions unnecessary.

20. PROBLEM.—Required a finite expression for the integral sum

$$\Sigma\int\phi\psi^{\frac{m}{n}}dx, \quad \ldots \ldots \ldots \quad (1.)$$

ϕ and ψ being any rational functions of x, the simultaneous values of x in the different integrals being determined by the equation

$$\psi^{\frac{m}{n}}=\chi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

χ also denoting any rational function of x of the form

$$\frac{a_0+a_1x+a_2x^2\,..+a_mx^m}{b_0+b_1x+b_2x^2\,..+b_nx^n}.$$

Our object is to determine (1.) as a function of $a_0, a_1, .. a_m, b_0, b_1, .. b_n$, which are arbitrary in value, whereas the coefficients in ϕ and ψ are definite in value and are usually numerical.

Representing ϕ, ψ and χ in the forms

$$\phi=\frac{p}{q}, \quad \psi=\frac{s}{t}, \quad \chi=\frac{u}{v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

p, q, s, t, u, and v being polynomials in x, the transforming equation (2.) assumes the form

$$\left(\frac{s}{t}\right)^{\frac{m}{n}}=\frac{u}{v}.$$

Whence

$$s^mv^n-t^mu^n=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

With this condition connecting the values of x in the several integrals with $a_0, a_1, .. b_0, b_1$, &c., we have to seek the value of the expression

$$\Sigma\int\frac{p}{q}\,\frac{u}{v}\,dx.$$

For to this form, rational with respect to x, the expression (1.) is reducible by virtue of the above relations.

Consider then $\Sigma\int\frac{pu}{qv}dx$ subject to (4.). To apply to this the general theorem, we must write therein

$$F=\frac{pu}{qv}, \; E=s^mv^n-t^mu^n.$$

We thus find

$$\Sigma\frac{pu}{qv}dx=\Theta\left[\frac{pu}{qv}\right]\delta\log(s^mv^n-t^mu^n)=n\Theta\left[\frac{pu}{qv}\right]\frac{s^mv^{n-1}\delta v-t^mu^{n-1}\delta u}{s^mv^n-t^mu^n}. \quad . \quad . \quad . \quad . \quad (5.)$$

We cannot in its present form integrate the second member, as the interpretation of Θ depends in part upon v, which contains some of the variables to which the integration has reference. As however the function which has to be developed in the performance of the operation Θ is a rational fraction relatively to u, viz.

$$\frac{pu}{qv}\,\frac{s^mv^{n-1}\delta v-t^mu^{n-1}\delta u}{s^mv^n-t^mu^n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

we can resolve it into partial fractions, and this resolution will, in virtue of the properties of Θ, enable us to effect the integration required.

The partial fraction which has v for its demominator will be $\frac{p}{q}\frac{\delta u}{v}$. Separating this, the entire fraction (6.) will assume the form

$$\frac{p}{q}\frac{\delta u}{v}+\frac{p}{q}\frac{s^m v^{n-1}\delta u-s^m u v^{n-2}\delta v}{t^m u^n-s^m v^n}.$$

Thus (5.) becomes

$$\Sigma\frac{p}{q}\frac{u}{v}dx=n\Theta\frac{p}{q}\frac{\delta u}{v}+n\Theta\frac{p}{q}\frac{s^m v^{n-1}\delta u-s^m u v^{n-2}\delta v}{t^m u^n-s^m v^n}, \quad \ldots \ldots \quad (7.)$$

Θ deriving its interpretation from q and v.

The first term in the second member, inasmuch as we may give to it the form

$$n\Theta\left[\frac{p\delta u}{qv}\right],$$

vanishes by virtue of (5.), art. 11. The second term may be reduced to the form

$$n\Theta\left[\frac{p}{q}\right]\frac{s^m v^{n-1}\delta u-s^m u v^{n-2}\delta v}{t^m u^n-s^m v^n}. \quad \ldots \ldots \ldots \quad (8.)$$

In proof of this, I observe that the function to which Θ is applied cannot have any of the simple factors of v in its denominator. No such factor is involved in q. For by supposition all the coefficients in q are definite, while those in v are arbitrary. Neither, again, can any of those simple factors be involved in $t^m u^n-s^m v^n$, for if so it will be involved in $t^m u^n$, and therefore either in t or u. But it is not involved in t, as the constants in t are definite; and it is not involved in u, for if it were $\frac{u}{v}$ would not be in its lowest terms.

The expression (8.) may be written in the form

$$n\Theta\left[\frac{p}{q}\right]\frac{\left(\frac{s}{t}\right)^m\frac{v\delta u-u\delta v}{v^2}}{\left(\frac{u}{v}\right)^n-\left(\frac{s}{t}\right)^m}=n\Theta[\phi]\frac{\psi^m\delta\chi}{\chi^n-\psi^m},$$

on replacing $\frac{p}{q}$, $\frac{s}{t}$ and $\frac{u}{v}$ by ϕ, ψ and χ.

Hence

$$\Sigma\phi\psi^{\frac{m}{n}}dx=n\Theta[\phi]\frac{\psi^m\delta\chi}{\chi^n-\psi^m}.$$

Therefore

$$\Sigma\int\phi\psi^{\frac{m}{n}}dx=n\int\Theta[\phi]\frac{\psi^m\delta\chi}{\chi^n-\psi^m}. \quad \ldots \ldots \ldots \quad (9.)$$

The symbols Θ and $\int$ in the second member are now independent and may be transposed. We thus have

$$\Sigma\int\phi\psi^{\frac{m}{n}}dx=n\Theta[\phi]\psi^m\int\frac{\delta\chi}{\chi^n-\psi^m}, \quad \ldots \ldots \ldots \quad (10.)$$

an expression of remarkable simplicity.

21. In applying this theorem we must effect the integration in the second member, regarding χ as the only variable, inasmuch as the variables a_0, b_0, &c. enter into the constitution of χ, but not into that of the other rational fractions ϕ and ψ. When the

integration is effected we must write for ϕ, ψ, and χ the several rational fractions for which they stand and then perform the operation Θ, as we are directed to do by the definition of that symbol.

On actual integration we have from (10.),

$$\Sigma\int\phi\psi^{\frac{m}{n}}dx=\Theta[\phi]\psi^{\frac{m}{n}}\Sigma\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\log\left\{\chi-\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}\right\}+n\Theta[\phi]\psi^{\frac{m}{n}}C, \quad (11.)$$

the summation in the second member extending from $r=0$ to $r=n-1$. The last term in that member is equivalent to an arbitrary constant. For ϕ and $\psi^{\frac{m}{n}}$ do not contain the variables a_0, b_0, &c., which are only found in χ. Hence the coefficients of terms in the developments of the function $\phi\psi^{\frac{m}{n}}$ will be determinate constants, and $n\Theta[\phi]\psi^{\frac{m}{n}}$ C, on account of the arbitrary factor C, will be itself an arbitrary constant.

22. We are thus led to the following theorem:—

THEOREM.—*The value of the expression* $\Sigma\int\phi\psi^{\frac{m}{n}}dx$, *where* ϕ *and* ψ *are any rational functions of* x, *and the simultaneous values of* x *in the several integrals under the sign* Σ *are determined by the algebraic equation* $\psi^{\frac{m}{n}}=\chi$ *in which* χ *is a rational function of* x, will be expressed by the formula

$$\Sigma\int\phi\psi^{\frac{m}{n}}dx=\Theta[\phi]\psi^{\frac{m}{n}}\Sigma\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\log\left\{\chi-\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}\right\}+C,$$

the summation in the second member extending from $r=0$ to $r=n-1$.

23. In the particular case in which $m=1$, $n=2$, we have

$$\Sigma\int\phi\sqrt{\psi}dx=\Theta[\phi]\sqrt{\psi}\log\frac{\chi-\sqrt{\psi}}{\chi+\sqrt{\psi}}. \quad \ldots\ldots \quad (12.)$$

Let us apply this theorem to the problem of Art. 17. We have

$$\phi=\frac{1}{x^2},\quad \psi=x^4-2m^2x^2\cos\theta+m^4,\quad \chi=vx+m^2;$$

$$\therefore\quad \Sigma\int\frac{\sqrt{x^4-2m^2x^2\cos\theta+m^4}}{x^2}dx$$

$$=\Theta\left[\frac{1}{x^2}\right]\sqrt{x^4-2m^2x^2\cos\theta+m^4}\log\frac{vx+m^2-\sqrt{x^4-2m^2x^2\cos\theta+m^4}}{vx+m^2+\sqrt{x^4-2m^2x^2\cos\theta+m^4}}.$$

First, then, we must develope the function in the second member in ascending powers of x, and seek the coefficient of $\frac{1}{x}$. Now

$$\sqrt{m^4-2m^2x^2\cos\theta+x^4}=m^2-x^2\cos\theta+\text{\&c.}$$

Substituting, the function becomes

$$\left(\frac{m^2}{x^2}-\cos\theta\ldots\right)\log\frac{vx+x^2\cos\theta\ldots}{2m^2+vx-x^2\cos\theta\ldots}.$$

But
$$\log(vx+x^2\cos\theta\,..)=\log x+\log v+\frac{x\cos\theta}{v}+\text{\&c.}$$
$$\log(2m^2+vx-x^2\cos\theta\,..)=\log 2m^2+\frac{vx-x^2\cos\theta}{2m^2}+\text{\&c.}$$
Substituting, we have
$$\left(\frac{m^2}{x^2}-\cos\theta\,..\right)\left\{\log x+\log v-\log 2m^2+\left(\frac{\cos\theta}{v}-\frac{v}{2m^2}\right)x\,..\right\},$$
wherein the coefficient of $\frac{1}{x}$ in the product is
$$\frac{m^2\cos\theta}{v}-\frac{v}{2}. \quad \ldots \ldots \quad (13.$$
In the second place, developing the function in descending powers of x, we have
$$\sqrt{x^4-2m^2x^2\cos\theta+m^4}=x^2-m^2\cos\theta,\ \text{\&c.},$$
which on substitution gives
$$\left(1-\frac{m^2\cos\theta}{x^2}\,..\right)\log\frac{-x^2+vx+m^2(1+\cos\theta)}{x^2+vx+m^2(1-\cos\theta)}$$
$$=\left(1-\frac{m^2\cos\theta}{x^2}\,..\right)\log\left(-1+\frac{2v}{x}\,..\right)$$
$$=\left(1-\frac{m^2\cos\theta}{x^2}\,..\right)\left(-\frac{2v}{x}\,..\right),$$
wherein the coefficient of $\frac{1}{x}$ is $-2v$. Hence, changing its sign and adding the result t (13.), we have
$$\Sigma\int\frac{\sqrt{x^4-2m^2x^2\cos\theta+m^4}}{x^2}\,dx=\frac{3}{2}v+\frac{m^2\cos\theta}{v}+\mathrm{C},$$
which agrees with (8.), art. 17.

It is, however, very much easier in the above problem, and perhaps in most others. apply at once the fundamental theorem of transformation, as already exemplified in solution.

24. ABEL'S theorem is of course included in that of Art. 23. To deduce it, we ı observe that its object is to determine the value of the expression
$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\varphi_1(x)\varphi_2(x)}}, \quad \ldots \ldots$$
the simultaneous values of x being connected by the following equation,—
$$\sqrt{\frac{\varphi_2(x)}{\varphi_1(x)}}=\frac{a_0+a_1x\,..+a_mx^m}{c_0+c_1x\,..+c_nx^n}.$$
To compare with the general theorem we must therefore write (1.) in the forr
$$\Sigma\int\frac{f(x)}{(x-a)\varphi_2(x)}\sqrt{\frac{\varphi_2(x)}{\varphi_1(x)}}\,dx.$$
Hence we must make in (1.),
$$\phi=\frac{f(x)}{(x-a)\varphi_2(x)},\quad \psi=\frac{\varphi_2(x)}{\varphi_1(x)},\quad \chi=\frac{a_0+a_1x\,..}{c_0+c_1x\,..}.$$

We thus find

$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}=\Theta\left[\frac{f(x)}{(x-a)\phi_2(x)}\right]\sqrt{\frac{\phi_2(x)}{\phi_1(x)}}\log\frac{\frac{a_0+a_1x\,..}{c_0+c_1x\,..}-\sqrt{\frac{\phi_2(x)}{\phi_1(x)}}}{\frac{a_0+a_1x\,..}{c_0+c_1x\,..}+\sqrt{\frac{\phi_2(x)}{\phi_1(x)}}}.$$

Here the function to be developed is

$$\frac{f(x)}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}\log\frac{(a_0+a_1x\,..)\sqrt{\phi_1(x)}-(c_0+c_1x\,..)\sqrt{\phi_2(x)}}{(a_0+a_1x\,..)\sqrt{\phi_1(x)}+(c_0+c_1x\,..)\sqrt{\phi_2(x)}},$$

the ascending developments having reference to $x-a$, and to the different simple factors, $x-h_1$, $x-h_2$.. of $\phi_2(x)$. The coefficient of $\frac{1}{x-a}$ in the first development is evidently

$$\frac{f(a)}{\sqrt{\phi_1(a)\phi_2(a)}}\log\frac{(a_0+a_1a\,..)\sqrt{\phi_1(a)}-(c_0+c_1a\,..)\sqrt{\phi_2(a)}}{(a_0+a_1a\,..)\sqrt{\phi_1 a}\;\;+(c_0+c_1a\,..)\sqrt{\phi_2(a)}}.$$

The coefficients of $\frac{1}{x-h_1}$, $\frac{1}{x-h_2}$ in the latter developments are 0. Hence we have

$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}=\frac{f(a)}{\sqrt{\phi_1(a)\phi_2(a)}}\log\frac{(a_0+a_1a\,..)\sqrt{\phi_1(a)}-(c_0+c_1a\,..)\sqrt{\phi_2(a)}}{(a_0+a_1a\,..)\sqrt{\phi_1(a)}+(c_0+c_1a\,..)\sqrt{\phi_2(a)}}$$
$$-C_{\frac{1}{x}}\frac{f(x)}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}\log\frac{(a_0+a_1x\,..)\sqrt{\phi_1(x)}-(c_0+c_1x\,..)\sqrt{\phi_2(x)}}{(a_0+a_1x\,..)\sqrt{\phi_1(x)}+(c_0+c_1x\,..)\sqrt{\phi_2(x)}},$$

which is ABEL's theorem. There is, however, nothing gained by the peculiar form in which it supposes the integral to be expressed. The resolution of the polynomial under the radical into two factors $\phi_1(x)$, $\phi_2(x)$ is only a substitute, and an inconvenient one, for the more general hypothesis of the theorem of art. 22, which permits the function under the radical sign to be any rational fraction.

25. The theorem of art. 22 is, I believe, more general than any which have been investigated with relation to the same well-marked class of transcendents. BROCH, JURGENSEN, and MINDING have given formulæ directly applicable to the case in which ψ is a polynomial*. Their results agree in substance with the above, under the particular restriction supposed, but they are far more complicated in expression. The introduction of the symbol Θ, definite in meaning and indicating the performance of operations which are always intelligible and always possible, greatly simplifies the expression of general theorems.

26. The most general form of the problem contemplated by ABEL in his theory of the comparison of transcendents may be thus expressed. Required a finite expression for $\Sigma\int f(x,y)dx$, $f(x,y)$ denoting a rational function of x and y, the latter of which quantities is itself an irrational function of x given by an algebraic equation of the form

$$p_0y^n+p_1y^{n-1}+p_2y^{n-2}..+p_n=0, \quad \ldots \quad (1.)$$

wherein p_0, p_1..p_n are rational and integral functions of x, the simultaneous values of x in the different integrals being determined by an equation of the form

$$y=r,$$

* CRELLE's Journal, vol. xxiii.

wherein r is a rational function of x, and of any new variables $a_1, a_2, .. a_r$ in terms of which the value $\Sigma\int f(x, y)dx$ is to be found*. JURGENSEN has remarked that this problem may be reduced to that of the determination of the value of the expression

$$\Sigma\int f(x)\phi(x, y)dx,$$

$f(x)$ being a rational function of x and $\phi(x, y)$ a rational and integral function of x and y; and under this form, adding also the restriction that p_0 the coefficient of the highest power of y in (1.) shall be unity, he has solved the problem. MINDING has investigated the solution when the above restriction is not imposed, but his analysis is in reality founded upon a transformation in which $p_0 y$ takes the place of y†.

We can, both with increased generality and with that gain of simplicity which results from the employment of the symbol Θ, solve the same problem by the method of this section. But as the comparison of the algebraical transcendents is not the most important object of this paper, I do not propose to enter here upon the investigation in its most general form, but shall demonstrate a theorem which, while it is sufficiently general for all practical ends, will at the same time serve to throw light upon a peculiarity in the theorem of art. 22 already demonstrated.

27. PROBLEM.—*Required, in finite terms, the value of the integral expression*

$$\Sigma\int\phi(x)ydx, \qquad (1.)$$

$\phi(\mathrm{x})$ *denoting a rational function of* x, *and* y *an irrational function of* x *determined by an equation of the* m*th degree*,

$$p_0y^m+p_1y^{m-1}+p_2y^{m-2}..+p_m=0, \qquad (2.)$$

where $\mathrm{p}_1..\mathrm{p}_m$ *are rational and integral functions of* x. *We shall suppose the different simultaneous values of* x *under the sign* Σ *determined by an equation*

$$y=\chi, \qquad (3.)$$

χ *being a rational function of* x *of the form* $\frac{\mathrm{u}}{\mathrm{v}}$, *in which* u *and* v *are polynomials.* The value of the integral expression (1.) is to be found in terms of the coefficients of those polynomials.

The equation (3.), cleared of the radicals contained in y, and arranged with respect to the powers of χ, will be

$$p_0\chi^m+p_1\chi^{m-1}..+p_m=0,$$

or writing for χ its value $\frac{u}{v}$, and clearing of fractions,

$$p_0u^m+p_1u^{m-1}v..+p_mv^m=0. \qquad (4.)$$

This equation, on substituting for u and v their values as polynomials, is rational and integral with respect to x, and occupies the place of the equation E=0 in the general theorem of transformation. We shall suppose it of the nth degree. It may of course

* ABEL's Works, vol. ii. p. 66.

† CRELLE's Journal, vol. xxiii.

be exhibited in the form

$$p_0(u-vy_1)(u-vy_2)\ldots(u-vy_m)=0, \quad \ldots \quad (5.)$$

$y_1, y_2 \ldots y_m$ being the different values of y, as determined by giving different signs to the radicals in its expression. Hence we have

$$\left.\begin{aligned}\Sigma\int\phi(x)y\,dx &= \Sigma\int\phi(x)\frac{u}{v}\,dx=\int\Theta\left[\phi(x)\frac{u}{v}\right]\delta\log p_0(u-vy_1)(u-vy_2)\ldots(u-vy_m)\\ &=\int\Theta\left[\phi(x)\frac{u}{v}\right]\Sigma\frac{\delta u-y_r\delta v}{u-y_r v}\end{aligned}\right\} \quad \ldots \quad (6.)$$

the summation in the second member extending from $r=1$ to $r=m$.

Here, transposing the symbol Σ, the function upon which the operation Θ, *whose interpretation is derived from* $\phi(x)\frac{u}{v}$, is to be performed, is

$$\phi(x)\frac{u}{v}\frac{\delta u-y_r\delta v}{u-y_r v},$$

which may be resolved into

$$\phi(x)y_r\frac{\delta u-y_r\delta v}{u-y_r v}+\frac{\phi(x)}{v}(\delta u-y_r\delta v).$$

Hence

$$\Sigma\int\phi(x)y\,dx=\Sigma\int\Theta\phi(x)y_r\frac{\delta u-y_r\delta v}{u-y_r v}+\Sigma\int\Theta\frac{\phi(x)}{v}(\delta u-y_r\delta v). \quad \ldots \quad (7.)$$

We are especially to remark, that while Σ in the first member has reference to the n different values of x furnished by the equation (3.) or (4.), Σ in the second member has reference to the different values of y furnished by the equation (2.), its numerical range being from $r=1$ to $r=m$.

Now considering the term $\quad \Sigma\int\Theta\phi(x)y_r\frac{\delta u-y_r\delta v}{u-y_r v}$,

all that part of the operation Θ which depends upon v produces no effect. For none of the factors of v can enter in any way into y_r, since those factors contain the variables $a_0, a_1 \ldots a_r$, from which y_r is wholly free. Again, they cannot enter into the denominator $u-y_r v$, for then they would enter into u, and the fraction $\frac{u}{v}$ would not be in its lowest terms. Hence the first term in the second member of (7) becomes

$$\Sigma\int\Theta[\phi(x)]y_r\frac{\delta u-y_r\delta v}{u-y_r v}.$$

We can now transpose the symbols $\int$ and Θ and integrate. The result is

$$\Theta[\phi(x)]\Sigma y_r\log(u-y_r v). \quad \ldots \quad (8.)$$

The last term of (7.) will, on account of the interpretation of Θ, be properly written in the form

$$\Sigma\Theta\left[\phi(x)\frac{1}{v}\right](\delta u-y_r\delta v),$$

which may be resolved into

$$\Sigma\Theta\left[\phi(x)\frac{1}{v}\right]\delta u-\Sigma\Theta\left[\phi(x)\frac{1}{v}\right]y_r\delta v.$$

The first term vanishes by Art. 11. The second may be reduced as follows. We have

$$-\Sigma\Theta\left[\phi(x)\frac{1}{v}\right]y_r\delta v=-\Theta\left[\phi(x)\frac{1}{v}\right]\Sigma y_r\delta v=\Theta\left[\phi(x)\frac{1}{v}\right]\frac{p_1}{p_0}\delta v.$$

Now $$\Theta\left[\phi(x)\frac{1}{v}\right]\frac{p_1}{p_0}\delta v+\Theta\left[\phi(x)\frac{p_1}{p_0}\right]\frac{1}{v}\delta v=\Theta\left[\phi(x)\frac{1}{v}\frac{p_1}{p_0}\right]\delta v+\Theta\left[\phi(x)\right]\frac{1}{v}\frac{p_1}{p_0}\delta v. \quad . \quad . \quad (9.)$$

This is evident if we collect the different parts of the interpretation of Θ from the terms in each member, observing that in all cases it is upon the same function that Θ operates. Now the first term in the second member vanishes by Art. 11. Hence

$$\Theta\left[\phi(x)\frac{1}{v}\right]\frac{p_1}{p_0}\delta v=\Theta\left[\phi(x)\right]\frac{p_1}{p_0}\frac{\delta v}{v}-\Theta\left[\phi(x)\frac{p_1}{p_0}\right]\frac{\delta v}{v}.$$

Attaching now the symbol of integration to the second member, and integrating, since $\int$ and Θ are now transposable, we have

$$\Theta[\phi(x)]\frac{p_1}{p_0}\log v-\Theta\left[\phi(x)\frac{p_1}{p_0}\right]\log v.$$

Writing in the first term of this expression $-\Sigma y_r$ for $\frac{p_1}{p_0}$ and adding the result to (8.), we have

$$\Theta[\phi(x)]\{\Sigma y_r\log(u-y_rv)-\Sigma y_r\log v\}-\Theta\left[\phi(x)\frac{p_1}{p_0}\right]\log v,$$

or

$$\Theta[\phi(x)]\Sigma y_r\log\left(\frac{u}{v}-y_r\right)-\Theta\left[\phi(x)\frac{p_1}{p_0}\right]\log v.$$

Hence replacing $\frac{u}{v}$ by χ and adding the constant of integration, we have

$$\Sigma\int\phi(x)ydx=\Theta[\phi(x)]\Sigma y_r\log(\chi-y_r)-\Theta\left[\phi(x)\frac{p_1}{p_0}\right]\log v+\mathrm{C}, \quad . \quad . \quad . \quad (10.)$$

the expression required. It will be observed that as $\phi(x)$ and $\frac{p_1}{p_0}$ are always rational functions, the operation Θ may always be performed on each term of the second member.

As the summation $\Sigma y_r\log(\chi-y_r)$ can only be effected by connecting the several terms included under Σ by the sign of addition, it will be most convenient to express the solution in the following form,

$$\Sigma\int\phi(x)ydx=\Sigma_{r=1}^{r=n}\Theta[\phi(x)]y_r\log(\chi-y_r)-\Theta\left[\phi(x)\frac{p_1}{p_0}\right]\log v+\mathrm{C}. \quad . \quad . \quad . \quad (11.)$$

28. If $p_1=0$, we have

$$\Sigma\int\phi(x)ydx=\Sigma_{r=1}^{r=n}\epsilon[\phi(x)]y_r\log(\chi-y_r)+\mathrm{C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

This includes as a particular case the theorem of Art. 22. For if $y=\psi^{\frac{m}{n}}$, we have

$$y^n-\psi^m=0,$$

of which any root y_r will be given by the formula

$$y_r=\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}.$$

Hence on substitution we have, since the number of values of y is n,

$$\Sigma\int\phi(x)\psi^{\frac{m}{n}}dx=\Sigma_{r=1}^{r=n}\Theta[\phi(x)]\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}\log\left\{\chi-\left(\cos\frac{2r\pi}{n}+\sqrt{-1}\sin\frac{2r\pi}{n}\right)\psi^{\frac{m}{n}}\right\},$$

which agrees with the result in question.

We now see why it is that, although in the investigation of the latter formula we had to take distinct account of the terms u and v in the fraction χ; in the final result they are recombined, and only present themselves implicitly as component parts of χ. It is due to the fact that the equation determining the irrational factor of the original integral wants a second term, *i. e.* that $p_1=0$.

If $p_0=1$, the term $-\Theta\left[\phi(x)\frac{p_1}{p_0}\right]\log v$ in (11.) becomes $-\Theta[\phi(x)]p_1\log v$, or $-\Sigma\Theta[\phi(x)]y_r\log v$, whence the theorem assumes the following form,

$$\Sigma\int\phi(x)ydx=\Sigma_{r=1}^{r=n}\Theta[\phi(x)]y_r\{\log(\chi-y_r)+\log v\}=\Sigma_{r=1}^{r=n}\Theta[\phi(x)]y_r\log(u-y_rv).$$

We may remark in concluding this section, that all general theorems like the above for the comparison of algebraical transcendents are difficult of application, from the necessity which they impose of developing logarithmic and irrational functions. This difficulty we avoid by employing directly the theorem of transformation exemplified in the earlier problems of this section. The application of that theorem requires only the development of rational fractions, and this can always be effected by the operations of multiplication and division. When this form of procedure is adopted there remains, however, an integration to be performed. Circumstances must decide which of these methods is preferable, but generally I conceive it will be the latter one. I will only add, that the interest attaching to the entire subject of the comparison of algebraical transcendents appears to me to be chiefly of a speculative character. It is to be valued rather as affording evidence of the powers, and at the same time of the limitations of analysis, than as offering any prospect of increased command over the problems of physical science. Such at least seems to be the tenor of present indications.

Application of the General Theorem of Transformation to the reduction of Functional Transcendents.

29. Let us first apply the general theorem to the reduction of the expression $\Sigma\int\phi f(\psi)dx$, where ϕ and ψ are any rational functions of x, and f a general functional symbol, the simultaneous values of x in the several integrals being determined by an equation,

$$\psi=v, \qquad (1.)$$

in which v is a new variable.

We have

$$\Sigma\int\phi f(\psi)dx=\Sigma\int f(v)\phi dx=\int\Sigma f(v)\phi dx. \qquad (2.)$$

We must now seek the value of the expression $\Sigma f(v)\phi dx$, subject to the condition (1.), and then integrate with respect to v.

Now ψ as a rational function of x may be represented under the form $\frac{\mathrm{P}}{\mathrm{Q}}$, where P and Q are polynomials in x not involving v. The equation of transformation then becomes

$$\frac{\mathrm{P}}{\mathrm{Q}}=v,$$

and in its rational and integral form $\mathrm{Q}v-\mathrm{P}=0$.

Hence by the general theorem of transformation,

$$\Sigma f(v)\phi dx=\Theta[f(v)\phi]\delta\log(\mathrm{Q}v-\mathrm{P})=\Theta[f(v)\phi]\frac{\mathrm{Q}\delta v}{\mathrm{Q}v-\mathrm{P}}.$$

Now the factor $f(v)$, inasmuch as it does not contain x, in nowise affects the interpretation of Θ. Hence it may be removed from within the brackets and the equation written in the form

$$\Sigma f(v)\phi dx=f(v)\Theta[\phi]\frac{\mathrm{Q}\delta v}{\mathrm{Q}v-\mathrm{P}}=f(v)\Theta[\phi]\frac{\delta v}{v-\psi},$$

on replacing $\frac{\mathrm{P}}{\mathrm{Q}}$ by ψ.

Hence, replacing the first member by the expression for which it is an equivalent, and attaching the sign of integration,

$$\Sigma\int\phi f(\psi)dx=\int f(v)\Theta[\phi]\frac{\delta v}{v-\psi}. \qquad (3.)$$

To this result we may also give the form

$$\Sigma\int\phi f(\psi)dx=\Theta[\phi]\int\frac{f(v)\delta v}{v-\psi}, \qquad (4.)$$

as the symbol Θ and the function ϕ are both independent of the variable v. And this would in fact be the best form if we could effect generally the integration in the second member. For the applications to which we shall proceed (3.), is, however, the form to be preferred.

We may express the results which have been arrived at in the following general theorem.

THEOREM.—*If* ϕ *and* ψ *are rational functions of* x, *and if the simultaneous values of* x *in the integrals included in the expression* $\Sigma\int\phi f(\psi)\mathrm{d}x$ *are roots of an equation*

$$\psi=v,$$

v *being a variable quantity, then*

$$\Sigma\int\phi f(\psi)dx=\int f(v)\Theta[\phi]\frac{\delta v}{v-\psi}.$$

30. Apparently this is the most general theorem which exists with relation to that class of transcendents in which a perfectly arbitrary symbol of functionality occurs under the sign of integration. If we specify the form of f, so as to meet the case of the particular transcendents discussed in the previous sections of this paper, we shall obtain results accordant with, but less general than, those which have been there obtained. But the most important feature of the theorem is, that, without restricting the generality

of the functional symbol f, we may so determine the form of ψ as to cause the several integrals included under the sign Σ in the first member to *close up*, if the expression is permitted, into a single definite integral whose actual value will then be given by that of a far more simple integral in the second member.

Let the limits of v in the second member be p and q, and let the transforming equation

$$\psi=v$$

give $p_1, p_2..p_{n+1}$ for the values of x when $v=p$, and $q_1, q_2..q_{n+1}$ for its corresponding values when $v=q$. Then we have

$$\int_{p_1}^{q_1}\varphi f(\psi)dx+\int_{p_2}^{q_2}\varphi f(\psi)dx..+\int_{p_{n+1}}^{q_{n+1}}\varphi f(\psi)dx=\int_p^q f(v)\Theta[\varphi]\frac{\delta v}{v-\psi}. \quad . \quad . \quad . \quad (5.)$$

Now let us give to ψ the form

$$x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}..-\frac{a_n}{x-\lambda_n},$$

where $\lambda_1, \lambda_2, ..\lambda_n$ are real, and $a_1, a_2..a_n$ real and positive. The transforming equation is then

$$x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}..-\frac{a}{x-\lambda_n}=v; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

whence representing still the first member by ψ,

$$\frac{d\psi}{dx}=1+\frac{a_1}{(x-\lambda_1)^2}+\frac{a_2}{(x-\lambda_2)^2}..+\frac{a_n}{(x-\lambda_n)^2};$$

and as this expression is always positive, it follows,—1st, that ψ regarded as a function of x is never a maximum or minimum; 2ndly, that whenever ψ varies continuously while x increases, it varies by way of increase.

From these properties, and from the form of the equation (6.), it readily follows that if $\lambda_1, \lambda_2, ..\lambda_n$ are arranged in the order of increasing magnitude, then whatever real value v may have, the roots of (6.) will be real and will be disposed in the following manner, viz. one root less than λ_1, one root between λ_1 and λ_2.., one between λ_{n-1} and λ_n, and finally one between λ_n and ∞. To prove this in detail, let x vary from $-\infty$ to λ_1, then ψ, as is evident from its form, varies from $-\infty$ to ∞, and it varies continuously by way of increase so as never to resume a former value. Once therefore in its course it will be equal to v. Wherefore one root of (6.) lies between $-\infty$ and λ_1. Supposing x to continue to increase, the value of ψ suddenly changes when x passes over the value λ_1 from ∞ to $-\infty$, and as x varies from λ_1 to λ_2, ψ again varies continuously, and by way of increase from $-\infty$ to ∞, and therefore again becomes equal to v once in its course; wherefore a value of x lies between λ_1 and λ_2. In like manner there is a value of x between λ_2 and λ_3.. λ_{n-1} and λ_n. Finally, as x varies from λ_n to ∞, ψ once more passes over the value v. Whence the proposition is manifest.

The reality of all the roots of (6.) may also be readily shown in the following manner. Let $x=p+q\sqrt{-1}$. Then substituting in (6.), and reducing that equation to the form

$$A+B\sqrt{-1}=v,$$

which entails as a necessary consequence $A=v$, $B=0$, we find as the form of the equation $B=0$,

$$q\left\{1+\frac{a_1}{(p-\lambda_1)^2+q^2}+\frac{a_2}{(p-\lambda_2)^2+q^2}\cdots+\frac{a_n}{(p-\lambda_n)^2+q^2}\right\}=0.$$

Now as the function within the brackets is essentially positive, the above equation can only be satisfied by making $q=0$. But this indicates that all the roots are real.

Resuming (5.), it is evident from what precedes, that if the lower limits of integration $p_1, p_2 \ldots p_{n+1}$, corresponding to $v=p$, are arranged in the ascending order of magnitude, the upper limits $q_1, q_2 \ldots q_{n+1}$ will also be ranged in the same order. Moreover p_1 and q_1 will both be less than λ_1; p_2 and q_2 will lie between λ_1 and λ_2; finally, p_{n+1} and q_{n+1} will lie between λ_n and ∞. Hence, then, the elements in the different integrals in the first member of (5.) will be all different, the superior limit of each integral being less than the inferior limit of the integral which follows it.

31. Let us now examine the case in which the integration relative to v in the second member of (5.) is from $-\infty$ to ∞.

Let $p=-\infty$, and $q=\infty$, we then have

$$p_1=-\infty,\quad p_2=\lambda_1,\quad p_3=\lambda_2 \ldots p_{n+1}=\lambda_n,$$
$$q_1=\lambda_1,\quad q_2=\lambda_2,\quad q_3=\lambda_3 \ldots q_n=\lambda_n,\quad q_{n+1}=\infty.$$

Thus $q_1=p_2$, $q_2=p_3 \ldots q_n=p_{n+1}$, or the upper limit of each integral coincides with the lower limit of the integral which follows it.

It is more strict, however, to regard p and q as *tending* to the respective limits $-\infty$ and ∞. The first inferior limit, p_1, then tends to $-\infty$, and the last superior limit, q_{n+1}, to ∞, while the superior limit of each integral but the last tends upward to the same limiting value to which the inferior limit of the integral following tends downward. The different integrals close up into a single definite integral taken between the limits $-\infty$ and ∞. Thus we have

$$\int_{-\infty}^{\infty}\varphi f(\psi)dx=\int_{-\infty}^{\infty}f(v)\Theta[\varphi]\frac{\delta v}{v-\psi}. \quad \text{(7.)}$$

The reasoning is evidently independent of the nature of the function symbolized by f. That function may either be continuous or discontinuous. We thus arrive at the following theorem, in the expression of which we shall restore to ψ its complete value, and shall replace the rational function φ by $\varphi(x)$, and the symbol δ, which is no longer necessary for distinction, by d.

32. Theorem.—*If* $\varphi(x)$ *denote a rational function of* x, *and if* f *be a general functional symbol, then*

$$\int_{-\infty}^{\infty}\varphi(x)f\left(x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}\cdots-\frac{a_n}{x-\lambda_n}\right)dx$$
$$=\int_{-\infty}^{\infty}dvf(v)\Theta[\varphi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}\cdots+\frac{a_n}{x-\lambda_n}}, \quad \text{(1.)}$$

provided that $a_1, a_2, \ldots a_n$ *are real and positive, and* $\lambda_1, \lambda_2 \ldots \lambda_n$ *real.*

This is the general theorem of definite integration to which reference has been made. The remainder of this paper will be devoted to its illustration.

The theorem is independent, as has been said, of the nature of the functional interpretation of f, and, even when the factor $f\left(x-\frac{a}{x-\lambda_1}\cdot\cdot-\frac{a}{x-\lambda_n}\right)$ becomes infinite for the limiting values $\lambda_1, \lambda_2 \ldots \lambda_n$, does not fail, but carries with it a correction for the discontinuity thence arising. We cannot otherwise attach a meaning to the expression $\int_a^b f(x)dx$, when for a value $x=\lambda$, included within the limits of integration, $f(x)$ becomes infinite, than by considering it as the limit to which the sum of the integrals

$$\int_a^{\lambda-e} f(x)dx+\int_{\lambda+e'}^b f(x)dx$$

tend as e and e' tend to 0. According to the nature of the function $f(x)$ and the modes in which e and e' tend to the limit 0, the integral may become, as CAUCHY has observed, finite or infinite, determinate or indeterminate. When $e'=e$, so that the approach on either side to the limiting value λ is made in the same manner, *i. e.* by equivalent infinitesimal variations of x, the value of the integral obtained will be that which CAUCHY terms its *principal* value. The equation (1.) will thus give the principal value of the integral in its first member, if we suppose v to approach by the same kind of variations to the limits $-\infty$ and ∞; in other words, if, representing the function under the sign of integration in the second member by $\mathrm{F}(v)$, we regard $\int_{-\infty}^{\infty} \mathrm{F}(v)dv$ as the limit of the value of $\int_{-a}^{a} \mathrm{F}(v)dv$, a becoming indefinitely great. For suppose x to be approaching the particular limit λ_1. The nearer its approach the more nearly (*vide* 6, art. 30) is the following equation realized, viz.—

$$\frac{-a_1}{x-\lambda_1}=v,$$

whence the more nearly have we

$$x=\lambda_1+\frac{a_1}{v};$$

and therefore, if v tend towards ∞ and $-\infty$ by equivalent variations, so also will x by equivalent variations approach from above and from below the limit λ_1.

Again, the larger x becomes the more nearly do x and v approach a ratio of equality, and therefore the mode of approach of x in the first member to the limits $-\infty$ and ∞ determines identically the mode of approach of v to $-\infty$ and ∞. Thus we may finally give to the theorem the following rigorous statement, viz.—

The two members of the equation

$$\int_{-a}^{a} dx\phi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdot\cdot-\frac{a_n}{x-\lambda_n}\right)=\int_{-a}^{a} dvf(v)\Theta[\phi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdot\cdot+\frac{a_n}{x-\lambda_n}}$$

approach a ratio of equality as a *approaches to infinity, provided that if*

$$f\left(x-\frac{a_1}{x-\lambda_1}\cdot\cdot-\frac{a_n}{x-\lambda_n}\right)$$

become infinite for the critical values $\lambda_1, \lambda_2 \ldots \lambda_n$, we suppose x *to approach each critical value by equivalent infinitesimal variations.*

The most important, perhaps the only important, cases are those in which $f(v)$ vanishes when v is infinite.

33. I shall begin with noticing some particular deductions from the theorem, among which will be included certain known formulæ of analysis. I shall show that it enables us to deduce from any known definite integral the values of an infinite number of other definite integrals of progressively increasing complexity. I shall show that when the arbitrary function under the sign of integration is regarded as discontinuous, the first member of the equation becomes resolved into a number of definite integrals of continuous functions, and that we thus arrive at the same theorem for the comparison of functional transcendents, (5.), art. 30, from which the above theorem was itself derived. Finally, I shall apply the theorem to the extension of the theory of definite multiple integrals.

Special deductions may be obtained by limiting either,—1st, the form of the rational function $\phi(x)$; or 2ndly, the interpretation of the functional sign f; or 3rdly, the number and value of the constants in the function under the sign f.

1st. Let $\phi(x)=1$. Then in the second member of (1.), art. 32,

$$\Theta[\phi(x)]\frac{dv}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}=-\mathrm{C}_{\frac{1}{x}}\frac{dv}{v-x+\frac{a}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}$$

$$=\mathrm{C}_{\frac{1}{x}}\frac{dv}{x-v-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}}$$

$$=dv.$$

Hence

$$\int_{-\infty}^{\infty}f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)dx=\int_{-\infty}^{\infty}f(v)dv. \quad \ldots \ldots \quad (2.)$$

This was the theorem, or rather the most important case of the theorem referred to in Art. 1, as published in the 'Cambridge and Dublin Mathematical Journal,' No. XIX. The following are special applications of it, chiefly selected from the paper in question.

Since we have

$$\int_{-\infty}^{\infty}f\left(x-\frac{a}{x}\right)dx=\int_{-\infty}^{\infty}f(v)dv, \quad \ldots \ldots \quad (3.)$$

and

$$x^2+\frac{a^2}{x^2}=\left(x-\frac{a}{x}\right)^2+2a,$$

we shall have

$$\int_{-\infty}^{\infty}f\left(x^2+\frac{a^2}{x^2}\right)dx=\int_{-\infty}^{\infty}f(v^2+2a)dv. \quad \ldots \ldots \quad (4.)$$

Hence

$$\int_{-\infty}^{\infty}\epsilon^{-\left(x^2+\frac{a^2}{x^2}\right)}dx=\int_{-\infty}^{\infty}\epsilon^{-2a}\times\epsilon^{-v^2}dv=\pi^{\frac{1}{2}}\epsilon^{-2a}. \quad \ldots \ldots \quad (5.)$$

whence, reducing the above expression for u, we find

$$\int_0^\infty \frac{dx.x^{n-\frac{1}{2}}}{(a+bx+cx^2)^n}=\frac{1}{c^{\frac{1}{2}}(b+2\sqrt{ac})^{n-\frac{1}{2}}}\cdot\frac{\Gamma(\frac{1}{2})\Gamma(n-\frac{1}{2})}{\Gamma(n)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

and hence on changing x into $\frac{1}{x}$, we find

$$\int_0^\infty \frac{dx.x^{n-\frac{3}{2}}}{(a+bx+cx^2)^n}=\frac{\Gamma(\frac{1}{2})\Gamma(n-\frac{1}{2})}{\Gamma(n)a^{\frac{1}{2}}(b+2\sqrt{ac})^{n-\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

The two last theorems were discovered independently and about the same time by Mr. CAYLEY, Professor THOMSON* and SCHLÖMILCH†. It is to be noted that a, b, and c must be positive.

The above examples have been selected in the first instance because they relate to known results. But there is not one of the results arrived at which may not be generalized to an indefinite degree.

Thus, since we have

$$\int_{-\infty}^\infty dv\varepsilon^{-v^n}=\frac{2}{n}\Gamma\left(\frac{1}{n}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

we have

$$\int_{-\infty}^\infty dx\varepsilon^{-\left(x-\frac{a}{x}\right)^n}=\frac{2}{n}\Gamma\left(\frac{1}{n}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

If $n=2$, this gives

$$\int_{-\infty}^\infty dx\varepsilon^{2a-\left(x^2+\frac{a^2}{x^2}\right)}=\pi^{\frac{1}{2}},$$

whence

$$\int_{-\infty}^\infty \varepsilon^{-\left(x^2+\frac{a^2}{x^2}\right)}dx=\pi^{\frac{1}{2}}\varepsilon^{-2a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

agreeing with (5.). But let $n=4$, and we find

$$\int_{-\infty}^\infty \varepsilon^{-\left(x^4+\frac{a^4}{x^4}\right)+4a\left(x^2+\frac{a^2}{x^2}\right)}dx=\frac{1}{2}\Gamma(\tfrac{1}{4})\varepsilon^{6a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.)$$

and so on indefinitely without even proceeding to employ the more general forms of (2.).

34. Let us examine the definite integral

$$u=\int_{-\infty}^\infty x^{2n}f\left(x-\frac{a}{x}\right)dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

By the general theorem (1.) we have

$$u=\int_{-\infty}^\infty dvf(v)\Theta[x^{2n}]\frac{1}{v-x+\frac{a}{x}}=\int_{-\infty}^\infty dvfv\mathrm{C}_{\frac{1}{x}}\frac{x^{2n+1}}{x^2-vx-a} \quad . \quad . \quad . \quad . \quad . \quad (18.)$$

For, x^{2n} not being fractional, the interpretation of Θ is reduced simply to $-\mathrm{C}_{\frac{1}{x}}$. It is obviously desirable to express the term $\mathrm{C}_{\frac{1}{x}}\frac{x^{2n+1}}{x^2-vx-a}$ in a series consisting of powers of v.

Now

$$\frac{x^{2n+1}}{x^2-vx-a}=x^{2n+1}\left\{\frac{1}{x^2-a}+\frac{vx}{(x^2-a)^2}+\frac{v^2x^2}{(x^2-a)^3}+\&c.\right\}=\frac{x^{2n+1}}{x^2-a}+v\frac{x^{2n+2}}{(x^2-a)^2}+v^2\frac{x^{2n+3}}{(x^2-a)^3}+\&c. \quad . \quad (1$$

* Cambridge and Dublin Mathematical Journal, vol. ii.

† CRELLE, vol. xxxii.

We are permitted to give this form to the expression before developing in descending powers of x, because, on thus developing the several terms in the second member, no higher power of x will present itself than would be obtained by developing the first member in descending powers of x.

The general term of (19.) is

$$\frac{v^i . x^{2n+i+1}}{(x^2-a)^{i+1}}.$$

If i be odd, there will be no terms of the form $\frac{A}{x}$ in the development of this function in descending powers of x. Let i be even and be expressed by $2m$. Then we have to develope in descending powers of x a series of functions of the form

$$\frac{v^{2m} x^{2n+2m+1}}{(x-a)^{2m+1}}.$$

The coefficient of $\frac{1}{x}$ in the development of this expression is easily found to be

$$\frac{(2m+1)(2m+2)..(n+m)}{1.2..(n-m)} a^{n-m} v^{2m}.$$

Therefore

$$C_{\frac{1}{x}} \frac{x^{2n+1}}{x^2-vx-a} = \Sigma \frac{(2m+1)(2m+2)..(n+m)}{1.2..(n-m)} a^{n-m} v^{2m},$$

the summation extending from $m=0$ to $m=n$. Hence

$$\int_{-\infty}^{\infty} x^{2n} f\left(x-\frac{a}{x}\right) dx = \Sigma_{m=0}^{m=n} \frac{(2m+1)(2m+2)..(n+m)}{1.2..(n-m)} a^{n-m} \int_{-\infty}^{\infty} dv v^{2m} f(v). \quad . \quad . \quad (20.)$$

In the particular case in which the function denoted by f is even, we have, on replacing $f(x)$ by $\varphi(x^2)$,

$$\int_{-\infty}^{\infty} x^{2n} \varphi\left\{\left(x-\frac{a}{x}\right)^2\right\} dx = \Sigma_{m=0}^{m=n} \frac{(2m+1)(2m+2)..(n+m)}{1.2..(n-m)} a^{n-m} \int_0^{\infty} dv v^{2m} \varphi(v^2). \quad . \quad (21.)$$

This, when $a=1$, is CAUCHY'S theorem referred to in Art. 1. Some valuable illustrations of it will be found in the Corollaries to the memoir of which it forms the subject.

We may employ (20.) or (21.) to generalize the results given in (11.) and (12.). We may thus finitely determine the values of the integrals

$$\int_0^{\infty} \frac{dx . x^{n+i-\frac{1}{2}}}{(a+bx+cx^2)^n} \quad \int_0^{\infty} \frac{dx x^{n-i-\frac{3}{2}}}{(a+bx+cx^2)^n},$$

i being an integer. For the former integral we shall have the expression

$$\Sigma_{m=0}^{m=i} \frac{\overline{2m+1}.\overline{2m+2}..\overline{i+m}}{1.2..(i-m)} \left(\frac{a}{c}\right)^{\frac{i-m}{2}} \times \frac{1}{(b+2\sqrt{ac})^{n-\frac{2i+1}{2}} c^{\frac{2i+1}{2}}} \times \frac{\Gamma\left(n-\frac{2i+1}{2}\right)\Gamma\left(\frac{2i+1}{2}\right)}{\Gamma(n)} \quad . \quad . \quad (22.)$$

For the latter integral we shall only have to change in the above, a into c and c into a.

The results in (11.) and (12.), and the more general conclusions just obtained, are of importance in some of the more difficult problems connected with the mathematical theory of electricity. It is probable that a result equivalent to (22.) may be obtained

by some formulæ of Mr. CAYLEY'S connected with the reduction of the integrals which occur in certain problems of this class. I have not, however, attempted a verification.

34. Although the list which I have given of results obtainable by other methods might be increased, it is still only in comparatively rare instances that the means of independent verification present themselves. We might by transformations such as CAUCHY has employed, verify the theorem

$$\int_{-\infty}^{\infty}\frac{dx\cos m\left(x-\frac{a}{x}\right)}{1+\left(x-\frac{a}{x}\right)^2}=\pi\varepsilon^{-m};$$

but it would not be easy by any such process to verify the theorem

$$\int_{-\infty}^{\infty}\frac{dx\cos\left\{m\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x_n-\lambda_n}\right)\right\}}{1+\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)^2}=\pi\varepsilon^{-m},$$

a_1 and a_2, &c. being any positive quantities, and λ_1, λ_2, &c. any real quantities whatever. I shall not, however, dwell any longer upon special results, but shall briefly state some of the general consequences which flow from the application of the primary theorem (1.).

1st. *The evaluation of any definite integral*

$$\int_{-\infty}^{\infty}\phi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)dx,$$

in which $\phi(x)$ *is a rational and integral function of* x, *is reducible to that of the definite integral*

$$\int_{-\infty}^{\infty}\psi(v)f(v)dv, \quad \ldots\ldots\ldots\ldots \quad (1.)$$

in which $\psi(v)$ *is a rational and integral function of an order not higher than the order of* $\phi(x)$.

For, by the general theorem,

$$\int_{-\infty}^{\infty}dx\phi(x)f\left(x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}dvf(v)\Theta[\phi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}$$

$$=\int_{-\infty}^{\infty}dvf(v)C_{\frac{1}{x}}\frac{\phi(x)}{x-v-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}},$$

since $\phi(x)$ is integral. If we develope the fractions $\frac{a_1}{x-\lambda_1}$, $\frac{a_2}{x-\lambda_2}$, &c. in descending powers of x, we shall have

$$\frac{\phi(x)}{x-v-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}}=\frac{\phi(x)}{x-v-\frac{\Sigma a_r}{x}-\frac{\Sigma a_r l_r}{x^2}\cdots}.$$

Suppose m to be the highest index of x in $\varphi(x)$, then the development of the right-hand member, in descending powers of x by division, will assume the following form,—

$$B_0x^{m-1}+B_1x^{m-2}+B_2x^{m-3}..+B_mx^{-1}+B_{m+1}x^{-2}..+\&c.,$$

B_0 not containing v, B_1 containing the first power of v, B_2 the first and second power, &c. Hence B_m, the coefficient of x^{-1}, will involve powers of v up to the mth. Let this function be represented by $\psi(v)$, and we have

$$\int_{-\infty}^{\infty}dx\varphi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdot\cdot-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}\psi(v)f(v)dv, \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

$\psi(v)$being of the same order with respect to v as $\varphi(x)$ with respect to x.

The following are particular examples:—

$$\int_{-\infty}^{\infty}dx.xf\left(x-\frac{a_1}{x-\lambda_1}\cdot\cdot-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}vf(v)dv \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$\int_{-\infty}^{\infty}dx.x^2f\left(x-\frac{a_1}{x-\lambda_1}\cdot\cdot-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}(v^2+a_1+a_2..+a_n)f(v)dv \quad . \quad . \quad . \quad . \quad (4.)$$

$$\int_{-\infty}^{\infty}dx.x^3f\left(x-\frac{a_1}{x-\lambda_1}\cdot\cdot-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}\left\{v^3+2(a_1..+a_n)v+a_1\lambda_1..+a_n\lambda_n\right\}f(v)dv. \quad . \quad . \quad (5.)$$

2ndly. *The evaluation of the definite integral*

$$\int_{-\infty}^{\infty}dx\varphi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdot\cdot-\frac{a_n}{x-\lambda_n}\right),$$

where $\varphi(\mathrm{x})$ *is a rational fraction, is reducible to that of the definite integral*

$$\int_{-\infty}^{\infty}dv\psi(v)f(v), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

where $\psi(\mathrm{v})$*is a rational fraction of the same order as* $\varphi(\mathrm{x})$.

By a rational fraction of the same order, I mean one whose numerator is of the same degree, and whose denominator involves the same number of simple factors elevated to the same powers, the only difference arising from the constant coefficients.

By the general theorem we have

$$\int_{-\infty}^{\infty}dx\varphi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdot\cdot-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}dvf(v)\Theta[\varphi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdot\cdot+\frac{a_n}{x-\lambda_n}}, \quad . \quad . \quad . \quad (7.)$$

in which, on account of the distributive character of the symbol Θ, we may resolve $\varphi(x)$ into its component terms, and give to Θ in succession the respective interpretations which they afford.

The component terms of $\varphi(x)$ will be of the forms ax^i and $\frac{a}{(x-p)^i}$, i being an integer. We have just considered the effect of the first class of terms, and it only remains to consider that of the second class.

Now
$$\Theta\left[\frac{a}{(x-h)^i}\right]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}}$$
$$=\frac{a}{1.2..(i-1)}\left(\frac{d}{dx}\right)^{i-1}\frac{1}{v-h+\frac{a_1}{h-\lambda_1}\cdots+\frac{a_n}{h-\lambda_n}}$$
$$+\mathrm{C}_{\frac{1}{x}}\frac{a}{(x-h)^i\left(x-v-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)},$$
the former of the two terms in the second member being the coefficient of $\frac{1}{x-h}$ in the development of the function in ascending powers of $x-h$.

It is evident that the latter of the terms in the second member vanishes; for the first term of the development in descending powers of x being $\frac{a}{x^{i+1}}$, there will be no term of the form $\frac{\mathrm{A}}{x}$. Hence we have merely to consider the term
$$\frac{a}{1.2..(i-1)}\cdot\left(\frac{d}{dh}\right)^{i-1}\frac{1}{v-\mathrm{H}}, \quad \dots \dots \dots \dots \quad (8.)$$
H standing for
$$h-\frac{a_1}{h-\lambda_1}\cdots-\frac{a_n}{h-\lambda_n}.$$
If $i=1$, (8.) becomes
$$\frac{a}{v-\mathrm{H}}.$$
Let $i=2$, and (8.) becomes
$$\frac{a\frac{d\mathrm{H}}{dh}}{(v-\mathrm{H})^2}.$$
If $i=3$, (8.) becomes
$$\frac{a}{1.2}\left\{\frac{\frac{d^2\mathrm{H}}{dh^2}}{(v-\mathrm{H})^2}+\frac{2\left(\frac{d\mathrm{H}}{dh}\right)^2}{(v-\mathrm{H})^3}\right\}.$$
Hence, generally,
$$\frac{a}{1.2..(i-1)}\left(\frac{d}{dh}\right)^{i-1}\frac{1}{v-\mathrm{H}}=\frac{\mathrm{H}_1}{v-\mathrm{H}}+\frac{\mathrm{H}_2}{(v-\mathrm{H})^2}\cdots+\frac{\mathrm{H}_i}{(v-\mathrm{H})^i},$$
$\mathrm{H}_1, \mathrm{H}_2, .. \mathrm{H}_i$ being independent of v. The second member of the above equation, on addition of its several terms, becomes a rational fraction whose denominator is $(v-\mathrm{H})^i$, and whose numerator is a rational and integral function of v of an order not higher than $i-1$. Hence the theorem is demonstrated.

35. The conclusion to which these investigations lead is a remarkable one, and may be thus expressed. *The evaluation of the definite integral*
$$\int_{-\infty}^{\infty}\frac{p_0+p_1x+p_2x^2..+p_ix^i}{q_0+q_1x+q_2x^2..+q_jx^j}f\left(x-\frac{a_1}{x-\lambda_1}-\frac{a_2}{x-\lambda_2}\cdots-\frac{a_n}{x-\lambda_n}\right)dx$$
is reducible to that of a definite integral of the form
$$\int_{-\infty}^{\infty}\frac{\mathrm{P}_0+\mathrm{P}_1v+\mathrm{P}_2v^2..+\mathrm{P}_iv}{\mathrm{Q}_0+\mathrm{Q}_1v+\mathrm{Q}_2v^2..+\mathrm{Q}_jv^j}f(v)dv,$$

P_0, P_1, .. Q_0, Q_1.. *being constants whose values in terms of* p_1, p_2, .. q_1, q_2.. *can always be finitely determined.*

As particular examples of the above, we should have

$$\int_{-\infty}^{\infty}\frac{dx}{x-h}f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_{-\infty}^{\infty}\frac{f(v)dv}{v-h+\frac{a_1}{h-\lambda_1}\cdots+\frac{a_n}{h-\lambda_n}}$$

$$\int_{-\infty}^{\infty}\frac{dx}{(x-h)^2}f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda^n}\right)=\left\{1+\frac{a_1}{(h-\lambda_1)^2}\cdots+\frac{a_n}{(h-\lambda_n)^2}\right\}\int_{-\infty}^{\infty}\frac{f(v)dv}{\left(v-h+\frac{a_1}{h-\lambda_1}\cdots+\frac{a_n}{h-\lambda_n}\right)^2}.$$

In the last theorem the particular case in which $h=\lambda_1$, the function $f(v)$ being at the same time supposed small for very large values of v, is interesting. We see that as h approaches λ_1 the terms $\frac{a_1}{(h-\lambda_1)^2}$ and $\frac{a_1}{h-\lambda_1}$ become large in comparison with those with which they are connected by addition. Thus the second member approaches the value

$$\frac{a_1}{(h-\lambda_1)^2}\int_{-\infty}^{\infty}\frac{f(v)dv}{\left(\frac{a_1}{h-\lambda_1}\right)^2}=\frac{1}{a_1}\int_{-\infty}^{\infty}f(v)dv,$$

so that in the limit

$$\int_{-\infty}^{\infty}\frac{dx}{(x-\lambda_1)^2}f\left(x-\frac{a_1}{(x-\lambda_1)}\cdots-\frac{a_n}{x-\lambda_n}\right)=\frac{1}{a_1}\int_{-\infty}^{\infty}f(v)dv.$$

It may be worth while to verify this theorem in a particular instance. We have from it

$$\int_{-\infty}^{\infty}\frac{dx}{(x-\lambda)^2}f\left(x-\frac{a}{x-\lambda}\right)=\frac{1}{a}\int_{-\infty}^{\infty}f(v)dv.$$

Now assume in the first member

$$\frac{-a}{x-\lambda}=y-\lambda.$$

The transformed integral is easily found to be

$$\frac{1}{a}\int dyf\left(y-\frac{a}{y-\lambda}\right).$$

As to the limits, when x varies from $-\infty$ to λ, y varies from λ to ∞; and when x varies from λ to ∞, y varies from $-\infty$ to λ. Thus, by mere transposition of the two portions of the integral, the limits of y become the same as those of x, and we have

$$\int_{-\infty}^{\infty}\frac{dx}{(x-\lambda)^2}f\left(x-\frac{a}{x-\lambda}\right)=\frac{1}{a}\int_{-\infty}^{\infty}dyf\left(y-\frac{a}{y-\lambda}\right)=\frac{1}{a}\int_{-\infty}^{\infty}f(v)dv \text{ by (2.), art. 32.}$$

As a particular deduction from the above we shall have.

$$\int_{-\infty}^{\infty}\frac{dx}{x^2}\epsilon^{-\left(x^2+\frac{a^2}{x^2}\right)}=\frac{\pi^{\frac{1}{2}}\epsilon^{-2a}}{a},$$

which may also be verified by differentiating (5.), art. 32, with respect to a. We are permitted thus to differentiate with respect to a, because the function under the sign of integration does not become infinite within the limits. This condition must be strictly attended to in all similar attempts at verification.

3rdly. *From the value of any known definite integral, we can, by the general theorem, deduce either the values of other definite integrals taken between the limits* $-\infty$ *and* ∞, *or relations among the values of those integrals taken between other limits.*

To accomplish the first object, we have only to transform the given integral into one whose limits are $-\infty$ and ∞, and then apply directly the general theorem. The method requires no illustration.

To accomplish the second object, we must express the function under the sign of integration, not as a continuous function taken between the given limits, but as a discontinuous function taken between the limits $-\infty$ and ∞, the character of its discontinuity being such, that for all values within the *given* limits of integration it shall assume the form specified, and for all values without the given limits shall vanish.

Thus let $\int_p^q f(v)dv$ be the definite integral whose value is given. We may extend the integration from $-\infty$ to ∞, provided that we regard $f(v)$ as vanishing when v falls without the limits p and q. We shall thus have

$$\int_{-\infty}^{\infty} f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)dx=\int_{-\infty}^{\infty} f(v)dv=\int_p^q f(v)dv,$$

provided that $f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)$ vanishes whenever $x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}$ falls without the limits p and q. Let the roots of the equation

$$x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}=p,$$

taken in ascending order of magnitude, be $p_1, p_2, \ldots p_{n+1}$, and the roots of

$$x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda^n}=q,$$

taken in the same order, be $q_1, q_2 \ldots q_{n+1}$. We suppose $\lambda_1, \lambda_2, \ldots \lambda_n$ also to be in ascending order of magnitude. Thus (art. 30) p_1 and q_1 lie between $-\infty$ and λ_1, p_2 and q_2 between λ_1 and λ_2, and so on. Also q_1 is greater than p_1, q_2 than p_2, &c. Hence we see that $x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}$ will only fall within the limits p and q when x falls either between p_1 and q_1, or between p_2 and q_2, &c. Thus we have in fact

$$\Sigma\int_{p_i}^{q_i} dx f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)=\int_p^q f(v)dv,$$

and still more generally, $\phi(x)$ being a rational function of x,

$$\Sigma\int_{p_i}^{q_i}\phi(x)f\left(x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}\right)dx=\int_p^q dv f(v)\Theta[\phi(x)]\frac{1}{v-x+\frac{a_1}{x-\lambda_1}\cdots+\frac{a_n}{x-\lambda_n}},$$

which is a reproduction of (5.), art. 30. I deem it, however, an important fact, that in the comparison of functional transcendents, formulæ involving the sign of summation

may be dispensed with; a more general conception of the nature of a function supplying their place.

36. One remarkable theorem must still be noticed. Since

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2\pi^2}\right)\left(1-\frac{x^2}{3\pi^2}\right)\cdots,$$

we have, on taking the logarithms of both sides and differentiating,

$$\cot x = \frac{1}{x}+\frac{1}{x+\pi}+\frac{1}{x-\pi}+\frac{1}{x+2\pi}+\frac{1}{x-2\pi}+\text{\&c.} \qquad (1.)$$

Hence

$$x-\cot x = x-\frac{1}{x}-\frac{1}{x+\pi}-\frac{1}{x-\pi}-\frac{1}{x+2\pi}-\text{\&c.}$$

Whence, by (2.), art. 32,

$$\int_{-\infty}^{\infty} dx f(x-\cot x) = \int_{-\infty}^{\infty} dv f(v). \qquad (2.)$$

The result may however be generalized. For from (1.),

$$a_1\cot(x-\lambda_1) = \frac{a_1}{x-\lambda_1}+\frac{a_1}{x-\lambda_1+\pi}+\frac{a_1}{x-\lambda_1-\pi}+\text{\&c.}$$

Taking the sum of any series of such terms, we shall evidently have

$$\begin{aligned} x-a_1\cot(x-\lambda_1)-a_2\cot(x-\lambda_2)\cdots-a_n\cot(x-\lambda_n) \\ = x-\frac{a_1}{x-\lambda_1}\cdots-\frac{a_n}{x-\lambda_n}-\frac{a_1}{x+\pi-\lambda_1}\cdots-\frac{a_n}{x+\pi-\lambda_n}\cdots, \end{aligned}$$

which agrees in form with the function under the sign f in (2.), art. 32. Hence

$$\int_{-\infty}^{\infty} dx f\{x-a_1\cot(x-\lambda_1)\cdots-a_n\cot(x-\lambda_n)\} = \int_{-\infty}^{\infty} dv f(v). \qquad (3.)$$

If we treat in the same way the theorem

$$\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\cdots,$$

we shall arrive at the theorem

$$\int_{-\infty}^{\infty} dx f\{x+a_1\tan(x-\lambda_1)\cdots+a_n\tan(x-\lambda_n)\} = \int_{-\infty}^{\infty} dv f(v). \qquad (4.)$$

Essentially, however, this result is involved in (3.), the analogy of which with (2.), Art. 32, will be most apparent if we place it in the form

$$\int_{-\infty}^{\infty} dx f\left\{x-\frac{a_1}{\tan(x-\lambda_1)}\cdots-\frac{a_n}{\tan(x-\lambda_n)}\right\} = \int_{-\infty}^{\infty} dv f(v).$$

As before, the quantities $a_1, a_2 \ldots a_n$ must be positive.

The verification of these theorems by some independent method seems desirable.

Application of the General Theorem to the evaluation of multiple definite integrals.

37. The form in which multiple definite integrals present themselves in the application of mathematics to natural philosophy, is usually the following. The value of a triple integral,

$$\iiint \phi(x, y, z)dx\,dy\,dz, \quad \ldots \ldots \ldots \quad (1.)$$

is required to be found, the integration extending, in some instances, to all positive values, but more generally to all values whatever of the variables x, y, z which satisfy a condition

$$\psi(x, y, z) \overline{<} 1. \quad \ldots \ldots \ldots \quad (2.)$$

The most general method of treating this class of problems is due to M. LEJEUNE DIRICHLET. It consists in converting $\phi(x, y, z)$ into a discontinuous function which vanishes whenever the variables x, y, z transcend the limits assigned in (2.), and which is equal to $\phi(x, y, z)$ whenever those variables satisfy the above condition. This transformation being effected, we are permitted to regard the integrations relatively to x, y, z as independent, and as individually taken between the limits $-\infty$ and ∞.

From this circumstance the progress of our knowledge of multiple definite integrals must be in some degree coordinate with the extension of our command over single definite integrals taken between the limits $-\infty$ and ∞. I propose in this section both to illustrate, by one or two examples, the theory of multiple integrals as above stated, and to show how it gains extension from the theorem of definite integration demonstrated in the preceding pages.

There are several different forms in which the application of the principle of discontinuity to the evaluation of multiple integrals may be presented. The form which I shall adopt in this paper is similar to one originally given by me in the Transactions of the Royal Irish Academy (vol. xxi. pt. 1), but is more convenient in application. It depends essentially upon the employment of FOURIER's theorem, viz.—

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} da.dv.\cos(av-xv)f(a).$$

If we write the cosine in its exponential form, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} dadv\left\{\epsilon^{(av-xv)\sqrt{-1}} + \epsilon^{-(av-xv)\sqrt{-1}}\right\}f(a). \quad \ldots \ldots \quad (3.)$$

Now by known theorems

$$\frac{1}{t^i} = \frac{\epsilon^{\frac{i\pi}{2}\sqrt{-1}}}{\Gamma(i)}\int_0^\infty dw.w^{i-1}\epsilon^{-tw\sqrt{-1}} = \frac{\epsilon^{-\frac{i\pi}{2}\sqrt{-1}}}{\Gamma(i)}\int_0^\infty dw.w^{i-1}\epsilon^{tw\sqrt{-1}}. \quad \ldots \ldots \quad (4.)$$

Multiply the terms of (3.) by those of (4.) taken in the same order, and, converting the exponentials into sines and cosines, we have

$$\frac{f(x)}{t^i} = \frac{1}{\pi\Gamma(i)}\int_{-\infty}^{\infty}\int_0^\infty\int_0^\infty dadvdw\cos\left(av-xv-tw+\frac{i\pi}{2}\right)w^{i-1}f(a), \quad \ldots \quad (5.)$$

the theorem employed in the Irish Transactions.

Now v remaining unchanged, let $w=vs$. Then transforming in the usual way, we have

$$dvdw=vdvds,$$

whence, on substitution,

$$\frac{f(x)}{t^i}=\frac{1}{\pi\Gamma(i)}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}dadvds\cos\left\{(a-x-ts)v+\frac{i\pi}{2}\right\}vs^{i-1}f(a). \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

In the application of this theorem to the reduction of multiple integrals, x and t will be replaced by functions of the variables involved in those integrals. Its advantages are the following. Like FOURIER's theorem, from which it is derived, it enables us to express any species of discontinuity in the function $f(x)$. Thus if $f(x)$ is to vanish for all values of x which lie without the limits p and q, we have only to substitute p and q for $-\infty$ and ∞ in the integration relative to a. At the same time the theorem presents x and t in a functional connexion, which in all the most important cases renders possible the subsequent integrations without any new transformation.

One subsidiary theorem remains to be noticed. Since we have

$$\int_{-\infty}^{\infty}dy\cos(a\pm cy^2)=\frac{\pi^{\frac{1}{2}}}{c^{\frac{1}{2}}}\cos\left(a\pm\frac{\pi}{4}\right),$$

we have, by successive applications of this theorem,

$$\int_{-\infty}^{\infty}dy_1dy_2\ldots dy_n\cos(a\pm\Sigma_{r=1}^{r=n}c_ry_r^2)=\frac{\pi^{\frac{n}{2}}}{c_1^{\frac{1}{2}}c_2^{\frac{1}{2}}\ldots c_n^{\frac{1}{2}}}\cos\left(a\pm\frac{n\pi}{4}\right). \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

38. Example 1st. Let it be required to evaluate the multiple definite integral

$$V=\int\ldots dx_1dx_2\ldots dx_n\frac{f(l_1x_1+l_2x_2\ldots+l_nx_n)}{\{h^2+(a_1-x_1)^2+(a_2-x_2)^2\ldots+(a_n-x_n)^2\}^i}, \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

the integration extending to all values of the variables which satisfy the condition

$$l_1x_1+l_2x_2\ldots+l_nx_n \begin{matrix}<q\\>p\end{matrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

If for convenience we represent $l_1x_1+l_2x_2\ldots+l_nx_n$ by Σl_rx_r and $(a_1-x_1)^2+(a_2-x_2)^2\ldots+(a_n-x_n)^2$ by $\Sigma(a_r-x_r)^2$, we have by the theorem (6.),

$$\frac{l_1x_1+l_2x_2\ldots+l_nx_n}{\{h^2+(a_1-x_1)^2+(a_2-x_2)^2\ldots+(a_n-x_n)^2\}^i}$$

$$=\frac{1}{\pi\Gamma i}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}dadvds\cos\left\{\left(a-\Sigma l_rx_r-(h^2+\Sigma\overline{a_r-x_r}^2.s)\right)v+\frac{i\pi}{2}\right\}v^is^{i-1}f(a).$$

The conditions relative to the limits will be fulfilled by introducing p and q for $-\infty$ and ∞ in the above. Effecting this change and extending, as we then may do, the integrations relative to x_1, $x_2\ldots x_n$ from $-\infty$ to ∞, we have

$$V=\frac{1}{\pi\Gamma(i)}\int_p^q\int_0^{\infty}\int_0^{\infty}dadvdsv^is^{i-1}f(a)T,$$

where $$T=\int_{-\infty}^{\infty}\ldots dx_1dx_2\ldots dx_n\cos\left\{\left(a-h^2s-\Sigma(l_rx_r+s.\overline{a_r-x_r}^2)\right)v+i\frac{\pi}{2}\right\}.$$

Now $l_rx_r+s(a_r-x_r)^2=s\left(x_r-a_r+\frac{l_r}{2s}\right)^2+l_ra_r-\frac{l_r^2}{4s}=sy_r^2+l_ra_r-\frac{l_r^2}{4s}$ if $y_r=x_r-a_r+\frac{l_r}{2s}$. Sub-

stituting and observing that the limits of y_r are $-\infty$ and ∞, we have

$$\mathrm{T}=\int_{-\infty}^{\infty}..dy_1dy_2..dy_n\cos\left\{\left(a-h^2s-\Sigma\left(sy_r^2+l_ra_r-\frac{l_r^2}{4s}\right)\right)v+i\frac{\pi}{2}\right\}$$

$$=\int_{-\infty}^{\infty}..dy_1dy_2..dy_n\cos\left\{\left(a-h^2s-\Sigma l_ra_r+\frac{\Sigma l_r^2}{4s}\right)v+\Sigma vsy_r^2+\frac{i\pi}{2}\right\}.$$

Let $\sigma=h^2s+\Sigma l_ra_r-\frac{\Sigma l_r^2}{4s}$. . . . . (3.)

Then $\quad \mathrm{T}=\int_{-\infty}^{\infty}..dy_1dy_2..dy_n\cos\left\{(a-\sigma)v+\Sigma vsy_r^2+i\frac{\pi}{2}\right\}=\frac{\pi^{\frac{n}{2}}}{(vs)^{\frac{n}{2}}}\cos\left\{(a-\sigma)v+i\frac{\pi}{2}-n\frac{\pi}{4}\right\},$

whence

$$\mathrm{V}=\frac{\pi^{\frac{n}{2}-1}}{\Gamma(i)}\int_p^q\int_0^\infty\int_0^\infty dadvds\,v^{i-\frac{n}{2}}s^{i-\frac{n}{2}-1}\cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\}f(a)=\frac{\pi^{\frac{n}{2}}}{\Gamma(i)}\int_0^\infty ds\,s^{i-\frac{n}{2}-1}\mathrm{Q},$$

where

$$\mathrm{Q}=\frac{1}{\pi}\int_p^q\int_0^\infty dadv\,v^{i-\frac{n}{2}}\cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\}f(a)$$

$$=\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}\frac{1}{\pi}\int_p^q\int_0^\infty dadv\cos\{(a-\sigma)v\}f(a)$$

$$=\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}f(\sigma)$$

by FOURIER's theorem, $f(\sigma)$ vanishing when σ does not fall within the limits p and q. Thus, finally,

$$\mathrm{V}=\frac{\pi^{\frac{n}{2}}}{\Gamma(i)}\int_0^\infty ds\,s^{i-\frac{n}{2}-1}\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}f(\sigma), \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (4.)$$

the fully expressed value of σ being

$$\sigma=h^2s+l_1a_1+l_2a_2..+l_na_n-\frac{l_1^2+l_2^2..+l_n^2}{4s}. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (5.)$$

We may remark, that as

$$\frac{d\sigma}{ds}=h^2+\frac{l_1^2+l_2^2..+l_n^2}{4s^2},$$

σ increases with s. As s varies from 0 to ∞, σ varies from $-\infty$ to ∞. Thus, whatever may be the values of p and q within which the variation of σ is confined, there will exist corresponding positive values within which the variation of s will be confined, and those limits must take the place of 0 and ∞ in the expression of (4.).

39. In strictness there is no need of referring to the limit in the statement of general theorems like the above involving an arbitrary symbol of functionality. The consistent interpretation of that symbol will suffice. Thus the results at which we have arrived are virtually included in the theorem

$$\int_{-\infty}^{\infty}..dx_1dx_2..dx_n\frac{f(l_1x_1+l_2x_2\cdot\cdot+l_nx_n)}{\{h^2+(a_1-x_1)^2+(a_2-x_2)^2\cdot\cdot+(a_n-x_n)^2\}^i}=\frac{\pi^{\frac{n}{2}}}{\Gamma(i)}\int_0^\infty ds.s^{i-\frac{n}{2}-1}\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}f(\sigma), \ (6.)$$

σ having the value given above. For if $l_1x_1+l_2x_2\cdot\cdot+l_nx_n$ is confined within given limits, $f(l_1x_1+l_2x_2\cdot\cdot+l_nx_n)$ may be regarded as vanishing whenever $l_1x_1+l_2x_2\cdot\cdot+l_nx_n$ transcends

those limits, and therefore, by consistency of interpretation, $f(\sigma)$ as vanishing whenever σ transcends the same limits.

40. I shall not enter into any discussion of the above solution, but shall briefly point out in what way it may be generalized by the theorem of definite integration of Art. 32. It is evident that if we change in (6.)

$$x_1 \text{ into } x_1-\frac{b_1}{x_1-\lambda_1}-\frac{b_2}{x_1-\lambda_2}\cdots-\frac{b_m}{x_1-\lambda_m},$$

$$x_2 \text{ into } x_2-\frac{c_1}{x_2-\mu_1}-\frac{c_2}{x_2-\mu_2}\cdots-\frac{c_m}{x_2-\mu_m},$$

&c. &c. &c.

or into any of the remarkable forms thence derived, Art. 36, leaving $dx_1dx_2\ldots dx_n$ unchanged, the actual value of the multiple integral will be unaltered. Thus, as a particular illustration, if we suppose

$$\mathrm{V}=\iiint dxdydz\frac{f\left\{l\left(x-\frac{1}{x}\right)+m\left(y-\frac{1}{y}\right)+n\left(z-\frac{1}{z}\right)\right\}}{\left\{h^2+\left(a-x+\frac{1}{x}\right)^2+\left(b-y+\frac{1}{y}\right)^2+\left(c-z+\frac{1}{z}\right)^2\right\}^i},$$

the integrations being limited only by the condition

$$l\left(x-\frac{1}{x}\right)+m\left(y-\frac{1}{y}\right)+n\left(z-\frac{1}{z}\right)\lesseqgtr{}^{1}_{0},$$

we should find

$$\mathrm{V}=\frac{\pi^{\frac{3}{2}}}{\Gamma(i)}\int_0^\infty ds\,.\,s^{i-\frac{3}{2}-1}\left(\frac{d}{d\sigma}\right)^{i-\frac{3}{2}}f(\sigma),$$

where

$$\sigma=la+mb+nc-\frac{l^2+m^2+n^2}{4s},$$

provided that $f(\sigma)=0$ when σ does not fall within the limits 0 and 1.

41. Example 2nd. Let

$$\mathrm{V}=\int\ldots dx_1dx_2\ldots dx_n\frac{f\left\{l_1^2\left(x_1^2+\frac{a_1^2}{x_1^2}\right)\cdots+l_n^2\left(x_n^2+\frac{a_n^2}{x_n^2}\right)\right\}}{\left\{h^2+m_1^2\left(x_1^2+\frac{b_1^2}{x_1^2}\right)\cdots+m_n^2\left(x_n^2+\frac{b_n^2}{x_n^2}\right)\right\}^i},$$

the integration extending to all values of the variables which satisfy the condition

$$l_1^2\left(x_1^2+\frac{a_1^2}{x_1^2}\right)\cdots+l_n^2\left(x_n^2+\frac{a_n^2}{x_n^2}\right)\leqq 1.$$

Here, after reductions similar to those which have been exemplified in the preceding problem, but more complicated, we find

$$\mathrm{V}=\frac{\pi^{\frac{n}{2}}}{\Gamma(i)}\int_0^\infty\frac{ds\,.\,s^{i-1}\left(-\frac{d}{d\sigma}\right)^{i-\frac{n}{2}}f(\sigma)}{(l_1^2+m_1^2s)^{\frac{1}{2}}\cdots(l_n^2+m_n^2s)^{\frac{1}{2}}}, \quad \ldots\ldots\ldots \quad (1.)$$

wherein $\sigma=h^2s+2(l_1^2+m_1^2s)^{\frac{1}{2}}(l_1^2a_1^2+m_1^2b_1^2s_1)^{\frac{1}{2}}+\cdots+2(l_n^2+m_n^2s)^{\frac{1}{2}}(l_n^2a_n^2+m_n^2b_n^2s)^{\frac{1}{2}}$,

$f(\sigma)$ being supposed to vanish when σ transcends the limits 0 and 1. This theorem admits of the same kind of generalization as the preceding one. It was communicated by me, some years ago, to Mr. CAYLEY, and published by him, at my desire, in LIOUVILLE'S 'Journal,' vol. xiii.

If in the above theorem we make $a_1=0,\ldots a_n=0$, we have an expression for the value of the multiple integral

$$\int\ldots dx_1dx_2\ldots dx_n\frac{f(l_1^2x_1^2+l_2^2x_2^2\cdots+l_n^2x_n^2)}{\left\{h^2+m_1^2\left(x_1^2+\frac{b_1^2}{x_1^2}\right)\cdots+m_n^2\left(x_n^2+\frac{b_n^2}{x_n^2}\right)\right\}^i},$$

the integrations being extended through the mass of the ellipsoid whose equation is

$$l_1^2x_1^2+\cdots+l_n^2x_n^2=1.$$

The following example, originally published by me, but without any intimation of its possible extension by means of the general theorem of definite integration, in the memoir already referred to in the 'Transactions of the Royal Irish Academy,' is of great practical importance.

42. Example 3rd. Required the value of the multiple integral

$$V=\int\ldots dx_1dx_2\ldots dx_n\frac{f\left(\frac{x_1^2}{h_1^2}+\frac{x_2^2}{h_2^2}\cdots+\frac{x_n^2}{h_n^2}\right)}{\{h^2+(a_1-x_1)^2+(a_2-x_2)^2\cdots+(a_n-x_n)^2\}^i}, \quad . \quad . \quad . \quad . \quad (1.)$$

the integrations extending to all values of the variables which satisfy the condition

$$\frac{x_1^2}{h_1^2}+\frac{x_2^2}{h_2^2}\cdots+\frac{x_n^2}{h_n^2}\overline{\overline{<}}1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Here V will be found by integrating with respect to $x_1, x_2, \ldots x_n$ between the limits $-\infty$ and ∞ the expression

$$\frac{1}{\pi\Gamma(i)}\int_0^1\int_0^\infty\int_0^\infty dadvds\ v^is^{i-1}\cos\left\{\left(a-\frac{\Sigma x_r^2}{h_r^2}-s\left(h^2+\Sigma(a_r-x_r)^2\right)\right)v+\frac{i\pi}{2}\right\}f(a).$$

Hence changing, as before, the order of integration,

$$V=\frac{1}{\pi\Gamma(i)}\int_0^1\int_0^\infty\int_0^\infty dadvds\ v^is^{i-1}f(a)T,$$

where

$$T=\int_{-\infty}^{\infty}\ldots dx_1dx_2\ldots dx_n\cos\left\{\left(a-h^2s-\Sigma\left(\frac{x_r^2}{h_r^2}+s(a_r-x_r)^2\right)\right)v+\frac{i\pi}{2}\right\}.$$

Now

$$\frac{x_r^2}{h_r^2}+s(a_r-x_r)^2=\frac{1+h_r^2s}{h_r^2}y_r^2+\frac{sa_r^2}{1+h_r^2s}$$

if we make

$$y_r=x_r-\frac{h_r^2sa_r}{1+h_r^2s}.$$

late. We can do little more than address ourselves patiently to follow the tracks which open before us, without attempting to prescribe their direction or conjecture their end.

The interpretation of the formulæ which have been arrived at in this section is so far distinct from the general course and design of the paper, that I have thrown into a note such observations as I have to offer upon the subject. And I have the rather taken this course, because those observations have been to some extent anticipated in a former publication. See Note B.

I have not attempted the further extension of the theory of multiple integrals, which would seem to be involved in the general theorem of Art. 32, when other values than unity are assigned to the function $\phi(x)$. Neither have I attempted to extend that theorem to the case in which $\phi(x)$ is rational—a case evidently of some importance from its formal connexion with the last member of (5.).

NOTE A.

On the Connexion between the Symbol Θ *and* CAUCHY'S *Symbol* $\mathcal{E}$, *employed in the Calculus of Residues.*

It has been explained that the operation denoted by the symbol $\mathcal{E}$ is equivalent to that portion of the operation represented by Θ which depends upon the ascending developments of the subject function. There is, however, a difference in the mode of statement.

Thus $\mathcal{E}\dfrac{x^2}{[(x-a)^2(x-b)]}$ would denote, according to CAUCHY'S definition, the sum of the coefficients of $\frac{1}{x}$ in the developments of the respective functions $\frac{(x+a)^2}{x^2(x+a-b)}$ and $\frac{(x+b)^2}{(x+b-a)^2x}$, in ascending powers of x. This would be the same as the sum of the coefficients of $\frac{1}{x-a}$ and $\frac{1}{x-b}$ in the respective ascending developments of the primitive function $\frac{x^2}{(x-a)^2(x-b)}$, in ascending powers of $x-a$ and $x-b$ respectively. The operation Θ would add to the above the coefficient with changed sign of $\frac{1}{x}$ in the development of the same function in descending powers of x.

We shall perhaps best exemplify the connexion thus established between the two symbols, by applying the new symbol Θ to the solution of some of the problems which CAUCHY has treated by the Calculus of Residues. I select for the purpose,—1st, the problem of the integration of linear differential equations with constant coefficients; 2ndly, the problem of the integration of rational fractions.

Both these applications depend on a transformation of the rational function $f(v)$.

1st. It follows from the subsidiary theorem (6.), Art. 13, that if we have

$$v-x=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

as an equation connecting x and v, then

$$\Sigma f(x)=\Theta[f(x)]\frac{1}{v-x}.$$

Whence, x having only one value in terms of v,

$$f(v)dv=\Theta[f(x)]\frac{dv}{v-x},$$

and integrating,

$$\int f(v)dv=\Theta[f(x)]\{\log(v-x)+\psi(x)\},$$

$\psi(x)$ being an arbitrary function of x. It is easily seen that $\Theta[f(x)]\psi(x)$ may be represented by C, whence

$$\int f(v)dv=\Theta[f(x)]\log(v-x)+\text{C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

CAUCHY has very extensively employed the Calculus of Residues in the evaluation of definite integrals taken between the limits 0 and ∞. These applications are among the most valuable portions of his writings. They have no connexion, however, with the researches of this paper, and I have not even examined whether they would be in any degree generalized by the adoption of the symbol Θ.

NOTE B.

On the Interpretation of the Formulæ for the Evaluation of Multiple Integrals.

The three principal formulæ, (4.) art. 38, (1.) art. 41, and (5.) art. 42, evidently possess a common type. In each of them we recognize under the sign $\int$, a function $f(\sigma)$ which may be discontinuous within the limits of integration, and which is at the same time subject to an operation of general differentiation. This is a combination which is at least unusual in analysis. I purpose to consider here some of the questions of interpretation which it suggests. To some extent, indeed, these questions have been considered in my previous memoir, already referred to; but one of the most important of them, the effect of discontinuity in the function $f(\sigma)$ upon the integral in which it is involved, admits of being presented in a more satisfactory light. I do not propose to enter upon a complete investigation of the latter question, but only to examine one or two special and well-marked cases, in the hope of directing the attention of others to the subject.

When there is but one variable, and the index of differentiation is 0, the formulæ reduce in effect to ordinary integral transformations. And it is quite worthy of observation, that in this way the formula (4.), Art. 38, leads to known *modular* transformations of the elliptic functions. Thus if we make $n=1$, $h=1$, $i=\frac{1}{2}$, $a_1=0$ and drop the suffix from x_1 and l_1, we have

$$\int\frac{dxf(lx)}{(1+x^2)^{\frac{1}{2}}}=\int\frac{ds}{s}f(\sigma), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

where $\sigma=s-\frac{l^2}{4s}$ and $f(\sigma)$ vanishes when σ transcends the limits of lx. But this amounts to saying that

$$\int\frac{dxf(lx)}{(1+x^2)^{\frac{1}{2}}}=\int\frac{dsf\left(s-\frac{l^2}{4s}\right)}{s},$$

provided that

$$lx=s-\frac{l^2}{4s},$$

a result easily verified. Now let $f(lx)=\frac{1}{\sqrt{1+l^2x^2}}$, we have

$$\int\frac{dx}{\sqrt{(1+x^2)(1+l^2x^2)}}=\int\frac{ds}{s\sqrt{1+\left(s-\frac{l^2}{4s}\right)^2}}=\int\frac{ds}{\sqrt{\frac{l^4}{16}+\left(1-\frac{l^2}{2}\right)s^2+s^4}}.$$

The second member is a function of the same kind as the first, differing only in the constants. If we assume $s=mt$, we can so determine m as to reduce the equation to the form

$$\int\frac{dx}{\sqrt{(1+x^2)(1+l^2x^2)}}=\mathrm{L}\int\frac{dt}{\sqrt{(1+t^2)(1+l'^2t^2)}}. \quad \text{. . . . . (2.)}$$

We shall find

$$\mathrm{L}=\frac{2}{1+\sqrt{1-l^2}},\ l'=\frac{1-\sqrt{1-l^2}}{1+\sqrt{1+l^2}};$$

the relation between x and t being

$$t=l'^{\frac{1}{2}}(x\sqrt{x^2+1}).$$

If in the above we make $x=\tan\phi$, $t=\tan\theta$, $1-l^2=h^2$, $1-l'^2=k^2$, we find

$$\int\frac{d\phi}{\sqrt{1-h^2\sin^2\phi}}=\frac{2}{1+h}\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad \text{. . . . . (3.)}$$

provided that

$$h=\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} \text{ and } \tan\theta=\sqrt{\frac{1+h}{1-h}}\,(\tan\phi+\sec\phi).$$

These are of course known relations.

In the following example, which is valuable only for the sake of the principle involved, the differential coefficient of a discontinuous function occurs under the sign of integration.

In the same equation (4.), Art. 38, let $h=0$, $i=\frac{3}{2}$, $n=1$, $l_1=1$; then dropping the suffix from the single variable retained, we have

$$\int\frac{dxf(x)}{(a-x)^3}=2\int_0^\infty ds\left(-\frac{d}{d\sigma}\right)f(\sigma), \quad \text{. . . . . (4.)}$$

wherein $\sigma=a-\frac{1}{4s}$ and $f(\sigma)$ vanishes whenever σ transcends the limits of x. Suppose those limits 0 and 1, and let a be greater than 1. Now σ and s increase together, and $s=\frac{1}{4(a-\sigma)}$. Hence $\sigma=0$ gives $s=\frac{1}{4a}$, and $\sigma=1$ gives $s=\frac{1}{4(a-1)}$. Therefore

$$\int_0^1\frac{dxf(x)}{(a-x)^3}=-2\int_0^\infty ds\frac{d}{d\sigma}f(\sigma), \quad \text{. . . . . (5.)}$$

provided that $f(\sigma)$ be regarded as a discontinuous function defined in the following manner, viz.—

From $s=0$ to $s=\frac{1}{4a}$ $\qquad f(\sigma)=0$. . . . . (6.)

From $s=\frac{1}{4a}$ to $s=\frac{1}{4a-1}$ $\qquad f(\sigma)=f\left(a-\frac{1}{4s}\right)$. . . . . (7.)

From $s=\frac{1}{4a-1}$ to $s=\infty$ $\qquad f(\sigma)=0$. . . . . (8.)

Let us now examine the corresponding values of the element $ds\frac{d}{d\sigma}f(\sigma)$ under the sign of integration in (4.).

1st. From $s=0$ to $s=\frac{1}{4a}$ that element is 0 by (6.).

2ndly. At the point $s=\frac{1}{4a}$ we have

$$ds\frac{d}{d\sigma}f(\sigma)=\frac{ds}{d\sigma}\frac{d}{ds}f(\sigma)ds. \quad \text{(9.)}$$

But $\frac{d}{ds}f(\sigma)\,ds$ is the increment of $f(\sigma)$ corresponding to an infinitesimal increment ds in the value of s. At the break, where $s=\frac{1}{4a}$, $f(\sigma)$ changes in value from 0 to $f(0)$, the initial value of $f\left(a-\frac{1}{4s}\right)$, and the increment of $f(\sigma)$ is $f(0)$, *whatever the value of* ds *may be*, provided only that it is infinitesimal. Hence at this point we have

$$ds\frac{d}{d\sigma}f(\sigma)=\frac{ds}{d\sigma}f(0)=\frac{1}{4(a-\sigma)^2}f(0)=\frac{f(0)}{4a^2}.$$

3rdly. From $s=\frac{1}{4a}$ to $s=\frac{1}{4(a-1)}$, $ds\frac{d}{d\sigma}f(\sigma)=dsf'\left(a-\frac{1}{4s}\right)$.

4thly. At the second break, where $s=\frac{1}{4(a-1)}$ and $\sigma=1$, we find $ds\frac{d}{d\sigma}f(\sigma)=\frac{-f(1)}{4(a-1)^2}$.

5thly. From $s=\frac{1}{4(a-1)}$ to $s=\infty$ we have $ds\frac{d}{d\sigma}f(\sigma)=0$. Thus on recapitulation

$$\int_0^\infty ds\frac{d}{d\sigma}f(\sigma)$$

comprises two finite elements whose sum is

$$\frac{f(0)}{4a^2}-\frac{f(1)}{4(a-1)^2},$$

and a series of infinitesimal elements which give by integration

$$\int_{\frac{1}{4a}}^{\frac{1}{4(a-1)}}dsf'\left(a-\frac{1}{4s}\right),$$

whence (5.) becomes

$$\int_0^1\frac{f(x)dx}{(a-x)^3}=2\left\{\frac{f(1)}{4(a-1)^2}-\frac{f(0)}{4a^2}-\int_{\frac{1}{4a}}^{\frac{1}{4(a-1)}}dsf'\left(a-\frac{1}{4s}\right)\right\}. \quad \text{(10.)}$$

Now this result may be verified by integrating the first member of the equation by parts, and transforming the integral which remains by assuming $x=a-\frac{1}{4s}$.

Now $f(\sigma)$ is to vanish when σ falls without the limits 0 and 1, and s as falling between the limits 0 and ∞ is to be positive. But from the expression for σ, it appears that when $s=0$ $\sigma=0$, and when $s=\infty$ $\sigma=\frac{a^2}{h_1^2}+\frac{b^2}{h_2^2}+\frac{c^2}{h_3^2}$; also σ increases with s (Art. 42.), and therefore passes over the value 1 when $\frac{a^2}{h_1^2}+\frac{b^2}{h_2^2}+\frac{c^2}{h_3^2}>1$, but reaches not that value otherwise.

The former condition is realized when the attracted point is external. Representing by η the value of s for which $\sigma=1$, we have

$$-\varrho\frac{dV}{da}=2h_1h_2h_3\pi\varrho a\int_0^{\eta}\frac{ds.s^{\frac{1}{2}}f\left(\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}\right)}{(1+h_1^2s)^{\frac{3}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

η being the positive root of the equation

$$\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

When the attracted point is internal we have only to substitute ∞ for η in the upper limit of integration; for all positive values of σ then satisfy the condition $\sigma<1$, and positive values only are admissible.

Both these cases may also be derived from the more general theorem deducible from (5.), Art. 42,

$$-\varrho\frac{d}{da}\iiint\frac{dxdydz}{\{h^2+(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{1}{2}}}=2h_1h_2h_3\pi\varrho a\int_0^{\eta}\frac{ds.s^{-\frac{1}{2}}f\left(h^2s+\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}\right)}{(1+h_1^2s)^{\frac{3}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}s^{\frac{1}{2}}}, \quad (16.)$$

where η is the positive root of the equation

$$h^2s+\frac{a^2s}{1+h_1^2s}+\frac{b^2s}{1+h_2^2s}+\frac{c^2s}{1+h_3^2s}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

When h approaches to 0 this root approaches to the positive root of the equation (15.) if $\frac{a^2}{h_1^2}+\frac{b^2}{h_2^2}+\frac{c^2}{h_3^2}$ is greater than 1, but tends to ∞ if $\frac{a^2}{h_1^2}+\frac{b^2}{h_2^2}+\frac{c^2}{h_3^2}$ is equal to or less than 1.

If $f(\sigma)=1$, the expressions are easily reducible to elliptic functions and agree with known results.

Lastly, let the law of force be that of the inverse fourth power of the distance and equal to ϱ. The expression for the attraction on an external point (a, b, c) is

$$-\frac{\varrho}{3}\frac{dV}{da},$$

where

$$V=\iiint\frac{dxdydz}{\{(a-x)^2+(b-y)^2+(c-z)^2\}^{\frac{3}{2}}}=2\pi h_1h_2h_3\int_0^{\infty}\frac{ds.s^{\frac{1}{2}}f(\sigma)}{(1+h_1^2s)^{\frac{1}{2}}(1+h_2^2s)^{\frac{1}{2}}(1+h_3^2s)^{\frac{1}{2}}},$$

therefore

$$-\frac{\varrho}{3}\frac{df(\sigma)}{da}=-\frac{\varrho}{3}\frac{d(\sigma)}{da}\frac{df(\sigma)}{d(\sigma)}=-\frac{2\varrho as}{3(1+h_1^2s)}\cdot\frac{df(\sigma)}{d(\sigma)}=-\frac{2\varrho as}{3(1+h_1^2s)}\cdot\frac{ds}{d(\sigma)}\cdot\frac{df(\sigma)}{ds}.$$

Therefore

$$-\frac{s}{3}\,\frac{dV}{da}=-\frac{4}{3}\pi\varrho a h_1 h_2 h_3\int_0^\infty \frac{s^{\frac{3}{2}}\frac{ds}{d\sigma}\cdot\frac{df(\sigma)}{ds}\,ds}{(1+h_1^2 s)^{\frac{3}{2}}(1+h_2^2 s)^{\frac{1}{2}}(1+h_3^2 s)^{\frac{1}{2}}}. \quad . \quad . \quad . \quad (18.)$$

Let η be the positive root of the equation

$$\frac{a^2 s}{1+h_1^2 s}+\frac{b^2 s}{1+h_2^2 s}+\frac{c^2 s}{1+h_3^2 s}=1,$$

and let the density be uniform; then $f(\sigma)=1$ or 0, according as s is less or greater than η. Before and after the break, therefore, $\frac{df(\sigma)}{ds}=0$. At the break we have, by the reasoning of a previous section, $\frac{df(\sigma)}{ds}ds=-1$. We must therefore substitute this value in (18.), and in the rest of the expression under the integral sign change s into η. Observing that this substitution converts $\frac{d\sigma}{ds}$ into $\frac{a^2}{(1+h_1^2 s)^2}+\frac{b^2}{(1+h_2^2 s)^2}+\frac{c^2}{(1+h_3^2 s)^2}$, and that the integral being reduced to a single finite element we may reject the integral sign, we have

$$-\frac{\rho}{3}\cdot\frac{dV}{da}=\frac{\frac{4}{3}\pi\rho a h_1 h_2 h_3 \eta^{\frac{3}{2}}}{\left\{\frac{a^2}{(1+h_1^2\eta)^2}+\frac{b^2}{(1+h_2^2\eta)^2}+\frac{c^2}{(1+h_3^2\eta)^2}\right\}(1+h_1^2\eta)^{\frac{3}{2}}(1+h_2^2\eta)^{\frac{1}{2}}(1+h_3^2\eta)^{\frac{1}{2}}}.$$

This result is due, I believe, to Mr. Cayley, but was originally obtained by an entirely different analysis.

It only remains to add, that when the index of differentiation is fractional we must revert to the first expression in (5.), Art. 42, and effect the integrations separately. The integration with respect to v may always be performed. The possibility of the two others will depend upon the nature of the problem under consideration. Thus writing the expression in the form

$$V=\frac{\pi^{\frac{n}{2}-1}h_1 h_2 \ldots h_n}{\Gamma(i)}\int_0^1\int_0^\infty \frac{da.ds.s^{i-1}f(a)}{(1+h_1^2 s)^{\frac{1}{2}}\ldots(1+h_n^2 s)^{\frac{1}{2}}}\int_0^\infty dv.v^{i-\frac{n}{2}}\cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\},$$

it is easily shown that, according as a is greater or less than σ,

$$\int_0^\infty dv.v^{i-\frac{n}{2}}\cos\left\{(a-\sigma)v+\left(i-\frac{n}{2}\right)\frac{\pi}{2}\right\}=\frac{\Gamma\left(i-\frac{n}{2}+1\right)\sin\left(i-\frac{n}{2}+1\right)\pi}{(a-\sigma)^{i-\frac{n}{2}+1}} \text{ or } 0,$$

$$=\frac{\pi}{\Gamma\left(\frac{n}{2}-1\right)(a-\sigma)^{i-\frac{n}{2}+1}} \text{ or } 0,$$

by a known property of the function Γ. The latter form gives

$$V=\frac{\pi^{\frac{n}{2}}h_1 h_2 \ldots h_n}{\Gamma(i)\Gamma\left(\frac{n}{2}-i\right)}\int_0^1\int_0^\infty \frac{da.ds.s^{i-1}f(a)}{(1+h_1^2 s)^{\frac{1}{2}}(1+h_2^2 s)^{\frac{1}{2}}\ldots(1+h_n^2 s)^{\frac{1}{2}}(a-\sigma)^{i-\frac{n}{2}+1}}.$$

This transformation, or one in effect equivalent to it, is due to Mr. Cayley *, who has applied it to obtain the value of a remarkable definite integral which occurs in the mathematical theory of electricity.

* Cambridge and Dublin Mathematical Journal, vol. ii. p. 219.

-6-

XIX. *On the Differential Equations which determine the form of the Roots of Algebraic Equations.*

By George Boole, *F.R.S., Professor of Mathematics in Queen's College, Cork.*

Received April 27,—Read May 26, 1864.

1. Mr. Harley* has shown that any root of the equation

$$y^n - ny + (n-1)x = 0$$

satisfies the differential equation

$$y - \frac{\left(D - \frac{2n-1}{n}\right)\left(D - \frac{3n-2}{n}\right)\cdot\cdot\left(D - \frac{n^2-n+1}{n}\right)}{D(D-1)\cdot\cdot(D-n+1)} e^{(n-1)\theta} y = 0, \quad . \quad . \quad . \quad . \quad (1)$$

in which $e^\theta = x$, and $D = \frac{d}{d\theta}$, provided that n be a positive integer greater than 2. This result, demonstrated for particular values of n, and raised by induction into a general theorem, was subsequently established rigorously by Mr. Cayley by means of Lagrange's theorem.

For the case of $n=2$, the differential equation was found by Mr. Harley to be

$$y - \frac{D - \frac{3}{2}}{D} e^\theta y = \tfrac{1}{2} e^\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving these differential equations for the particular cases of $n=2$ and $n=3$, Mr. Harley arrived at the actual expression of the roots of the given algebraic equation for these cases. That all algebraic equations up to the fifth degree can be reduced to the above trinomial form, is well known.

A solution of (1) by means of definite triple integrals in the case of $n=4$ has been published by Mr. W. H. L. Russell; and I am informed that a general solution of the equation by means of a definite single integral has been obtained by the same analyst.

While the subject seems to be more important with relation to differential than with reference to algebraic equations, the connexion into which the two subjects are brought must itself be considered as a very interesting fact. As respects the former of these subjects, it may be observed that it is a matter of quite fundamental importance to ascertain for what forms of the function ϕ (D), equations of the type

$$u + \phi(D) e^{m\theta} u = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

admit of finite solution. We possess theorems which enable us to deduce from each known integrable form an infinite number of others. Yet there is every reason to think

* Proceedings of the Literary and Philosophical Society of Manchester, No. 12, Session 1861–62.

that the number of really primary forms—of forms the knowledge of which, in combination with such known theorems, would enable us to solve all equations of the above type that are finitely solvable—is extremely small. It will, indeed, be a most remarkable conclusion, should it ultimately prove that the forms in question stand in absolute and exclusive connexion with the class of algebraic equations here considered.

The following paper is a contribution to the general theory under the aspect last mentioned. In endeavouring to solve Mr. HARLEY'S equation by definite integrals, I was led to perceive its relation to a more general equation, and to make this the subject of investigation. The results will be presented in the following order:—

First, I shall show that if u stand for the mth power of any root of the algebraic equation

$$y^n - xy^{n-1} - 1 = 0,$$

then u, considered as a function of x, will satisfy the differential equation

$$[\mathrm{D}]^n u + \left[\frac{n-1}{n}\mathrm{D} + \frac{m}{n} - 1\right]^{n-1} \left(\frac{\mathrm{D}}{n} - \frac{m}{n} - 1\right) e^{n\theta} u = 0,$$

in which $e^\theta = x$, $\mathrm{D} = \frac{d}{d\theta}$, and the notation

$$[a]^b = a(a-1)(a-2)\ldots(a-b+1)$$

is adopted.

Secondly, I shall show that for particular values of m, the above equation admits of an immediate first integral, constituting a differential equation of the $n-1$th order, and that the results obtained by Mr. HARLEY are particular cases of this depressed equation, their difference of form arising from difference of determination of the arbitrary constant.

Thirdly, I shall solve the general differential equation by definite integrals.

Fourthly, I shall determine the arbitrary constants of the solution so as to express the mth power of that real root of the proposed algebraic equation which reduces to 1 when $x=0$.

The differential equation which forms the chief subject of these investigations certainly occupies an important place, if not one of exclusive importance, in the theory of that large class of differential equations of which the type is expressed in (3). At present, I am not aware of the existence of any differential equations of that particular type which admit of finite solution at all, otherwise than by an ultimate reduction to the form in question, or by a resolution into linear equations of the first order. It constitutes, in fact, a generalization of the form

$$u + \frac{a(\mathrm{D}-2)^2 \pm n^2}{\mathrm{D}(\mathrm{D}-1)} e^{2\theta} u = 0$$

given in my memoir "On a General Method in Analysis" (Philosophical Transactions for 1844, Part II.).

Formation of the Differential Equation.—General finite integral.

2. Proposition.—*If* u *represent the* mth *power of any root of the algebraic equation*

$$y^n - xy^{n-1} - 1 = 0,$$

then u, *considered as a function of* x, *satisfies the linear differential equation*

$$[\mathrm{D}]^n u + \left[\frac{n-1}{n}\mathrm{D} + \frac{m}{n} - 1\right]^{n-1} \left(\frac{\mathrm{D}}{n} - \frac{m}{n} - 1\right) e^{n\theta} u = 0,$$

in which $e^\theta = x$, *and* $\mathrm{D} = \frac{d}{d\theta}$.

And the complete integral of the above differential equation will be

$$u = \mathrm{C}_1 y_1^m + \mathrm{C}_2 y_2^m \ldots + \mathrm{C}_n y_n^m,$$

$y_1, y_2, \ldots y_n$ *being the* n *roots of the given algebraic equation.*

Representing y^n by z, we may give to the proposed algebraic equation the form

$$z = b + xz^{\frac{n-1}{n}}, \quad \ldots \quad (1)$$

in which $b=1$. Hence by Lagrange's theorem

$$u = z^{\frac{m}{n}} = b^{\frac{m}{n}} + b^{\frac{n-1}{n}} \frac{d}{db}\left(b^{\frac{m}{n}}\right)x + \frac{d}{db}\left(b^{\frac{2(n-1)}{n}} \frac{d}{db} b^{\frac{m}{n}}\right)\frac{x^2}{1.2} + \&\text{c.}, \quad \ldots \quad (2)$$

the general term of the expansion being

$$\left(\frac{d}{db}\right)^{r-1} \left\{b^{\frac{r(n-1)}{n}} \frac{d}{db} b^{\frac{m}{n}}\right\} \frac{x^r}{1.2..r}, \quad \ldots \quad (3)$$

which, on effecting the operations indicated, becomes

$$\frac{m\left[\frac{m+r(n-1)}{n} - 1\right]^{r-1} b^{\frac{m-r}{n}}}{n[r]^r} x^r \quad \ldots \quad (4)$$

We see then that u is expanded in a series of the form

$$u_0 + u_1 x + u_2 x^2 + \&\text{c. } ad\ inf.,$$

in which, since $b=1$,

$$u_r = \frac{m\left[\frac{m+(n-1)r}{n} - 1\right]^{r-1} \times (1)^{\frac{m-r}{n}}}{n[r]^r}; \quad \ldots \quad (5)$$

and this expression will represent the first term as well as the succeeding coefficients of the Lagrangian development, provided that we interpret the form $[p]^0$ by 1, and $[p]^{-1}$ by $\frac{1}{p+1}$.

As $1^{\frac{1}{n}}$ admits of n distinct values, the above development may be made to represent the mth power of any one of the n roots of the given algebraic equation. In particular,

if we give to $1^{\frac{1}{n}}$ the particular value 1, we have

$$u_r = \frac{m\left[\frac{m+(n-1)r}{n}-1\right]^{r-1}}{n[r]^r},$$

and the expansion then represents the mth power of that particular root which, when $x=0$, reduces to 1. The law of the series upon which the formation of the differential equation depends is, as we shall perceive, independent of these determinations.

Changing r into $r-n$, we have

$$u_{r-n} = \frac{m\left[\frac{m+(n-1)r}{n}-n\right]^{r-n-1}\times 1^{\frac{m-r}{n}+1}}{n[r-n]^{r-n}},$$

whence the law of the series is seen to be

$$[r]^r u_r + \left[\frac{m+(n-1)r}{n}-1\right]^{n-1}\left(\frac{r}{n}-\frac{m}{n}-1\right)u_{n-r}=0, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and therefore, by what is shown in my memoir "On a General Method in Analysis," the differential equation defining u will be

$$[\mathrm{D}]^n u + \left[\frac{m+(n-1)\mathrm{D}}{n}-1\right]^{n-1}\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-1\right)\epsilon^{n\theta}u=0, \quad . \quad . \quad . \quad . \quad (\mathrm{I})$$

in which $\epsilon^\theta = x$ and $\mathrm{D}=\frac{d}{d\theta}$.

3. As u may here represent the mth power of *any* of the roots of the given equation, it is evident that the general integral of the above differential equation will be

$$u = \mathrm{C}_1 y_1^m + \mathrm{C}_2 y_2^m \ldots + \mathrm{C}_n y_n^m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

exception arising, however, in the case in which for a particular value of m the n particular integrals y_1^m, y_2^m, .. y_n^m cease to be independent. In such cases the above value of u constitutes an integral, but not the general integral of the differential equation.

For instance, if $m=-1$, and if we reduce the given algebraic equation to the form

$$(y^{-1})^n + xy^{-1} - 1 = 0,$$

it is evident that, except when $n=2$, we shall have

$$y_1^{-1} + y_2^{-1} \ldots + y_n^{-1} = 0.$$

Here then

$$u = \mathrm{C}_1 y_1^{-1} + \mathrm{C}_2 y_2^{-1} \ldots + \mathrm{C}_n y_n^{-1}$$

may be reduced to the form

$$u = (\mathrm{C}_1 - \mathrm{C}_n) y_1^{-1} + (\mathrm{C}_2 - \mathrm{C}_n) y_2^{-1} \ldots + (\mathrm{C}_{n-1} - \mathrm{C}_n) y_{n-1}^{-1},$$

virtually involving but $n-1$ arbitrary constants.

Such cases of failure may, however, be treated by giving to the integral a form which for the particular value of m shall become indeterminate, and then seeking the limiting

which is satisfied by $u=y_1^{-1}$ and by $u=y_2^{-1}$, but, as is evident from its unhomogeneous form, not by $u=C_1y_1^{-1}+C_2y_2^{-1}$. In this case, in fact, the condition $y_1^{-1}+y_2^{-1}=0$ not being fulfilled, the primary differential equation (I) suffers no change in the form of its general solution.

Mr. HARLEY's results are in effect transformations of (10) and (11). Since $u=y^{-1}$, it is seen that u will satisfy the algebraic equation

$$u^n+xu-1=0.$$

Transform this by assuming

$$x=-n(1-n)^{\frac{1-n}{n}}x'^{\frac{1-n}{n}}, \quad u=(1-n)^{-\frac{1}{n}}x'^{-\frac{1}{n}}u',$$

and we have

$$u'^n-nu'+(n-1)x'=0,$$

which is Mr. HARLEY's algebraic equation in form. Hence, if $x'=e^{\theta'}$ and $D'=\frac{d}{d\theta'}$, we shall have

$$e^\theta=-n(1-n)^{\frac{1-n}{n}}e^{\frac{1-n}{n}\theta'}, \quad u=(1-n)^{-\frac{1}{n}}e^{-\frac{1}{n}\theta'}u', \quad D=\frac{n}{1-n}D'.$$

And (10) will become

$$\left[\frac{nD'}{1-n}\right]^{n-1}e^{-\frac{1}{n}\theta'}u'+\frac{1}{n}\left[-D'-\frac{1}{n}-1\right]^{n-1}(-n)^n(1-n)^{1-n}e^{\left(1-n-\frac{1}{n}\right)\theta'}u'=0.$$

Multiply by $e^{\left(n-1+\frac{1}{n}\right)\theta'}$, and we have

$$\left[\frac{n\left(D'-n+1-\frac{1}{n}\right)}{1-n}\right]^{n-1}e^{(n-1)\theta'}u'-[-D'+n-2]^{n-1}(-n)^{n-1}(1-n)^{1-n}u'=0.$$

Now

$$\left[\frac{n\left(D'-n+1-\frac{1}{n}\right)}{1-n}\right]^{n-1}=\left[\frac{n}{1-n}D'+n-\frac{1}{1-n}\right]^{n-1}=(-1)^{n-1}\left[\frac{n}{n-1}D'-\frac{2n-1}{n-1}\right]^{n-1},$$

and

$$[-D'+n-2]^{n-1}=(-1)^{n-1}[D']^{n-1}.$$

Hence

$$\left[\frac{n}{n-1}D'-\frac{2n-1}{n-1}\right]^{n-1}e^{(n-1)\theta'}u'-\left(\frac{n}{n-1}\right)^{n-1}[D']^{n-1}u'=0,$$

or

$$[D']^{n-1}u'-\left(\frac{n-1}{n}\right)^{n-1}\left[\frac{n}{n-1}D'-\frac{2n-1}{n-1}\right]^{n-1}e^{(n-1)\theta'}u'=0,$$

which is Mr. HARLEY's equation (1), art. 1. When $n=2$, we obtain from (11), by the same transformations, Mr. HARLEY's second equation (2), art. 1.

Not only for the particular value $m=-1$, but apparently for all integer values of m, the general differential equation (I) admits of one integration. It may be said that while the differential equation determining the form of the mth power of a root of the algebraic equation is in general of the nth order, this equation may, when m is an integer, be reduced to an equation of the $n-1$th order; not, however, like the higher equation,

unvarying in its type. I have thus verified some other particular forms obtained by Mr. HARLEY.

Solution of the Differential Equation by Definite Integrals.

5. On account of the difficulty of the investigation, I propose to employ two distinct methods leading to coincident results.

First Method.—Operating on both sides of the given differential equation (I) with $\{[D]^n\}^{-1}$, we have

$$u+\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{n-1}\left(\frac{D}{n}-\frac{m}{n}-1\right)}{[D]^n}e^{n\theta}u=C_0+C_1e^{\theta}\cdots+C_{n-1}e^{(n-1)\theta}, \quad . \quad . \quad (1)$$

$C_0, C_1, \ldots C_{n-1}$ being arbitrary constants. Let us represent

$$\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{n-1}\left(\frac{D}{n}-\frac{m}{n}-1\right)}{[D]^n}e^{n\theta}U,$$

whatever the nature of the subject U, by ϱU, then the differential equation becomes

$$u+\varrho\, u=C_0+C_1e^{\theta}\cdots+C_{n-1}e^{(n-1)\theta},$$

or

$$(1+\varrho)u=C_0+C_1e^{\theta}\cdots+C_{n-1}e^{(n-1)\theta};$$

$$\therefore \quad u=(1+\varrho)^{-1}\{C_0+C_1e^{\theta}\cdots+C_{n-1}e^{(n-1)\theta}\}=\Sigma_iC_i(1+\varrho)^{-1}e^{i\theta},$$

the summation extending from $i=0$ to $i=n-1$.

Now

$$(1+\varrho)^{-1}e^{i\theta}=(1-\varrho+\varrho^2-\varrho^3\ldots)e^{i\theta}.$$

But if

$$\phi(D)=\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{n-1}\left(\frac{D}{n}-\frac{m}{n}-1\right)}{[D]^n},$$

we have

$$\varrho e^{i\theta}=\phi(D)e^{n\theta}e^{i\theta},$$

$$\varrho^2e^{i\theta}=\phi(D)e^{n\theta}\phi(D)e^{(n+i)\theta}=\phi(D)\phi(D-n)e^{(2n+i)\theta},$$

$$\varrho^pe^{i\theta}=\phi(D)\phi(D-n)\ldots\phi\big(D-(p-1)n\big)e^{(pn+i)\theta}.$$

But from the form of $\phi(D)$ it is easily seen that

$$\phi(D)\phi(D-n)=\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{2(n-1)}\left[\frac{D}{n}-\frac{m}{n}-1\right]^2}{[D]^{2n}},$$

and generally

$$\phi(D)\phi(D-n)\ldots\phi\big(D-(p-1)n\big)=\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{p(n-1)}\left[\frac{D}{n}-\frac{m}{n}-1\right]^p}{[D]^{pn}};$$

$$\therefore\ \varrho^pe^{i\theta}=\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{p(n-1)}\left[\frac{D}{n}-\frac{m}{n}-1\right]^p}{[D]^{pn}}e^{(pn+i)\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now

$$[a]^b = a(a-1)\ldots(a-b+1)$$
$$=\frac{\Gamma(a+1)}{\Gamma(a-b+1)},$$

provided that $a+1$ and $a-b+1$ are positive. This law we can extend symbolically to expressions in which D appears, provided that, in the application of the symbolic forms thence arising, D shall admit of an interpretation which shall effectively make the subjects of the symbol Γ to be positive numerical magnitudes. Under this condition we have then

$$\varrho^p e^{i\theta} = \frac{\dfrac{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}\right)}{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-p(n-1)\right)} \times \dfrac{\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}\right)}{\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-p\right)}}{\dfrac{\Gamma(\mathrm{D}+1)}{\Gamma(\mathrm{D}-pn+1)}} e^{(pn+i)\theta}$$

$$=\Phi(\mathrm{D})\Psi(\mathrm{D})e^{(pn+i)\theta},$$

where

$$\Phi(\mathrm{D}) = \frac{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}\right)\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}\right)}{\Gamma(\mathrm{D}+1)},$$

$$\Psi(\mathrm{D}) = \frac{\Gamma(\mathrm{D}-pn+1)}{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-p(n-1)\right)\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-p\right)}.$$

Now

$$\Psi(\mathrm{D})e^{(pn+i)\theta} = \Psi(pn+i)e^{(pn+i)\theta}$$
$$=\frac{\Gamma(i+1)}{\Gamma\left(\frac{n-1}{n}i+\frac{m}{n}\right)\Gamma\left(\frac{i-m}{n}\right)}e^{(pn+i)\theta}.$$

We see then that the conditions

$$(n-1)i+m \overline{\gtrless} 0, \quad i-m \overline{\gtrless} 0 \quad \ldots \ldots \ldots \quad (3)$$

must be satisfied. For $i=0$ these conditions are inconsistent, and the proposed employment of Γ therefore unlawful. For values of i greater than 0 the conditions will be found to amount to this, viz. that m must lie between the limits $-(n-1)$ and 1. We shall suppose m thus conditioned, and shall consider first the case in which $i>0$.

Here then we have, interpreting D by $pn+i$ in $\Psi(\mathrm{D})$, but leaving it unchanged in $\Phi(\mathrm{D})$,

$$\varrho^p e^{i\theta} = \frac{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}\right)\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}\right)}{\Gamma(\mathrm{D}+1)} \quad \frac{\Gamma(i+1)}{\Gamma\left(\frac{n-1}{n}i+\frac{m}{n}\right)\Gamma\left(\frac{i-m}{n}\right)}e^{(pn+i)\theta}, \quad \ldots \quad (4)$$

it being seen that if we similarly interpreted D in $\Phi(\mathrm{D})$ the conditions relative to Γ would be satisfied throughout.

Hence if we write

$$\frac{\Gamma(i+1)C_i}{\Gamma\left(\frac{n-1}{n}i+\frac{m}{n}\right)\Gamma\left(\frac{i-m}{n}\right)}=A_i,$$

we shall have

$$(1-\varrho+\varrho^2-\varrho^3+\&c.)C_ie^{i\theta}=\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}A_i(e^{i\theta}-e^{(n+i)\theta}+e^{(2n+i)\theta}-\&c.)$$

$$=\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\frac{A_ie^{i\theta}}{1+e^{n}},$$

and therefore

$$\Sigma_iC_i(1+\varrho)^{-1}e^{i\theta}=\Sigma_i\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\frac{A_ie^{i\theta}}{1+e^{n\theta}}, \quad \ldots \ldots \quad (5)$$

the summation extending from $i=1$ to $i=n-1$.

Consider next the case in which $i=0$. We have, when p is not less than 1,

$$\begin{aligned}\varrho^pC_0&=\varrho^{p-1}\varrho C_0\\&=\varrho^{p-1}\phi(D)e^{n\theta}C_0\\&=C_0\phi(n)\varrho^{p-1}e^{n\theta}.\end{aligned}$$

But changing in (4) p into $p-1$, and i into n,

$$\varrho^{p-1}e^{i\theta}=\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\frac{\Gamma(n+1)}{\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(\frac{n-m}{n}\right)}e^{pn\theta}.$$

Hence, if we write

$$C_0\phi(n)\frac{\Gamma(n+1)}{\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(\frac{n-m}{n}\right)}=-A_n, \quad \ldots \ldots \quad (5')$$

we have for all positive integral values of p,

$$\varrho^pC_0=-A_n\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}e^{pn\theta},$$

and therefore

$$(1-\varrho+\varrho^2-\varrho^3+\&c.)C_0=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}(A_ne^{n\theta}-A_ne^{2n\theta}+A_ne^{3n\theta}-\&c.);$$

$$\therefore\ (1+\varrho)^{-1}C_0=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\frac{A_ne^{n\theta}}{1+e^{n\theta}}.$$

Combining this with (5), we find

$$u=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\left\{\frac{A_1e^\theta+A_2e^{2\theta}\ldots+A_ne^{n\theta}}{1+e^{n\theta}}\right\}$$

$$=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D)}D^{-1}\left\{\frac{A_1e^\theta+A_2e^{2\theta}\ldots+A_ne^{n\theta}}{1+e^{n\theta}}\right\}.$$

Now, resolving the rational fraction, we have

$$D^{-1}\frac{A_1e^\theta+A_2e^{2\theta}\ldots+A_ne^{n\theta}}{1+e^{n\theta}}=D^{-1}\left\{\frac{N_1e^\theta}{1-\alpha_1e^\theta}+\frac{N_2e^\theta}{1-\alpha_2e^\theta}\ldots+\frac{N_ne^\theta}{1-\alpha_ne^\theta}\right\}$$

$$=B_1\log(1-\alpha_1e^\theta)+B_2\log(1-\alpha_2e^\theta)\ldots+B_n\log(1-\alpha_ne^\theta),$$

$\alpha_1, \alpha_2, \ldots \alpha_n$ being the nth roots of -1, and $B_i=\frac{-N_i}{\alpha_i}$. Hence

$$u=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D)}\{B_1\log(1-\alpha_1e^\theta)\ldots+B_n\log(1-\alpha_ne^\theta)\}. \quad . \quad . \quad (6)$$

In this expression $B_1, \ldots B_n$, being generated from the arbitrary constants $C_0, C_1, \ldots C_{n-1}$, may themselves be regarded as arbitrary constants. And this being done, C_0 will become a dependent constant, the form of which it will be necessary to determine.

First, however, let us endeavour to interpret by a definite integral the symbolic function of D.

We know that a and b being positive quantities,

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 dt\, t^{a-1}(1-t)^{b-1}=\int_0^\infty \frac{dt\, t^{a-1}}{(1+t)^{a+b}}.$$

If we employ the second of these forms, we shall have

$$\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D)}\phi(e^\theta)=\int_0^\infty dt\,\frac{t^{\frac{n-1}{n}D+\frac{m}{n}-1}}{(1+t)^D}\phi(e^\theta)$$

$$=\int_0^\infty dt\, t^{\frac{m}{n}-1}\left(\frac{t^{\frac{n-1}{n}}}{1+t}\right)^D\phi(e^\theta)$$

$$=\int_0^\infty dt\, t^{\frac{m}{n}-1}\phi\left(\frac{t^{\frac{n-1}{n}}e^\theta}{1+t}\right)$$

by a known symbolical form of TAYLOR's theorem. Hence if

$$\frac{t^{\frac{n-1}{n}}}{1+t}=T,$$

we have

$$u=C_0+B_1\int_0^\infty dt\, t^{\frac{m}{n}-1}\log(1-\alpha_1Te^\theta)\cdots+B_n\int_0^\infty dt\, t^{\frac{m}{n}-1}\log(1-\alpha_nTe^\theta). \quad . \quad . \quad (7)$$

6. In determining C_0 the following theorem will be of use, viz.:—

If r *be a positive integer, and* a *positive and less than* r, *then*

$$\Gamma(a)\Gamma(r-a)=\frac{[r-a-1]^{r-1}\pi}{\sin(a\pi)}. \qquad (8)$$

This may be proved as follows:—

Let i be the greatest integer in a, and let $a-i=a'$. Then

$$\Gamma(a)\Gamma(r-a)=[a-1]^i\Gamma(a')\times[r-a-1]^{r-i-1}\Gamma(1-a').$$

But a' being a positive proper fraction,

$$\Gamma(a')\Gamma(1-a')=\frac{\pi}{\sin(a'\pi)},$$

and

$$[a-1]^i=(a-1)(a-2)\,..\,(a-i)$$
$$=(-1)^i(i-a)(i-a-1)\,..\,(1-a),$$
$$[r-a-1]^{r-i-1}=(r-a-1)(r-a-2)\,..\,(i-a+1);$$
$$\therefore\quad [r-a-1]^{r-i-1}[a-1]^i=(-1)^i(r-a-1)\,..\,(i-a+1)\times(i-a)\,..\,(1-a)$$
$$=(-1)^i[r-a-1]^{r-1}.$$

Hence

$$\Gamma(a)\Gamma(r-a)=(-1)^i[r-a-1]^{r-1}\frac{\pi}{\sin(a'\pi)}.$$

But

$$\sin(a'\pi)=\sin(a\pi-i\pi)=(-1)^i\sin(a\pi),$$

$$\therefore\quad \Gamma(a)\Gamma(r-a)=\frac{[r-a-1]^{r-1}\pi}{\sin(a\pi)},$$

as was to be proved.

Now in the instance before us we have by (5′)

$$C_0=-A_n\frac{\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(\frac{n-m}{n}\right)}{\Gamma(n+1)\phi(n)},$$

where

$$\phi(n)=\frac{\left[n+\frac{m}{n}-2\right]^{n-1}\left(-\frac{m}{n}\right)}{[n]^n}.$$

Hence, since $\Gamma(n+1)=[n]^n$,

$$C_0=A_n\frac{\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right)}{\left[n+\frac{m}{n}-2\right]^{n-1}\times\frac{m}{n}};$$

wherefore $1-\frac{m}{n}$ being a positive quantity, and n a positive integer, we have, by the

above theorem,

$$\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right)=\frac{\left[n-2+\frac{m}{n}\right]^{n-1}\pi}{\sin\left(1-\frac{m}{n}\right)\pi}$$

$$=\frac{\left[n-2+\frac{m}{n}\right]^{n-1}\pi}{\sin\left(\frac{m}{n}\pi\right)}.$$

Accordingly

$$C_0=\frac{A_n\pi}{\frac{m}{n}\sin\frac{m\pi}{n}}.$$

But since

$$\frac{N_1e^\theta}{1-\alpha_1e^\theta}\cdots+\frac{N_ne^\theta}{1-\alpha_ne^\theta}=\frac{A_1e^\theta\cdots+A_ne^{n\theta}}{1+e^{n\theta}},$$

we have

$$A_n=(-1)^{n-1}\left(\frac{N_1}{\alpha_1}\cdots+\frac{N_n}{\alpha_n}\right)\alpha_1\alpha_2\ldots\alpha_n$$

$$=(-1)^n\quad(B_1\cdots+B_n)\times(-1)^n$$

$$=\qquad B_1\cdots+B_n.$$

Therefore, finally,

$$C_0=\frac{B_1+B_2\cdots+B_n}{\frac{m}{n}\sin\frac{m\pi}{n}}.$$

Substituting in (7), and replacing e^θ by x, we have

$$u=\frac{(B_1+B_2\cdots+B_n)\pi}{\frac{m}{n}\sin\frac{m\pi}{n}}+B_1\int_0^\infty dt\ t^{\frac{m}{n}-1}\log(1-\alpha_1x\mathrm{T})\cdots+B_n\int_0^\infty dt\ t^{\frac{m}{n}-1}\log(1-\alpha_nx\mathrm{T}),\quad .\quad (9)$$

wherein, it must be remembered, that $\alpha_1, \alpha_2, \ldots \alpha_n$ are the several nth roots of -1, and

$$\mathrm{T}=\frac{t^{\frac{n-1}{n}}}{1+t}.$$

And this is the general integral of (I), $B_1, B_2, \ldots B_n$ being the arbitrary constants of the solution.

Second Method.—7. For the *finite* solution of differential equations of the form

$$f_0(D)u+f_1(D)e^{n\theta}u=0,$$

it is usually convenient to reduce them to the form

$$u+\frac{f_1(D)}{f_0(D)}e^{n\theta}u=\{f_0(D)\}^{-1}0,$$

which falls under the general type

$$u+\phi(\mathrm{D})e^{n\theta}u=\mathrm{U}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

U being a function of θ when the inverse operation $\{f_0(\mathrm{D})\}^{-1}0$ has been performed.

The theory of equations of the above type has been discussed fully in my memoir "On a General Method in Analysis." In particular it is there shown that the above equation can be converted into another of the same type,

$$v+\psi(\mathrm{D})e^{n\theta}v=\mathrm{V},$$

by assuming

$$u=\mathrm{P}_n\frac{\phi(\mathrm{D})}{\psi(\mathrm{D})}v; \quad \mathrm{V}=\left\{\mathrm{P}_n\frac{\phi(\mathrm{D})}{\psi(\mathrm{D})}\right\}^{-1}\mathrm{U}, \quad \ldots \ldots \ldots \quad (2)$$

where

$$\mathrm{P}_n\frac{\phi(\mathrm{D})}{\psi(\mathrm{D})}=\frac{\phi(\mathrm{D})\phi(\mathrm{D}-n)\phi(\mathrm{D}-2n)\ldots ad\ inf.}{\psi(\mathrm{D})\psi(\mathrm{D}-n)\psi(\mathrm{D}-2n)\ldots ad\ inf.}.$$

This theory I shall apply here, not to the ordinary finite solution, but to the solution by definite integrals of the differential equation (I). In doing this I shall give to U and V the particular values 0. We are justified in doing this by the canons relating to the arbitrary constants which are laid down in the memoir; but it will suffice here to direct attention to the fact that while the processes employed are strictly speaking particular, they lead to a solution involving the requisite number of arbitrary constants, and at the same time of the proper *form*, as manifested by the succession of the indices in its development.

Giving then to (I) the form

$$u+\frac{\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]^{n-1}\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-1\right)}{[\mathrm{D}]^n}e^{n\theta}u=0,$$

assume as the transformed equation

$$v+\frac{1}{[\mathrm{D}]^n}e^{n\theta}v=0.$$

Then by (2)

$$u=\mathrm{P}_n\left\{\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-1\right)\right\}v.$$

Now

$$\mathrm{P}_n\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]^{n-1}=\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right)\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-2\right)\ldots ad\ inf.,$$

since representing $\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]^{n-1}$ by $\phi(\mathrm{D})$, the first term in the factorial expression of $\phi(\mathrm{D}-n)$ will so follow the last term in that of $\phi(\mathrm{D})$ as to leave the law of factorial succession unbroken. Again, if $\mathrm{A}_i e^{i\theta}$ be any term in the development of v, we have, i being a positive integer,

$$\begin{aligned}&\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right)\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-2\right)\ldots\mathrm{A}_i e^{i\theta}\\&=\mathrm{A}_i\left(\frac{n-1}{n}i+\frac{m}{n}-1\right)\left(\frac{n-1}{n}i+\frac{m}{n}-2\right)\ldots e^{i\theta}\\&=\mathrm{A}_i\mathrm{C}\Gamma\left(\frac{n-1}{n}i+\frac{m}{n}\right)e^{i\theta},\end{aligned}$$

C being a constant, the value of which does not change with i. Hence we may write

$$P_n\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{n-1}=C\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right),$$

and in like manner

$$P_n\left(\frac{D}{n}-\frac{m}{n}-1\right)=C'\Gamma\left(\frac{D}{n}-\frac{m}{n}\right).$$

The legitimacy of the introduction of Γ depends upon the condition

$$\frac{n-1}{n}i+\frac{m}{n}>0,\ \frac{i}{n}-\frac{m}{n}>0,$$

so that the value $i=0$ is inadmissible, as we have already assumed. Moreover m must lie between the limits $-(n-1)$ and 1.

Since $e^\theta=x$, the equation for v is equivalent to

$$\frac{d^nv}{dx^n}+v=0,$$

whence

$$v=c_1e^{\alpha_1x}+c_2e^{\alpha_2x}\ldots+c_ne^{\alpha_nx},$$

in which $\alpha_1, \alpha_2, \ldots \alpha_n$ are the nth roots of -1. This value of v can be expanded in ascending powers of x in the form

$$\begin{aligned}v&=v_0+v_1x+v_2x^2+\&c.\\&=v_0+v_1e^\theta+v_2e^{2\theta}+\&c.\end{aligned}$$

Hence $u-u_0$ representing that part of u which contains positive and integral powers of x, we shall have

$$u-u_0=CC'\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)(v-v_0).$$

Now

$$\begin{aligned}v-v_0&=C_1(e^{\alpha_1x}-1)+C_2(e^{\alpha_2x}-1)\ldots+C_n(e^{\alpha_nx}-1)\\&=\Sigma C_i(e^{\alpha_ix}-1),\end{aligned}$$

the summation extending from $i=1$ to $i=n$. Hence, merging CC' in the arbitrary constants $C_1, \ldots C_n$, we have

$$u=u_0+\Sigma C_i\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)(e^{\alpha_ix}-1), \quad \ldots \ldots \quad (3)$$

in which $x=e^\theta$. This expression we now propose to interpret by definite integrals.

Now

$$e^{\alpha_ix}-1=\int_0^x\alpha_ie^{\alpha_ih}dh.$$

Substituting and merging α_i in the arbitrary constant C_i, we have

$$\begin{aligned}u&=u_0+\Sigma C_i\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)\int_0^x e^{\alpha_ih}dh\\&=u_0+\Sigma C_i\int_0^\infty ds e^{-s}s^{\frac{n-1}{n}D+\frac{m}{n}-1}\int_0^\infty dt e^{-t}t^{\frac{D}{n}-\frac{m}{n}-1}\int_0^x e^{\alpha_i}\,dh\end{aligned}$$

on interpreting the Γ functions in the usual manner. We may therefore write

$$u=u_0+\Sigma C_i\int_0^\infty\int_0^\infty dsdte^{-(s+t)}s^{\frac{m}{n}-1}t^{-\frac{m}{n}-1}\int_0^{xs^{\frac{n-1}{n}}t^{\frac{1}{n}}}e^{\alpha_i h}dh,$$

since by the symbolical form of TAYLOR's theorem

$$s^{\frac{n-1}{n}\mathrm{D}}t^{\frac{\mathrm{D}}{n}}\phi(x)=\left(s^{\frac{n-1}{n}}t^{\frac{1}{n}}\right)^{\mathrm{D}}\phi(x)=\phi\left(xs^{\frac{n-1}{n}}t^{\frac{1}{n}}\right).$$

Let us now transform the double integral relative to s and t by assuming

$$s=vt,$$

and making v and t the new system of variables. We shall have

$$dsdt=tdvdt,$$

while the limits of v and t will be 0 and ∞. Hence

$$u=u_0+\Sigma C_i\int_0^\infty\int_0^\infty dvdte^{-(1+v)t}v^{\frac{m}{n}-1}t^{-1}\int_0^{xv^{\frac{n-1}{n}}t}e^{\alpha_i h}dh.$$

Again, transform the double integral relative to t and h, by assuming $h=ty$. We shall have $dh=tdy$, and the limits of y will be 0 and $xv^{\frac{n-1}{n}}$. Whence

$$u=u_0+\Sigma C_i\int_0^\infty\int_0^\infty\int_0^{xv^{\frac{n-1}{n}}}dvdtdye^{-(1+v-\alpha_i y)t}v^{\frac{m}{n}-1}.$$

Integrating with respect to t, we have

$$u=u_0+\Sigma C_i\int_0^\infty dv\int_0^{xv^{\frac{n-1}{n}}}dy\,\frac{v^{\frac{m}{n}-1}}{1+v-\alpha_i y}.$$

Now integrating with respect to y, and merging $\frac{-1}{\alpha_i}$ in the arbitrary constants,

$$\left.\begin{aligned}u&=u_0+\Sigma C_i\int_0^\infty dv\,v^{\frac{m}{n}-1}\left\{\log\left(1+v-\alpha_i xv^{\frac{n-1}{n}}\right)-\log(1+v)\right\}\\&=u_0+\Sigma C_i\int_0^\infty dv\,v^{\frac{m}{n}-1}\log\left(1-\frac{\alpha_i xv^{\frac{n-1}{n}}}{1+v}\right)\end{aligned}\right\}\quad\ldots\quad(4)$$

It remains to determine u_0.

Developing the function under the sign of integration in ascending powers of x, and effecting the integration for each term separately, we find, for the coefficient of x^n, the expression

$$u_n=\Sigma C_i\frac{\Gamma\left(\frac{m}{n}+n-1\right)\Gamma\left(1-\frac{m}{n}\right)}{n\Gamma(n)};$$

but from the law of the series as expressed in (6), art. 2,

$$u_n = -\frac{\left[\frac{m}{n}+n-2\right]^{n-1}\times\left(-\frac{m}{n}\right)}{[n]^n}u_0.$$

Equating these values,

$$u_0 = \Sigma C_i \frac{\Gamma\left(\frac{m}{n}+n-1\right)\Gamma\left(1-\frac{m}{n}\right)}{\frac{m}{n}\left[\frac{m}{n}+n-2\right]^{n-1}}$$

$$= \Sigma C_i \frac{\pi}{\frac{m}{n}\cdot\sin\frac{m\pi}{n}}$$

by the reductions of art. 6.

Hence, finally,

$$u = \Sigma\frac{C_i\pi}{\frac{m}{n}\sin\frac{m\pi}{n}} + \Sigma C_i\int_0^\infty dv\, v^{\frac{m}{n}-1}\log\left(1-\alpha_i\frac{xv^{\frac{n-1}{n}}}{1+v}\right), \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

which agrees with the previous result.

Determination of the Constants.

8. I propose here to determine the constants of the general integral (II), so as to obtain an expression for the mth power of that particular (real) root of the equation

$$y^n - xy^{n-1} - 1 = 0$$

which becomes unity when $x=0$.

We have

$$u = \Sigma C_i\frac{\pi}{\frac{m}{n}\cdot\sin\frac{m\pi}{n}} + \Sigma C_i\int_0^\infty dv\, v^{\frac{m}{n}-1}\log(1-\alpha_i xV), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $V=\frac{v^{\frac{n-1}{n}}}{1+v}$, and α_i represents in succession the different nth roots of -1.

The coefficient of x^r in the expansion of this value of u in ascending powers of x will be found to be

$$-\Sigma C_i\alpha_i^r\frac{\Gamma\left(\frac{m+(n-1)r}{n}\right)\Gamma\left(\frac{r-m}{n}\right)}{r\Gamma(r)},$$

and its coefficient in the expansion of y^m by LAGRANGE'S theorem is, for the particular root in question,

$$\frac{m\left[\frac{m+(n-1)r}{n}-1\right]^{r-1}}{n[r]^r},$$

equating which we have

$$\Sigma C_i\alpha_i^r = -\frac{m\left[\frac{m+(n-1)r}{n}-1\right]^{r-1}}{n\Gamma\left(\frac{m+(n-1)r}{n}\right)\Gamma\left(\frac{r-m}{n}\right)}.$$

But by the theorem of art. 6,

$$\Gamma\left(\frac{m+(n-1)r}{n}\right)\Gamma\left(\frac{r-m}{n}\right)=\Gamma\left(\frac{r-m}{n}\right)\Gamma\left(r-\frac{r-m}{n}\right)$$

$$=\left[r-\frac{r-m}{n}-1\right]^{n-1}\frac{\pi}{\sin\left(\frac{r-m}{n}\pi\right)}$$

$$=\frac{\left[\frac{m+(n-1)r}{n}-1\right]^{r-1}\pi}{\sin\left(\frac{m-r}{n}\pi\right)}.$$

Hence

$$\Sigma_i C_i \alpha_i^r = -\frac{m\sin\left(\frac{r-m}{n}\pi\right)}{n\pi} \quad . \quad . \quad . \quad (2)$$

Giving, in this equation, to r any particular system of n values, we shall obtain a system of n linear equations for the determination of the n constants C_1, C_2, .. C_n. We shall form this system by giving to r the values 1, 2, .. n.

Now α_j representing any *particular* root selected from the series α_1, α_2, .. α_n, multiply the above typical equation by α_j^{n-r}, and then, giving to r the successive values 1, 2 .. n, form the sum of the equations thus arising. The result may be expressed in the form

$$\Sigma_i C_i \Sigma_r \alpha_i^r \alpha_j^{n-r} = -\frac{m}{n\pi}\Sigma_r \sin\left(\frac{r-m}{n}\pi\right)\alpha_j^{n-r}, \quad . \quad . \quad . \quad (3)$$

the summations with respect to i and r being both from 1 to n inclusive.

But

$$\Sigma_r \alpha_i^r \alpha_j^{n-r} = \alpha_i \alpha_j^{n-1} + \alpha_i^2 \alpha_j^{n-2} \ldots + \alpha_i^n$$

$$= \alpha_i(\alpha_j^{n-1} + \alpha_i \alpha_j^{n-2} \ldots + \alpha_i^{n-1})$$

$$= \alpha_i \frac{\alpha_j^n - \alpha_i^n}{\alpha_j - \alpha_i}.$$

Now when α_i is not equal to α_j, this expression vanishes, since $\alpha_i^n = \alpha_j^n = -1$. When, however, $\alpha_i = \alpha_j$, the fraction $\frac{\alpha_j^n - \alpha_i^n}{\alpha_j - \alpha_i}$ becomes indeterminate, and its true limiting value is seen to be $n\alpha_j^n = -n$. Hence (3) becomes

$$-nC_j = -\frac{m}{n\pi}\Sigma_r \sin\left(\frac{r-m}{n}\pi\right)\alpha_j^{n-r},$$

$$\therefore \quad C_j = \frac{m}{n^2\pi}\Sigma_r \sin\left(\frac{r-m}{n}\pi\right)\alpha_j^{n-r}. \quad . \quad . \quad . \quad (4)$$

We have thus solved the linear system of equations. We have still to reduce this solution by effecting the summation in the second member.

Now to α_j we may give the form $e^{\frac{2j+1}{n}\pi\sqrt{-1}}$, which will represent all the nth roots of -1 in succession if we give to j the series of values 1, 2, .. n. Hence substituting for α_j

the above value, and giving to $\sin\left(\frac{m-r}{n}\pi\right)$ its exponential form, we have

$$\Sigma_r \sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=\Sigma_r\frac{e^{\frac{m-r-(2j+1)r}{n}\pi\sqrt{-1}}-e^{\frac{-(m-r)-(2j+1)r}{n}\pi\sqrt{-1}}}{2\sqrt{-1}}$$

$$=\frac{e^{\frac{m\pi}{n}\sqrt{-1}}\,\Sigma_r e^{\frac{-2(j+1)r\pi}{n}\sqrt{-1}}-e^{\frac{m-\pi}{n}\sqrt{-1}}\,\Sigma_r e^{\frac{-2jr\pi}{n}\sqrt{-1}}}{2\sqrt{-1}}.$$

Now in general

$$\Sigma_r e^{kr\pi\sqrt{-1}}=e^{k\pi\sqrt{-1}}+e^{2k\pi\sqrt{-1}}\ldots+e^{nk\pi\sqrt{-1}}$$

$$=\frac{e^{(n+1)k\pi\sqrt{-1}}-e^{k\pi\sqrt{-1}}}{e^{k\pi\sqrt{-1}}-1}$$

$$=e^{\frac{k(n+1)\pi}{2}\sqrt{-1}}\times\frac{e^{\frac{kn\pi}{2}\sqrt{-1}}-e^{\frac{-kn\pi}{2}\sqrt{-1}}}{e^{\frac{k\pi}{2}\sqrt{-1}}-e^{\frac{-k\pi}{2}\sqrt{-1}}}$$

$$=e^{\frac{k(n+1)}{2}\pi\sqrt{-1}}\times\frac{\sin\frac{kn\pi}{2}}{\sin\frac{k\pi}{2}}.$$

Putting therefore $$k=-\frac{2(j+1)}{n},$$

we have $$\Sigma_r e^{\frac{-2(j+1)r\pi}{n}\sqrt{-1}}=e^{\frac{-(j+1)(n+1)}{n}\pi\sqrt{-1}}\frac{\sin(j+1)\pi}{\sin\frac{(j+1)\pi}{n}},$$

and putting $$k=e\frac{-2j}{n},$$

$$\Sigma_r e^{\frac{-2jr\pi}{n}\sqrt{-1}}=e^{\frac{-j(n+1)}{n}\pi\sqrt{-1}}\frac{\sin j\pi}{\sin\frac{j\pi}{n}}.$$

Hence $$\Sigma_r \sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=\frac{1}{2\sqrt{-1}}\left\{\begin{array}{l} e^{\frac{m-(j+1)(n+1)}{n}\pi\sqrt{-1}}\dfrac{\sin(j+1)\pi}{\sin\frac{(j+1)\pi}{n}} \\ -e^{\frac{-m-j(n+1)}{n}\pi\sqrt{-1}}\dfrac{\sin j\pi}{\sin\frac{j\pi}{n}}\end{array}\right\}$$

Now $$\frac{\sin(j+1)\pi}{\sin\frac{(j+1)\pi}{n}}=0$$

for all values of j taken from the series 1, 2, .. n except the value $n-1$, for which the expression becomes indeterminate in form, and has for its true value

$$\frac{\pi\cos n\pi}{\frac{\pi}{n}\cos\frac{n\pi}{n}}=\frac{n\cos n\pi}{\cos\pi}=\pm n,$$

as n is odd or even.

So too $$\frac{\sin j\pi}{\sin\frac{j\pi}{n}}=0$$

for all values of j taken from the series 1, 2, .. n except the value n, for which its true value is

$$\frac{n\cos n\pi}{\cos\pi}=\pm n$$

as n is odd or even.

Hence when j stands for any of the integers 1, 2, .. $n-2$, we have

$$\Sigma_r \sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=0.$$

When $j=n-1$, we have

$$\Sigma_r \sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=\pm\frac{n}{2\sqrt{-1}}e^{\frac{m-n(n+1)}{n}\pi\sqrt{-1}},$$

the upper or lower sign being taken according as n is odd or even. To the second member we may give the form

$$\pm\frac{n}{2\sqrt{-1}}e^{\frac{m\pi}{n}\sqrt{-1}}\left(\cos(n+1)\pi-\sqrt{-1}\sin(n+1)\pi\right)=\frac{n}{2\sqrt{-1}}e^{\frac{m\pi}{n}\sqrt{-1}},$$

since $\sin(n+1)\pi=0$, $\cos(n+1)\pi=\pm1$, as n is odd or even.

Thus when $j=n-1$, we have

$$\Sigma_r \sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=\frac{n}{2\sqrt{-1}}e^{\frac{m\pi}{n}\sqrt{-1}}.$$

In the same way when $j=n$, we find

$$\Sigma_r \sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=-\frac{n}{2\sqrt{-1}}e^{\frac{-m\pi}{n}\sqrt{-1}}.$$

It results therefore that, according as j is less than $n-1$, equal to $n-1$, or equal to n, we shall have

$$C_j=0,\ \text{ or }\ \frac{m}{n\pi}\,\frac{e^{\frac{m}{n}\pi\sqrt{-1}}}{2\sqrt{-1}},\ \text{ or }\ \frac{-m}{n\pi}\,\frac{e^{\frac{-m}{n}\pi\sqrt{-1}}}{2\sqrt{-1}}.$$

In the general integral (II), art. 7, we shall therefore have

$$\Sigma C_i=\frac{m}{n\pi}\left(\frac{e^{\frac{m\pi}{n}\sqrt{-1}}-e^{-\frac{m\pi}{n}\sqrt{-1}}}{2\sqrt{-1}}\right)=\frac{m}{n\pi}\sin\frac{m\pi}{n},$$

$$u=1+\frac{m}{n\pi}\left\{\frac{e^{\frac{m\pi}{n}\sqrt{-1}}}{2\sqrt{-1}}\int_0^\infty dv\,v^{\frac{m}{n}-1}\log(1-\alpha_{n-1}x\mathrm{V})-\frac{e^{\frac{-m\pi}{n}\sqrt{-1}}}{2\sqrt{-1}}\int_0^\infty dv\,v^{\frac{m}{n}-1}\log(1-\alpha_n x\mathrm{V})\right\}, \quad (5)$$

where $\mathrm{V}=\frac{v^{\frac{n-1}{n}}}{1+v}$.

Now $$\alpha_{n-1}=e^{(2(n-1)+1)\pi\sqrt{-1}}=e^{(2n-1)\pi\sqrt{-1}}=e^{-\pi\sqrt{-1}},$$

$$\alpha_n\ =e^{(2n+1)\pi\sqrt{-1}}\quad =e^{\pi\sqrt{-1}};$$

therefore, finally,

$$u=1+\frac{m}{2n\pi\sqrt{-1}}\left\{e^{\frac{m\pi}{n}\sqrt{-1}}\int_0^\infty dv\, v^{\frac{m}{n}-1}\log\left(1-e^{\frac{-\pi}{n}\sqrt{-1}}x\mathrm{V}\right)-e^{\frac{-m\pi}{n}\sqrt{-1}}\int_0^\infty dv\, v^{\frac{m}{n}-1}\log\left(1-e^{\frac{\pi}{n}\sqrt{-1}}x\mathrm{V}\right)\right\}. \quad (6)$$

It is seen, from the form of this expression, that it represents a *real* value.

If we substitute v for $v^{\frac{1}{n}}$, a change which does not affect the limits, there results

$$u=y^m=1+\frac{m}{2\pi\sqrt{-1}}\left\{e^{\frac{m\pi}{n}\sqrt{-1}}\int_0^\infty dv\, v^{m-1}\log\left(1-e^{\frac{-\pi}{n}\sqrt{-1}}x\mathrm{V}\right)-e^{-\frac{m\pi}{n}\sqrt{-1}}\int_0^\infty dv\, v^{m-1}\log\left(1-e^{\frac{\pi}{n}\sqrt{-1}}x\mathrm{V}\right)\right\}$$

in which $\mathrm{V}=\frac{v^{n-1}}{1+v^n}$. This expression we shall now reduce to an equivalent *real* form.

Reduction of the expression for y^m.

9. We shall somewhat simplify the general expression above found for y^m by integrating by parts. The integrated portion will be found to vanish at both limits.

Representing $\frac{d\mathrm{V}}{dv}$ by V', we have

$$\int mv^{m-1}\log\left(1-e^{\frac{\pm\pi}{n}\sqrt{-1}}x\mathrm{V}\right)dv=v^m\log\left(1-e^{\frac{\pm\pi}{n}\sqrt{-1}}x\mathrm{V}\right)+xe^{\frac{\pm\pi}{n}\sqrt{-1}}\int\frac{v^m\mathrm{V}'dv}{1-e^{\frac{\pm\pi}{n}\sqrt{-1}}x\mathrm{V}}.$$

Now, expanding the logarithm in the integrated portion, and putting for V its value $\frac{v^{n-1}}{1+v^n}$, we see that that portion will consist of a series of terms of the form

$$\frac{\mathrm{A}v^{m+(n-1)r}}{(1+v^n)^r};$$

r being for each such term a positive integer.

All these terms vanish when $v=0$, since, by the conditions to which m is subject, $m+(n-1)r$ is positive.

Again, they vanish when v is made infinite, since in this case

$$\frac{\mathrm{A}v^{m+(n-1)r}}{(1+v^n)^r}=\mathrm{A}v^{m-r},$$

and, by the conditions relative to m, the index $m-r$ is negative.

We have, then, on applying the above reduction to the terms of the general value of y^m,

$$y^m=1+\frac{1}{2\pi\sqrt{-1}}\left\{e^{\frac{(m-1)\pi}{n}\sqrt{-1}}\int_0^\infty\frac{xv^m\mathrm{V}'dv}{1-x\mathrm{V}e^{-\frac{\pi}{n}\sqrt{-1}}}-e^{\frac{-(m-1)\pi}{n}\sqrt{-1}}\int_0^\infty\frac{xv^m\mathrm{V}'dv}{1-x\mathrm{V}e^{\frac{\pi}{n}\sqrt{-1}}}\right\}.$$

Now substitute for the imaginary exponentials their trigonometrical value, and there results

$$y^m=1+\frac{x}{\pi}\int_0^\infty \frac{\left(\sin\left(\frac{m-1}{n}\pi\right)-xV\sin\frac{m\pi}{n}\right)v^m V' dv}{1-2xV\cos\frac{\pi}{n}+x^2V^2}.$$

As x enters this expression only in combination with V, it is suggested to us to represent xV by V. If we do this the final theorem will be

THEOREM. *If* y^m *represent the* m*th power of that real root of the equation*

$$y^n-xy^{n-1}-1=0$$

which reduces to 1 *when* $x=0$, *then, supposing* m *to be between the limits* 1 *and* $-n+1$, *the value of* y^m *will be*

$$y^m=1+\frac{1}{\pi}\int_0^\infty \frac{\left(\sin\left(\frac{m-1}{n}\pi\right)-V\sin\frac{m\pi}{n}\right)v^m\frac{dV}{dv}dv}{1-2V\cos\frac{\pi}{n}+V^2}, \quad . \; . \; . \; . \; . \; . \; . \quad \text{(IV)}$$

in which

$$V=\frac{xv^{n-1}}{1+v^n}.$$

10. Hence too we have the value of a remarkable definite integral, viz.

$$\int_0^\infty \frac{\left(\sin\frac{m-1}{n}\pi-V\sin\frac{m\pi}{n}\right)\frac{dV}{dv}v^m dv}{1-2V\cos\frac{\pi}{n}+V^2}=\pi(y^m-1) \quad . \; . \; . \; . \; . \quad \text{(V)}$$

under the above conditions and with the above interpretations.

It may be desirable to verify this result.

Since $$V=\frac{xv^{n-1}}{1+v^n},$$

we shall have $$\frac{dV}{dv}=\frac{(n-1)V}{v}-\frac{nV^2}{x},$$

so that the definite integral is resolvable into

$$(n-1)\int_0^\infty \frac{v^{m-1}V\left(\sin\frac{(m-1)\pi}{n}-V\sin\frac{m\pi}{n}\right)V'dv}{1-2V\cos\frac{\pi}{n}+V^2}$$

$$\frac{-n}{x}\int_0^\infty \frac{v^mV^2\left(\sin\frac{(m-1)\pi}{n}-V\sin\frac{m\pi}{n}\right)dv}{1-2V\cos\frac{\pi}{n}+V^2}.$$

Now it may be shown that

$$\frac{\sin\left(\frac{m-1}{n}\pi\right)-\mathrm{V}\sin\frac{m\pi}{n}}{1-2\mathrm{V}\cos\frac{\pi}{n}+\mathrm{V}^2}=\Sigma_r\sin\left(\frac{m-r-1}{n}\pi\right)\mathrm{V}^r,$$

the summation with respect to r extending from $r=0$ to $r=\infty$. Hence the first member of (V) may be developed in the form

$$(n-1)\Sigma_r\int_0^\infty v^{m-1}\sin\left(\frac{m-r-1}{n}\pi\right)\mathrm{V}^{r+1}dv$$

$$\frac{-n}{x}\Sigma_r\int_0^\infty v^m\sin\left(\frac{m-r-1}{n}\pi\right)\mathrm{V}^{r+2}\,dv.$$

Now $$\int_0^\infty v^{m-1}\mathrm{V}^{r+1}dv=x^{r+1}\int_0^\infty\frac{v^{m+(r+1)(n-1)}dv}{(1+v^n)^{r+1}}$$

$$=\frac{\Gamma\left(\frac{m+(r+1)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+1)}x^{r+1},$$

and $$\int_0^\infty v^m\mathrm{V}^{r+1}dv=x^{r+2}\frac{\Gamma\left(\frac{m+1+(r+2)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+2)}x^{r+2}$$

$$=\frac{m+(r+1)(n-1)}{n(r+1)}\;\frac{\Gamma\left(\frac{m+(r+1)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+1)}x^{r+2}.$$

Hence the total coefficient of x^{r+1} in (V) is

$$\sin\frac{(m-r-1)\pi}{n}\;\frac{\Gamma\left(\frac{m+(r+1)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+1)}\left\{n-1-n\times\frac{m+(r+1)(n-1)}{n(r+1)}\right\}$$

$$=\frac{\sin\frac{(m-r-1)\pi}{n}\Gamma\left(\frac{m+(r+1)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+1)}\times\frac{-m}{r+1},$$

and therefore that of x^r is

$$\frac{\sin\left(\frac{m-r}{n}\pi\right)\Gamma\left(\frac{m+r(n-1)}{n}\right)\Gamma\left(\frac{r-m}{n}\right)}{n\Gamma(r)}\times\frac{-m}{r}.$$

Now

$$\Gamma\left(\frac{m+r(n-1)}{n}\right)\Gamma\left(\frac{r-m}{n}\right)=\Gamma\left(\frac{r-m}{n}\right)\Gamma\left(r-\frac{r-m}{n}\right)$$

$$=\left[r-1-\frac{r-m}{n}\right]^{r-1}\frac{\pi}{\sin\left(\frac{r-m}{n}\pi\right)}.$$

Therefore the coefficient of x^r is

$$\frac{m\left[r-1-\frac{r-m}{n}\right]^{r-1}\pi}{n[r]^r};$$

and this is, by art. 2, equal to πu_r in the expansion of y^m in ascending powers of x. Hence, the lowest value of r in the expansion of the definite integral being unity, we see that the value of that integral will be expressed by $\pi(y^m-1)$, as was to be shown.

It will be observed that the function under the sign of definite integration does not become infinite within the limits. Ordinary methods of approximation might therefore be applied. I apprehend, however, that it is not in this direction that the value of such results is to be sought.

- 1 -

From the Author

XII. *On the Theory of Probabilities.*

By George Boole, *F.R.S.*, *Professor of Mathematics in Queen's College, Cork.*

Received June 19,—Read June 19, 1862.

This paper has for its object the investigation of the general analytical conditions of a Method for the solution of Questions in the Theory of Probabilities, which was proposed by me in a work entitled "An Investigation of the Laws of Thought" (London, Walton and Maberly, 1854).

The application of this method to particular problems has been illustrated in the work referred to, and yet more fully in a 'Memoir on the Combination of Testimonies and of Judgments' published in the Transactions of the Royal Society of Edinburgh (vol. xxi. Part 4). Some observations, too, on the general character of the solutions to which the method leads, founded upon induction from particular cases, were contained in the original treatise, and the outlines, still in some measure conjectural, of their general theory were given in an Appendix to the Memoir. But the complete development of that theory was attended with analytical difficulties which I have only lately succeeded in overcoming. It involves discussions relating to the properties of a certain functional determinant, and to the possible solutions of a system of algebraic equations of peculiar form—discussions which will, I trust, be thought to possess a value, as contributions to Mathematical Analysis, independent of their present application.

As concerns the nature of the problems to which the method is applicable, it may be stated that they are such that the numerical elements, both given and sought, are the probabilities of events or states of things the definitions of which, and the connexions of which, are capable of expression by logical propositions. There is ground for believing that all questions whatever involving probability are ultimately reducible to this general form. This point, however, I do not purpose to discuss here. It has been already in some degree considered in the Memoir referred to.

In order to explain more fully the necessity for the present investigation, it will be requisite to state the fundamental principles upon which the method in question rests. There are only two of them which can possibly afford matter for discussion.

1st. The expression in language of the data of a problem in the Theory of Probabilities is to a certain extent arbitrary, because it depends upon the extent of meaning of the primary simple terms employed to express the events the conceptions of which it involves. But the choice of simple terms is, if we consider it with respect to our absolute power of choice, arbitrary. Any complex combination of events can be contemplated as a single whole in thought, and expressed by a single term. The invention of new simple terms to express what was before expressed by a combination of terms is a normal phenomenon in the growth of language.

Now the first principle upon which the method rests is the following:—

Principle I.—The different forms which a problem may be made to assume by different elections with respect to the simple terms of its expression are mutually equivalent.

For instance, if the following data were given,

The probability of rain is p,
The probability of rain with snow is q,

the form which the problem would assume in a language in which there was no word for snow, but in which the combination of snow with rain was called sleet, would be

The probability of rain is p,
The probability of sleet is q,

with the added condition, expressed as a logical proposition, that sleet always implies rain. And this as a statement of the data would, it is affirmed, be equivalent to the former statement. If these were the data of an actual problem, the event of which the probability is sought would require similar translation.

I desire to guard here against a possible misapprehension. I have said that the choice of simple terms, if considered with respect to our power of choice, is arbitrary. I do not mean by this to affirm that the actual growth of language is arbitrary. We know that it is far otherwise. Unity of sensuous impression in the early stages of its growth, unity of thought in the latter, seems to govern the invention and introduction of simple terms. It has indeed been said that there is a λόγος in the constitution of things of which language in its varied forms is the human reflexion, but never without the inseparable human element of choice and voluntary power.

It is then affirmed that whatever the grounds of fitness or propriety (and the existence of such grounds is fully conceded) may be, which have governed the actual choice of the simple terms of language, those grounds have nothing whatever to do with the calculation of probability. This depends upon the *information* contained in the data, information supposed to be derived from actual experience, or at least to be of such a nature that experience might have furnished it.

The different forms in which a problem is capable of being expressed, though differing in consequence of the different arbitrary elections which are possible with respect to its simple terms, are not independent of each other. They are connected together by the Laws of Thought, and pass one into the other by the processes of the Calculus of Logic, which is an organized expression of those Laws.

Among these forms there is one which presents exclusive advantages. It is that in which those events, however originally expressed, the probabilities of which constitute the data, are assumed as the simple events of the problem, and expressed by logical symbols corresponding to the simple terms of ordinary language; the event of which the probability is sought being also expressed logically by means of the same symbols. The Calculus of Logic enables us to do this, determining at the same time in an *explicit* form, *i. e.* in a form capable of expression in ordinary language by definite logical pro-

properties in the actual urn is the same as it is conceived to be in the ideal urn of free balls, but the hypothesis that it is so, involves an equal distribution of our actual knowledge, and enables us to construct the problem from ultimate hypotheses which reduce it to a calculation of combinations.

I pass from the particular and material to the general problem. In the form to which this is brought by the Calculus of Logic, the probabilities are those of events of which certain combinations are, as a logical consequence of the original definitions of those events, impossible. It might, at first sight, appear that this establishes a fundamental difference between this problem and that of the urn, in which certain combinations are prevented from issuing by a material hindrance. In the one case the restriction appears as logically necessary, in the other as only actual.

Upon this I remark, that the data of the problem in its ultimate reduced form *might* result from the same kind of dependence as in the actual data; that they, in fact, *would* thus result if the mind of the observer were capable of contemplating, and were in a position to contemplate, each of the events in this ultimate translated form simply as a whole, and of recording, through an approximately infinite series of observations, what combinations of those wholes come into being, and what do not, in the actual universe. What appears as necessary in the translated data would now appear as actual—as a result of observation; what is impossible would be received as non-existent. The question is, then, whether the difference between the conception of what is impossible from involving a logical contradiction, and the conception of what in the actual constitution of things never exists, is of a kind to affect expectation. I do not hesitate to say that it is not. We are concerned with events in so far as they are capable of happening or not happening, of combining or not combining; but we are not concerned with the reasons in virtue of which they happen or do not happen, combine or do not combine. If we went beyond this, we should enter upon a metaphysical question to which I presume that no answer can, upon rational grounds, be given, viz. upon the question whether, when two things or events are in the actual constitution of things incapable of happening together, it would, if our knowledge were sufficiently extended, be found that the resulting conceptions of them were logically inconsistent.

I have but one further observation on Principle II. to make. It is that in the general problem we are not called upon to interpret the ideal events. The whole procedure is, like every other procedure of abstract thought, formal. We do not say that the ideal events exist, but that the events in the translated form of the actual problem are to be considered to have such relations with respect to happening or not happening as a certain system of ideal events would have if conceived first as free, and then subjected, without their freedom being otherwise affected, to relations formally agreeing with those to which the events in the translated problem are subject.

---

tion through arbitrary hypotheses, coupled with the assumption that any result thus obtained is necessarily *the* true one. The application of the principle employed in the text, and founded upon the general theorem of development in Logic, I hold to be *not* arbitrary.

probabilities of the alternatives which it involves, we shall have a system of equations connecting λ, μ, ν, &c. with p_1, p_2, ... p_n, the probabilities supposed given. Again, λ, μ, ν ..., as probabilities, are subject to the conditions

$$\lambda \geqq 0,\ \mu \geqq 0,\ \nu \geqq 0, \ldots \text{\&c.},$$

and, as alternatives mutually excluding each other, to the condition

$$\lambda+\mu+\nu+\ldots=1,$$

or the condition

$$\lambda+\mu+\nu+\ldots \leqq 1,$$

according as the alternatives in question together make up certainty or not.

Thus we have a system consisting of equations and inequations from which λ, μ, ν, &c. must be eliminated. To effect this elimination we must determine as many of the quantities λ, μ, ν ... as possible from the equations, substitute their values in the inequations, and then eliminate the remainder of the quantities λ, μ, ν ... by means of the theorem that if we have simultaneously

$$\lambda \leqq a_1,\ \lambda \leqq a_2, \ldots \lambda \leqq a_m,$$
$$\lambda \geqq b_1,\ \lambda \geqq b_2, \ldots \lambda \geqq b_n,$$

then we have the system of conditions of which the type is

$$a_i \geqq b_j,$$

a_i representing any one of the set a_1, a_2, ... a_m, and b_j any one of the set b_1, b_2, ... b_n. Thus there are mn conditions in all.

This method is illustrated in the following problem, in the expression and solution of which it is to be noticed, that when in the Calculus of Logic an event is represented by x, the event which consists in its not happening is denoted by $1-x$, or for brevity by $\bar{x}$; that when two events are represented by x and y, their concurrence is denoted by xy, the happening of the first without the second by $x\bar{y}$, and so on.

Problem. Given that the probability of the concurrence of the events x and y is p, of the events y and z, q, and of the events z and x, r. Required the conditions to which p, q, and r must be subject in order that the above data may be consistent with a possible experience.

Resolving the events xy, yz, xz into the possible alternations out of which they are formed, let us write

$$\text{Prob. } xyz=\lambda,\ \text{Prob. } xy\bar{z}=\mu,\ \text{Prob. } x\bar{y}z=\nu,\ \text{Prob. } \bar{x}yz=\varrho.$$

Then we have the equations

$$\lambda+\mu=p,\ \lambda+\varrho=q,\ \lambda+\nu=r,$$

together with the inequations

$$\lambda \geqq 0,\ \mu \geqq 0,\ \nu \geqq 0,\ \varrho \geqq 0,$$
$$\lambda+\mu+\nu+\varrho \leqq 1.$$

From the equations we find

$$\mu=p-\lambda,\ \varrho=q-\lambda,\ \nu=r-\lambda,$$

which, substituted in the inequations, give

$$\lambda \geqq 0,\ p-\lambda \geqq 0,\ q-\lambda \geqq 0,\ r-\lambda \geqq 0,$$
$$p+q+r-2\lambda \leqq 1;$$

Statement of the Method for the Solution of Questions in the Theory of Probabilities.

For the general demonstration of this method the reader is referred to the 'Laws of Thought,' chap. xvii. For the purpose of the analytical investigation the statement of the method will suffice.

Let s, t, v, &c. represent the events of which the probabilities are given, p, q, r, &c. those probabilities, and w the event of which the probability is sought; then, whatever the definitions of $s, t \ldots$ and w may be, and whatever connecting relations may exist, it is always possible by the Calculus of Logic to determine the logical dependence of w upon s, t, &c. in the following most general form, viz.

$$w=\mathrm{A}+0\mathrm{B}+\frac{0}{0}\mathrm{C}+\frac{1}{0}\mathrm{D}.$$

Here A, B, C, D are logical combinations of the events s, t, &c., and the connexion in which these stand to the event w and to each other is the following: A expresses those combinations of s, t, &c. which are entirely included in w, i. e. which cannot happen without our being permitted to say that w happens. B represents those combinations which may happen but are not included under w; so that when they happen we may say that w does not happen. C represents those combinations the happening of which leaves us in doubt whether w happens or not. D those combinations the happening of which would involve logical contradiction.

It follows from the above that the *translated* form of the problem is

Given Prob. $s=p$, Prob. $t=q$, Prob. $v=r$, &c., s, t, $v \ldots$ being regarded as events subject to the explicit logical condition

$$\mathrm{A}+\mathrm{B}+\mathrm{C}=1.$$

Required the probability u of the event of which the logical expression is

$$w=\mathrm{A}+\frac{0}{0}\mathrm{C};$$

and it is shown (Laws of Thought, p. 265), upon grounds essentially the same as those expressed in Principles I. and II. of this paper, that the solution of the problem is involved in the following *algebraic* equations, viz.

$$\frac{\mathrm{V}_s}{p}=\frac{\mathrm{V}_t}{q}\ldots=\frac{\mathrm{A}+c\mathrm{C}}{u}=\mathrm{V}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I.})$$

in which the functions V, V_s, $\mathrm{V}_t \ldots$ are formed in the following manner, viz.,—

1st. V is derived from A+B+C without change of form by interpreting s, t, &c. no longer as logical symbols, but as symbols of quantity. They represent the probabilities of the ideal events of Principle II.

2ndly. V_s is the sum of those terms in V which contain s as a factor, V_t the sum of those which contain t as a factor, &c.

The quantity c is an arbitrary constant, admitting of any value between 0 and 1.

To effect the solution, the quantities s, t, &c. are to be eliminated from the system (I.), and u then determined as a function of p, q, $r \ldots$ and c. The arbitrary constant c may

tities in the one case, will eliminate the corresponding single quantities, or sums of single quantities, in the other.

Simplification of the General Equations for the Solution of Questions in the Theory of Probabilities.

Let us express the system (I.) in the form

$$\frac{V_s}{V}=p, \quad \frac{V_t}{V}=q, \text{ \&c.},$$

$$u=\frac{A+cC}{V},$$

and let us suppose the quantities p, q... (and therefore s, t...) to be n in number. Then all the terms in V will be composed of products of s, t... $\bar{s}$, $\bar{t}$..., each term involving either s or $\bar{s}$, either t or $\bar{t}$, &c., but not the combinations $s\bar{s}$, $t\bar{t}$, &c. Each term is therefore homogeneous and of the nth degree.

It follows, therefore, that if we divide the numerator and denominator of each of the first members of the above system by $\bar{s}\,\bar{t}\,\bar{v}$..., and then make

$$\frac{s}{\bar{s}}=x_1, \quad \frac{t}{\bar{t}}=x_2, \quad \frac{v}{\bar{v}}=x_3, \text{ \&c.},$$

and if at the same time we, for symmetry, change p, q, r... into p_1, p_2..p_n, the system will assume the following form,—

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n,$$

$$u=\frac{A+cC}{V},$$

in which V, A, C are formed from their former values by suppressing $\bar{s}$, $\bar{t}$, $\bar{v}$, &c., or, which is the same thing, changing each of them into unity, and then changing s, t, v... into x_1, x_2, x_3..., while V_1 consists of those terms of V which contain x_1, V_2 of those which contain x_2, and so on.

In its new form V is a rational and entire function of x_1, x_2, ...x_n not involving powers of those quantities, and with all its coefficients equal to unity. Again, as s, t, &c. are from the theory of their origin required to be positive proper fractions, x_1, x_2, ... x_n are, from the nature of their connexion with s, t..., required to be positive quantities. And it is sufficient that x_1, x_2, ... x_n be determinable as positive quantities in order that s, t... may be determinable as positive fractions.

Now we shall proceed to show that x_1, x_2, ... x_n are determinable as positive quantities precisely when p_1, p_2, ...p_n satisfy the conditions of possible experience. We shall further show, as a consequence of this, that the value of the probability sought, when determined by the General Rule, will, under the same conditions, lie within such limits as if it were itself given by the same experience. In the order of this proof, we shall first demonstrate the theorems of pure Analysis upon which the conclusions depend, then in a distinct section make the particular application.

Analytical Theorems relating to Functional Determinants and Systems of Algebraic Equations.

A symmetrical determinant may be conveniently expressed in the form

$$\begin{vmatrix} A_1 & A_{12} \ldots A_{1n} \\ A_{21} & A_2 \ldots A_{2n} \\ . \quad . & . \quad . \quad . \quad . \\ A_{n1} & A_{n2} \ldots A_n \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

the conditions of symmetry being

$$A_{ij}=A_{ji},\ A_{ii}=A_i.$$

It is desirable to employ fixed language in referring to this. We shall therefore call the quantities A_1, $A_2 \ldots A_n$ the 'principal elements,' and the diagonal series of terms which they form the 'principal diagonal.' The elements A_{ij}, when i and j differ, we shall call 'subordinate elements.' The element A_i, together with all the subordinate elements which occur upon the same horizontal or vertical line of the determinant, we shall designate the 'i-system of elements.' Lastly, in comparing two rows or two columns of elements together, those elements will be said to correspond which occupy the same numerical place in their respective rows or columns.

The following Lemma will next be established.

Lemma.—A symmetrical determinant expressed in the form (I.) will be unaltered in value, if from each subordinate element of its i-system we subtract the corresponding element of its j-system multiplied by a quantity λ, which is invariable for the same system,—and for the principal element A_i substitute $A_i-2\lambda A_{ij}+\lambda^2 A_j$.

It is known that a determinant vanishes if two of its lines or columns are identical, and it is known as a consequence of this that if from a particular line or column of a determinant the corresponding elements of another line or column, multiplied each by the same constant, are subtracted, the determinant is unaltered in value. From the ith line of the above symmetrical determinant subtract, term by term, λ times the jth line, and then from the ith column of the resulting determinant subtract λ times the jth column. As respects any subordinate element, the result will obviously accord with the statement in the Lemma. But the element A_i will be successively converted into

$$A_i-\lambda A_{ji}$$

$$(A_i-\lambda A_{ji})-\lambda(A_{ij}-\lambda A_j).$$

The last expression, since $A_{ji}=A_{ij}$, is reducible to

$$A_i-2\lambda A_{ij}+\lambda^2 A_j.$$

Upon this property the demonstration of the following general proposition will be founded:

PROPOSITION I.

Let the symmetrical determinant (I.) *possess the following properties, viz.*:—

1st. *That all its elements are linear homogeneous rational functions of certain quantities a, b, c, &c., unlimited in number.*

2ndly. *That if the coefficients of any one of these quantities a in the elements of any particular line or column taken in order are $\alpha_1, \alpha_2, \ldots \alpha_n$, and in any other line or column $\beta_1, \beta_2, \ldots \beta_n$, then these two series of quantities are respectively proportional.*

3rdly. *That the principal terms $A_1, A_2, \ldots A_n$ are positive*, i. e. *that the coefficients of all the quantities a, b, c, &c. which appear in these terms are positive.*

Then the developed determinant will be itself positive, and will consist of products of the quantities a, b, c, &c. without powers, each product affected by a positive sign.

First, it may be observed that any letter a of the set $a, b, c\ldots$ which appears in the subordinate term A_{ij} will appear in both the principal terms A_i, A_j.

For let m be the coefficient of a in A_{ij}, and therefore also in A_{ji}; let l be the coefficient of a in A_i, and n its coefficient in A_j. Thus to the elements A_i, A_{ji} in the i-column correspond A_{ji}, A_j in the j-column. Hence, by the definition of the determinant,

$$l : m :: m : n,$$
$$\therefore \quad m^2 = ln,$$

which implies that neither l nor n vanishes, so that a appears in A_i and A_j.

Secondly, we shall show that the determinant can, without alteration of its final developed value, be reduced to a form in which any letter a of the system $a, b, c\ldots$ shall appear in only one system of elements, and therefore only in the principal term of that system, since every subordinate term is common to *two* systems.

Let us suppose a to be contained in two at least of the systems of elements, and for convenience of expression, let these be the 1-system and the n-system. Let, then, $\alpha_1, \alpha_2, \ldots \alpha_n$ be the successive coefficients of a in $A_1, A_{21}, \ldots A_{n1}$, and therefore, by definition of the determinant, $\lambda\alpha_1, \lambda\alpha_2, \ldots \lambda\alpha_n$, its coefficients in $A_{n1}, A_{n2}, \ldots A_n$. Any of the quantities $\alpha_1, \alpha_2, \ldots \alpha_n$ may be 0. But by the Lemma above demonstrated the determinant may, without alteration of value, be reduced to the following form, viz.:—

$$\begin{vmatrix} A_1, \ A_{12}, \ldots A_{1n} - \lambda A_1 \\ A_{21}, \ A_2, \ldots A_{2n} - \lambda A_{21} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_{n1} - \lambda A_1, A_{n2} - \lambda A_{12} \ldots A_n - 2\lambda A_{n1} + \lambda^2 A_1 \end{vmatrix} \quad \ldots \ldots \quad \text{(B.)}$$

Now in the determinant thus transformed the quantity a will no longer occur in the n-system.

This is obvious with respect to the subordinate elements of that system. With respect to the principal element, we observe that the coefficient of a is

in A_1, equal to α_1,
in A_{n1}, equal to $\lambda\alpha_1$,
. . . .
in A_n, equal to $\lambda \times \lambda\alpha_1$ or $\lambda^2\alpha_1$,

whence the coefficient of a in $A_n - 2\lambda_{n1} + \lambda^2 A_1$ is equal to 0.

Thus a has been eliminated from the n-system, and as the process has not affected any elements but those which belong to the n-system, it will not affect the relations under which a enters into the other systems.

Consider then any other quantity b in the set a, b, c, then by hypothesis the coefficients of b in any line or column of elements

$$A_{i1}, A_{i2}, \ldots A_{in}, \text{ or } A_{1i}, A_{2i}, \ldots A_{ni}$$

may be represented by

$$\mu_i\beta_1, \ \mu_i\beta_2, \ldots \mu_i\beta_n,$$

$\beta_1, \beta_2, \ldots \beta_n$ being an arbitrary set of quantities which are the same for all lines or columns, while μ_i differs for different lines or columns, and vanishes for those in which b does not enter.

It is to be noted that as $A_{ij}=A_{ji}$, we have in general

$$\mu_i\beta_j=\mu_j\beta_i,$$

while as the principal elements of the determinant (I.) are positive, we have always $\mu_i\beta_i=$a positive quantity.

Now reverting to the derived determinant (B.), we see that its ith line or column of elements will be

$$A_{i1}, \ A_{i2}, \ \ldots \ A_{in}-\lambda A_{i1},$$

and its jth line or column

$$A_{j1}, \ A_{j2}, \ \ldots \ A_{jn}-\lambda A_{j1},$$

supposing i and j to be both less than n.

In these lines or columns the successive coefficients of b will therefore be

$$\mu_i\beta_1, \ \mu_i\beta_2, \ \ldots \ \mu_i\beta_n-\lambda\mu_i\beta_1,$$
$$\mu_j\beta_1, \ \mu_j\beta_2, \ \ldots \ \mu_j\beta_n-\lambda\mu_j\beta_1,$$

which stand to each other in the constant ratio $\mu_i : \mu_j$.

Now let $j=n$. The coefficients of b in the nth line or column of (B.) are obviously

$$\mu_n\beta_1-\lambda\mu_1\beta_1, \ \mu_n\beta_2-\lambda\mu_1\beta_2, \ \ldots \ \mu_n\beta_n-2\lambda\mu_1\beta_n+\lambda^2\mu_1\beta_1,$$

of which the last term may be reduced as follows,

$$\mu_n\beta_n-2\lambda\mu_1\beta_n+\lambda^2\mu_1\beta_1=\mu_n\beta_n-\lambda\mu_1\beta_n-\lambda\mu_n\beta_1+\lambda^2\mu_1\beta_1=(\mu_n-\lambda\mu_1)(\beta_n-\lambda\beta_1);$$

so that the series of coefficients of b becomes

$$(\mu_n-\lambda\mu_1)\beta_1, \ (\mu_n-\lambda\mu_1)\beta_2 \ \ldots \ (\mu_n-\lambda\mu_1)(\beta_n-\lambda\beta_1),$$

and they are now seen to stand to those of b in the i-line and column in the constant ratio $\mu_n-\lambda\mu_1 : \mu_i$.

We have, lastly, to prove that the new principal element $A_n-2\lambda A_{1n}+\lambda^2 A_1$ is positive.

Let N be the coefficient of any one of the quantities $a, b, c \ldots$ in the above element, L its coefficient in the principal element A_i, and M its coefficient in each of the subordinate elements common to the two systems of which the above are the respective

principal elements, viz. in $A_{in}-\lambda A_{i1}$ and $A_{ni}-\lambda A_{1i}$. Then, by what has already been proved,

$$L:M::M:N,$$

$$\therefore\ M^2=LN;$$

but L is positive; therefore N is so, and the principal element in question consists wholly of positive terms.

The above demonstration shows that the elimination of a from the n-system produces a new determinant equivalent to the original one, and in which the characters noted in the original one still remain. Should a occur in any other system or systems of elements of the new determinant beside the 1-system, it can, by repetitions of the same process, be eliminated thence. Ultimately, then, it will only remain in the 1-system, and therefore only in the principal term of that system. Again, as it enters that term in the first degree, it follows that the developed determinant will involve only the first power of a. Hence, as a may represent any of the quantities a, b, c,..., it is seen that no powers, but only products of these quantities, will appear in the developed determinant.

Let us represent the determinant, after the elimination of a from all the elements but A_1, in the form

$$\begin{vmatrix} A_1 & B_{12} & \dots & B_{1n} \\ B_{21} & C_2 & \dots & C_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ B_{n1} & C_{n2} & \dots & C_n \end{vmatrix}.$$

Now let ah_1 represent that term in A_1 which involves a. Then the portion of the determinant which involves a will be

$$ah_1 \begin{vmatrix} C_2 & \dots & C_{2n} \\ \cdot & \cdot & \cdot \\ C_{n2} & \dots & C_n \end{vmatrix}.$$

And here it is to be observed that ah_1 is positive, while the new determinant to which it is attached as a coefficient possesses all the characters of the old one. This determinant we can therefore transform in the same way, so as to eliminate any other letter b from all but a single principal element, which we shall suppose to contain it in a term bh_2. That portion of the original determinant which involves ab will therefore assume the form

$$abh_1h_2 \begin{vmatrix} D_3 & \dots & D_{3n} \\ \cdot & \cdot & \cdot \\ D_{n3} & \dots & D_n \end{vmatrix}.$$

Ultimately, then, as the result of such processes continued, the portion of the original determinant which involves any particular combination of n letters selected from a, b, c ... will consist of the product of a series of positive terms, each of which has appeared in some residual principal element. Every such combination being positive, it follows that the determinant itself consists solely of positive terms.

PROPOSITION II.

If V *be any rational entire function of the n variables* $x_1, x_2, \ldots x_n$, *but involving no powers of those variables above the first, and if, further, all the different terms of* V *have positive signs, then the determinant*

$$\begin{vmatrix} V & V_1 & V_2 & \ldots & V_n \\ V_1 & V_1 & V_{12} & \ldots & V_{1n} \\ V_2 & V_{21} & V_2 & \ldots & V_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ V_n & V_{n1} & V_{n2} & \ldots & V_n \end{vmatrix}$$

in which V_i *denotes the sum of the terms in* V *which contain* x_i, *and* V_{ij} *the sum of the terms in* V *which contain* x_i, x_j, *will, when developed as a rational and entire function of* $x_1, x_2, \ldots x_n$, *consist wholly of terms with positive coefficients.*

From the definition it is plain that in general

$$V_{ij}=V_{ji}, \quad V_{ii}=V_i,$$

whence the above determinant is symmetrical.

Again, all its elements are homogeneous linear functions of the terms in V.

Again, if $\alpha, \alpha_1, \alpha_2, \ldots \alpha_n$ represent the successive coefficients of any one of the terms of V in any row or column of the determinant, and $\beta, \beta_1, \beta_2 \ldots \beta_n$ the successive corresponding coefficients of the same term in any other row or column of the determinant, the one series of coefficients shall be proportional to the other.

Let us compare the first column and the i-column headed with the element V_i. Selecting any term in V, suppose it to contain x_i, then in whatever element of the first column that term is found, it will be found in a corresponding element of the i-column, and in each case with unity for its coefficient, since all the elements are mere collections of terms from V. But when it is not found in a particular element of the first column, it will not be found in the corresponding element of the i-column. The entire series of coefficients in the one being then the same as that in the other, the common ratio of the corresponding terms is unity.

Suppose, secondly, that the proposed term is found in V and not in V_i; then in all the elements of the i-column its coefficient is 0, so that the series of coefficients in the i-column might be formed from those in the first column by multiplying the latter successively by 0. This again represents a common ratio.

The same reasoning may be applied to the comparison of any two columns of the determinant. Thus in comparing the i-column and the j-column:—terms of V which contain both x_i and x_j will be found in corresponding elements of both columns—terms which contain x_i but not x_j will be wholly absent from the j-column. Thus in all cases if $\alpha, \alpha_1, \alpha_2, \ldots \alpha_n$ represent the coefficients of a term of V in one column, its coefficients in any other column, taken in the same order, will be of the form $\lambda\alpha, \lambda\alpha_1, \lambda\alpha_2 \ldots \lambda\alpha_n$, the coefficient λ being either 1 or 0.

Lastly, the principal elements consist, as do all the elements, of positive terms.

Therefore by the last proposition the developed determinant will consist of products (without powers higher than the first) of different terms of V, and the coefficients of all such products will be positive.

Therefore the determinant will be expressible as a rational entire function of x_1, x_2, ... x_n with positive coefficients.

The rapidity with which the complexity of the determinant increases as the number of variables increases is remarkable. For example, if $n=2$ and $V=axy+bx+cy+d$, the determinant is

$$\begin{vmatrix} axy+bx+cy+d & axy+bx & axy+cy \\ axy+bx & axy+bx & axy \\ axy+cy & axy & axy+cy\,; \end{vmatrix}$$

and its calculated value will be found to be

$$abcx^2y^2+abdx^2y+acdxy^2+bcdxy,$$

consisting of four positive terms.

But if $n=3$ and

$$V=axyz+byz+cxz+dxy+ex+fy+gz+h,$$

the developed determinant will consist of fifty-eight positive terms. Its calculated value will be found in the Memoir on Testimonies and Judgments.

PROPOSITION III.

The functions V, V_1, V_2, ... V_n *being defined as above, if* V *be complete in form*, i. e. *if it consist of all the terms which according to definition it can contain, each with a positive coefficient, then the system of equations*

$$\frac{V_1}{V}=p_1,\ \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n \quad \ldots \ldots \ldots \quad (1.)$$

will, when $p_1, p_2, \ldots p_n$ *are proper fractions, admit of one solution, and only one, in positive values of* x_1, x_2, ... x_n.

We shall show, first, that the above proposition is true when $n=1$, secondly, that on the hypothesis that it is true for $n-1$ variables, it is true for n variables. Hence it will follow that it is true generally.

Suppose $n=1$. Then $V=ax_1+b$, whence the system (1.) reduces to the single equation

$$\frac{ax_1}{ax_1+b}=p_1,$$

$$\therefore \qquad x_1=\frac{bp_1}{a(1-p_1)},$$

whence, since a and b are positive, and p is a positive fraction, x_1 is positive.

Thus the proposition is true when $n=1$.

Now, let $x_1=0$, and let x_2, x_3 ... x_n be determined to satisfy the last $n-1$ equations of the system (1.). These $n-1$ equations will, when $x_1=0$, form a system of the same

nature with respect to the $n-1$ variables $x_2, x_3 \ldots x_n$, as (1.) is with respect to the n variables $x_1, x_2, \ldots x_n$. This will be at once seen by taking any particular example. Hence by hypothesis $x_2, x_3 \ldots x_n$ will be determinable as positive quantities, and their values substituted in the first member of the first equation of (1.) will reduce it to the form

$$\frac{Ax_1}{Ax_1+B},$$

A and B being finite and positive. Hence the function $\frac{V_1}{V}$ will become 0.

Secondly, let any finite positive value be assigned to x_1. The last $n-1$ equations of the system (1.) will again form a system of the same nature as before, and will by hypothesis determine a set of finite positive values for $x_2, x_3, \ldots x_n$. These values again substituted in $\frac{V_1}{V}$, will give to it again the form

$$\frac{Ax_1}{Ax_1+B},$$

A and B being finite and positive. Hence as x_1 is finite and positive, $\frac{V_1}{V}$ will be a positive fraction.

Lastly, let x_1 be infinite. Still the last $n-1$ equations of the system (1.) will assume the same form as before. Determining thence $x_2, x_3 \ldots x_n$, and substituting in $\frac{V_1}{V}$, we have

$$\frac{V_1}{V}=\frac{Ax_1}{Ax_1+B},$$

in which A and B are finite and positive and x_1 is infinite. Hence $\frac{V_1}{V}=1$. It is seen then that as x_1 varies from 0 to infinity, $x_2, x_3, \ldots x_n$ being at the same time always by hypothesis determined to satisfy the last $n-1$ equations of the system (1.), the function $\frac{V_1}{V}$ will vary from 0 through positive fractional values to unity. It is manifest, too, that it varies continuously. If then it vary by continuous *increase*, it will once, and only once in its change, become equal to p_1, and the whole system of equations thus be satisfied together. I shall show that it does vary by continuous *increase*.

If it vary continuously from 0 to 1 and not by continuous increase, it must in the course of its variation assume at least once a maximum or minimum value. Let us then seek the condition of possibility of

$$\frac{V_1}{V} = \text{a maximum or minimum},$$

the variables being subject to the relations

$$\frac{V_2}{V}=p_2, \quad \frac{V_3}{V}=p_3 \ldots \frac{V_n}{V}=p_n.$$

Here, proceeding in the usual way by differentiation, we have

$$\frac{VdV_1-V_1dV}{V^2}=0, \quad \frac{VdV_2-V_2dV}{V^2}=0, \ldots \frac{VdV_n-V_ndV}{V^2}=0,$$

or

$$\frac{dV}{V}=\frac{dV_1}{V_1}=\frac{dV_2}{V_2}\ldots=\frac{dV_n}{V_n}.$$

Let the common value of these fractions be represented by $-dt$, then we have a system of $n+1$ equations of which the first is

$$Vdt+dV=0,$$

while the n others are of the type

$$V_i dt+dV_i=0.$$

The complete system, therefore, on effecting the total differentiations, becomes

$$Vdt+\frac{dV}{dx_1}dx_1\ldots+\frac{dV}{dx_n}dx_n=0,$$

$$V_1dt+\frac{dV_1}{dx_1}dx_1\ldots+\frac{dV_1}{dx_n}dx_n=0,$$

$$\cdots\cdots\cdots\cdots$$

$$V_ndt+\frac{dV_n}{dx_1}dx_1\ldots+\frac{dV_n}{dx_n}dx_n=0.$$

Now from the nature of the function V we have

$$\frac{dV}{dx_i}=\frac{V_i}{x_i},\quad \frac{dV_i}{dx_i}=\frac{V_i}{x_i},\quad \frac{dV_i}{dx_j}=\frac{V_{ij}}{x_j},$$

so that the above equations become

$$Vdt+V_1\frac{dx_1}{x_1}+V_2\frac{dx_2}{x_2}\ldots+V_n\frac{dx_n}{x_n}=0,$$

$$V_1dt+V_1\frac{dx_1}{x_1}+V_{12}\frac{dx_2}{x_2}\ldots+V_{1n}\frac{dx_n}{x_n}=0,$$

$$V_2dt+V_{21}\frac{dx_1}{x_1}+V_2\frac{dx_2}{x_2}\ldots+V_{2n}\frac{dx_n}{x_n}=0,$$

$$\cdots\cdots\cdots\cdots$$

$$V_ndt+V_{n1}\frac{dx_1}{x_1}+V_{n2}\frac{dx_2}{x_2}\ldots+V_n\frac{dx_n}{x_n}=0,$$

and the elimination of dt, $\frac{dx_1}{x_1}$, $\frac{dx_2}{x_2}\ldots\frac{dx_n}{x_n}$ from these equations gives the sought condition of possibility of a maximum value of $\frac{V_1}{V}$, consistently with the satisfaction of the last $n-1$ equations of the system (1.).

This condition is therefore expressed by the equation

$$\begin{vmatrix} V & V_1 & V_2 \ldots V_n \\ V_1 & V_1 & V_{12}\ldots V_{1n} \\ V_2 & V_{21} & V_2 \ldots V_{2n} \\ \ldots & \ldots & \ldots \\ V_n & V_{n1} & V_{n2}\ldots V_n \end{vmatrix}=0.$$

But we have already seen (Prop. II.) that the first member of this equation is essentially positive for positive values of $x_1, x_2 \ldots x_n$. Hence the function $\frac{V_1}{V}$ varies by continuous increase, and on the hypothesis that the proposition to be proved is true for $n-1$ variables, it is true for n variables.

Therefore, connecting this with the former result, the proposition is true universally.

PROPOSITION IV.

If V *be an incomplete function, some of the terms belonging to the complete form being wanting, but the terms present having their coefficients positive, it will in general be necessary not only that the quantities* $p_1, p_2, \ldots p_n$ *should be positive fractions, but also that they should satisfy certain inequations of the form*

$$a_1p_1+a_2p_2\ldots+a_np_n+b \overline{\gtrless} 0,$$

in order that the system

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n \quad \ldots \quad (1.)$$

may admit of a solution in positive values of $x_1, x_2 \ldots x_n$.

For let $Ax_r x_s x_t \ldots$ be any term in V, A being a constant which is positive in all the terms, but which may be different in the different terms. Suppose that in V_i there exist e terms like the above, and let the several ratios of these terms to V be denoted by $\lambda_1, \lambda_2 \ldots \lambda_e$. Then the ith equation of the system (1.) will become

$$\lambda_1+\lambda_2\ldots+\lambda_e=p_i, \quad \ldots \quad (2.)$$

and the system (1.) will be converted into a system of n equations of this nature. We will suppose that there exist m distinct quantities of the nature of $\lambda_1, \lambda_2 \ldots \lambda_e$ in the first members of this transformed system, and we will represent these by $\lambda_1, \lambda_2 \ldots \lambda_m$. Then, if these constitute all the ratios of the separate terms of V to V itself, we have a new equation,

$$\lambda_1+\lambda_2\ldots+\lambda_m=1. \quad \ldots \quad (3.)$$

If they do not constitute all those separate ratios, we have, on the contrary, an inequation,

$$\lambda_1+\lambda_2\ldots+\lambda_m \overline{\lessgtr} 1. \quad \ldots \quad (4.)$$

Lastly, the condition that $\lambda_1, \lambda_2 \ldots \lambda_m$ are positive fractions, gives the inequations

$$\lambda_1 \overline{\gtrless} 0, \ \lambda_2 \overline{\gtrless} 0 \ldots \lambda_m \overline{\gtrless} 0. \quad \ldots \quad (5.)$$

The conditions $\lambda_1 \overline{\lessgtr} 1$, &c. are already implied in (3.) or (4.).

The λ quantities are thus subject to a system of united *equations* and *inequations*, from which they must be eliminated by the method already explained.

The result of such elimination will be a final system of inequations connecting $p_1, p_2, \ldots p_n$. Equations connecting these quantities can only present themselves when the equations of the original system are not independent, or, which really falls under

the same hypothesis, when one or more of the variables $x_1, x_2 \ldots x_n$ is wholly absent from that system. Thus if x_1 were a common factor of all the terms of V, it would divide out from the numerators and denominators of the system, which would thus become a system of n simultaneous equations connecting the $n-1$ variables $x_2, x_3 \ldots x_n$. Considered with reference to these variables, therefore, the equations of the system would not be independent.

All resulting inequations will be capable of expression under the one general form,

$$a_1p_1+a_2p_2 \ldots +a_np_n+b \overline{\gtrless} 0,$$

the coefficients $a_1, a_2, \ldots a_n$ and b being positive, negative, or vanishing, numerical constants. For any inequation which presents itself in the form

$$a'_1p_1+a'_2p_2 \ldots +a'_np_n+b \overline{\lessgtr} 1$$

may be transformed into

$$-a'_1p_1 \ldots -a'_np_n+1-b \overline{\gtrless} 0.$$

Again, the general inequation

$$a_1p_1+a_2p_2 \ldots +a_np_n+b \overline{\gtrless} 0$$

determines an inferior limit of p_1 when a_1 is positive, and a superior limit of p_1 when a_1 is negative.

For in the former case we have

$$p_1 \overline{\gtrless} -\left(\frac{a_2}{a_1}p_1 \ldots +\frac{a_n}{a_1}p_n+\frac{b}{a_1}\right),$$

the second member of which is an inferior limit of p_1; and it will be observed that the calculated value of this member may be positive, as there is no general restriction on the signs of $a_2, \ldots a_n, b$.

In the latter case, changing a_1 into $-a'_1$, and observing that a'_1 is positive, we have

$$p_1 \overline{\lessgtr} \frac{a_2}{a'_1}p_2+\frac{a_3}{a'_1}p_3 \ldots +\frac{a_n}{a'_1}p_n+\frac{b}{a'_1},$$

the second member of which is a superior limit of p_1.

Lastly, the final system of inequations is totally independent of the numerical value of the coefficients of V. The only restriction is that these coefficients are positive.

PROPOSITION V.

Let V be incomplete in form; then, provided that the equations

$$\frac{V_1}{V}=p_1, \ \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n \quad \ldots \ldots \ldots \quad (1.)$$

are independent with respect to the quantities $x_1, x_2, \ldots x_n$, and that the inequations of condition deducible by the last proposition are satisfied, the equations will admit of one solution, and only one, in positive finite values of $x_1, x_2, \ldots x_n$.

The proof of this proposition will, in its general character, resemble the proof of

$x_2, x_3, \ldots, x_n$ corresponding to the assumed positive finite value of x_1. And these values together make $\frac{V_1}{V}$ a positive proper fraction. We may notice that, representing $\frac{V_1}{V}$ under the form

$$\frac{Ax_1}{Ax_1+B},$$

it cannot be that either A or B is wanting so as to reduce $\frac{V_1}{V}$ to the value 0 or 1. For if A were wanting, V would not contain x_1 at all, as by hypothesis it does; and if B were wanting, V would contain x_1 in every term. Thus x_1 would divide out from the system (1.), which would thus become a system of n equations between $n-1$ variables, and would cease to be independent, as by hypothesis it is.

But when $x_1=0$, or $x_1=$ infinity, the form of V, considered as a function of $x_2, x_3 \ldots x_n$, will not generally be the same as in the case last considered; and the conditions connecting $p_2, p_3, \ldots p_n$ will no longer be such that we can affirm the possibility of deducing from the last $n-1$ equations of the system (1.), as transformed, positive finite values of $x_2, x_3, \ldots x_n$.

The theory of this case depends upon a remarkable transformation.

The most general form of the inequations of condition connecting $p_1, p_2, \ldots p_n$, as determined by Proposition IV., is

$$a_1p_1+a_2p_2 \ldots +a_np_n+b \overline{\gtrless} 0. \quad \ldots \quad (3.)$$

Hence, from the nature of the system (1.), it follows that the function

$$a_1V_1+a_2V_2 \ldots +a_nV_n+bV \quad \ldots \quad (4.)$$

must consist wholly of positive terms. Therefore V must consist of terms which would either appear in the development of the above function with positive signs, or not appear in it at all. Let $Ax_rx_sx_t \ldots$ be any term of V. Then, as the coefficient of this term in (4.) would be

$$a_rA+a_sA+a_tA \ldots +bA,$$

and as A is positive, we have

$$a_r+a_s+a_t \ldots +b \overline{\gtrless} 0,$$

a general condition which determines not what terms have actually entered, but what could alone possibly have entered into the constitution of V.

From the system (1.) we have

$$\frac{a_1V_1+a_2V_2 \ldots +a_nV_n+bV}{V}=a_1p_1+a_2p_2 \ldots +a_np_n+b.$$

Hence if we write

$$a_1V_1+a_2V_2 \ldots +a_nV_n+bV=H,$$

we have

$$\frac{H}{V}=a_1p_1+a_2p_2 \ldots +a_np_n+b, \quad \ldots \quad (5.)$$

an equation by which we may replace any one of the equations of the system (1.), and

Now let $$a_1p_1+a_2p_2\ldots+a_np_n+b=0.$$
This, as a_1 is positive, is to suppose that p_1 coincides with one of its own inferior limits. We must suppose this to be the highest of those limits, as otherwise some of the other limiting conditions would be violated. Now, since all the terms in H are affected with positive powers of x_1, while those in K do not contain x_1, the first equation of the system (6.) will be satisfied by $x_1=0$, provided that the remaining $n-1$ equations give finite positive values for $y_2\ldots y_n$. But the vanishing of x_1 reduces these equations to the form

$$\frac{\mathrm{K}_2}{\mathrm{K}}=p_2,\ \frac{\mathrm{K}_3}{\mathrm{K}}=p_3,\ldots\ \frac{\mathrm{K}_n}{\mathrm{K}}=p_n. \quad \ldots\ldots\ldots \quad (7.)$$

It is therefore necessary to show that $p_2,\ p_3\ldots p_n$ in this system are actually subject to the conditions to which the application of the method of Proposition IV. to the system itself would lead.

The n quantities $p_1,\ p_2\ldots p_n$ are by hypothesis subject to the conditions furnished by the application of the method of Proposition IV. to the original system (1.). In applying this method each of the original equations yields an equation of the form

$$\lambda_1+\lambda_2\ldots+\lambda_s=p_i; \quad \ldots\ldots\ldots\ldots \quad (8.)$$

and to the equations thus formed are added the inequations

$$\lambda_1+\lambda_2\ldots+\lambda_m\overline{\overline{<}}1,$$
$$\lambda_1\overline{\overline{>}}0,\ \lambda_2\overline{\overline{>}}0,\ \ldots\lambda_m\overline{\overline{>}}0;$$

$\lambda_1,\ \lambda_2\ldots\lambda_m$ having reference to the whole system of original equations.

Now the satisfaction of the equation

$$\frac{\mathrm{H}}{\mathrm{H}+\mathrm{K}}=0$$

by the value $x_1=0$, involves the vanishing of all those quantities of the system $\lambda_1,\ \lambda_2\ldots\lambda_m$, which are derived from terms in V that are also found in H. Hence the λ quantities that do not vanish are those derived from terms in V which appear in K.

Again, the condition

$$a_1p_1+a_2p_2\ldots+a_np_n+b=0$$

shows that the system of equations of which (8.) is the type are not independent. They must, under the particular circumstances of the case, be such that the above equation shall be derivable from them. Hence one of these equations may be rejected. If we reject the first, viz. the one which contains p_1, and then reduce the others by making the λ quantities which are not derived from K to vanish, the system typified by (8.) evidently reduces to the system which we should have to employ if we applied the method of Proposition IV. directly to the system of $n-1$ equations (7.). Hence the quantities $p_2,\ p_3,\ \ldots p_n$ satisfy the final conditions to which that application would lead, and therefore by hypothesis the equations (7.) admit of solution by a single system of finite positive values of $y_2,\ y_3,\ \ldots y_n$.

Now in general

$$x_i=x_1^{\frac{a_i}{a_1}}y_i.$$

1st. The probability determined is not precisely of the same nature as the probabilities given.

For the data are supposed to be derived from experience; and therefore, on the supposition that the future will resemble the past, the events of which the probabilities are given will in the long run recur with a frequency proportioned to their probability.

But the probability determined is always an intellectual rather than a material probability. We cannot affirm that in the long run an event will occur with a frequency proportional to its calculated probability; but we can affirm that it is more likely to occur with this than with any other precise degree of frequency; that if it do not occur with this degree of frequency, the data are in some measure *one-sided*.

At the same time the limits of possible deviation are determined.

2ndly. General solutions obtained by the method do sometimes, but not always, admit* of being verified by other methods. I believe that this is solely because it is not often possible to solve the problem by other methods without introducing hypotheses which are of the nature of additional data, and, in effect, limit the problem. Every general solution, however, admits of a number of particular verifications by necessary consequence from the theorems established in this paper.

3rdly. It has been seen that a calculated probability is not necessarily a definite numerical value. It may be of the form $A+cC$, in which c is an arbitrary positive fraction. Here it is implied that the probability admits of any value between A and $A+C$. If, further, $A=0$ and $C=1$, it is implied that the probability may have any value between 0 and 1,—is therefore quite indefinite. This would really arise if we applied the method to a case in which the event of which the probability is sought had absolutely no connexion with those of which the probabilities are given.

Hence in the present theory the numerical expression for the probability of an event about which we are totally ignorant is not $\frac{1}{2}$, but c†. Hence, also, when all the probabilities given are measured by $\frac{1}{2}$, it is not to be concluded (upon the ground of *e nihilo nihil*) that the probability sought will also be $\frac{1}{2}$.

4thly. While extending the real power of the theory of probabilities, the method tends in some cases to diminish the apparent value of its results. For all problems in which the data admit of logical expression can be solved by it; but the resulting solutions, founded upon the bare data, may be of an indeterminate character, in place of the determinate results to which ordinary methods, aided by hypotheses not really involved in the data, lead.

This is the case with the problem of the combination of different grounds of belief or opinion. The general solution is indefinite. In two limiting cases, however, it assumes a definite form; one of these, which agrees with the formula generally accepted, representing the extreme cumulative force of testimonies, the other the mean weight of

* Professor DONKIN has verified a general solution (Laws of Thought, p. 362).

† See on this subject a paper by Bishop TERROT, Edinburgh Transactions, vol. xxi. part 3.

judgments. Both these, however, occur as limiting cases, and they can only be applied with confidence under extreme circumstances, such as probably never occur in human affairs. (*Edinburgh Memoir*, pp. 630–645.)

5thly. I have, in effect, remarked that there is reason to suppose that all questions in the theory of probabilities can ultimately be reduced to questions in which the immediate subjects of probability are *logical*, i. e. involve no other essential relations than those of genus and species, whole and part. This is a question of theoretical rather than of practical interest. For instance, whether the formula of the arithmetical mean, which is the basis of the theory of astronomical observations, is self-evident, or whether it rests upon an ultimate logical basis, or whether, as I am inclined to believe, it may lawfully be regarded in either of these distinct but not conflicting lights, the superstructure remains the same.

-8-

VII. *On the Theory of Local Probability, applied to Straight Lines drawn at random in a plane; the methods used being also extended to the proof of certain new Theorems in the Integral Calculus. By* MORGAN W. CROFTON, *B.A., of the Royal Military Academy, Woolwich; late Professor of Natural Philosophy in the Queen's University, Ireland. Communicated by* J. J. SYLVESTER, *F.R.S.*

Received February 5,—Read February 27, 1868.

1. THE new Theory of Local or Geometrical Probability, so far as it is known, seems to present, in a remarkable degree, the same distinguishing features which characterize those portions of the general Theory of Probability which we owe to the great philosophers of the past generation. The rigorous precision, as well as the extreme beauty of the methods and results, the extent of the demands made on our mathematical resources, even by cases apparently of the simplest kind, the subtlety and delicacy of the reasoning, which seem peculiar to that wonderful application of modern analysis—*ce calcul délicat*, as it has been aptly described by LAPLACE—reappear, under new forms, in this, its latest development. The first trace which we can discover of the Theory of Local Probability seems to be the celebrated problem of BUFFON, the great naturalist*—a given rod being placed at random on a space ruled with equidistant parallel lines, to find the chance of its crossing one of the lines. Although the subject was noticed so early, and though BUFFON's and one or two similar questions have been considered by LAPLACE, no real attention seems to have been bestowed upon it till within the last few years, when this new field of research has been entered upon by several English mathematicians, among whom the names of SYLVESTER and WOOLHOUSE† are particularly

* The mathematical ability evinced by BUFFON may well excite surprise; that one whose life was devoted to other branches of science should have had the sagacity to discern the true mathematical principles involved in a question of so entirely novel a character, and to reduce them correctly to calculation by means of the integral calculus, thereby opening up a new region of inquiry to his successors, must move us to admiration for a mind so rarely gifted.

† Many remarkable propositions on the subject, by these eminent mathematicians, have appeared in the mathematical columns of the 'Educational Times' and other periodicals. A very important principle has been introduced by Professor SYLVESTER, which may be termed *decomposition of probabilities*. For instance, he has shown that the probability of a group of three points, taken at random within a given triangle, fulfilling a given *intrinsic* condition (*i. e.* one depending solely on the internal relations of the points among each other), may be expressed as a linear function of two simpler probabilities; viz. that of the same condition being fulfilled (1) when one of the points is fixed at a vertex of the triangle, and a second restricted to the opposite side; (2) when all three points are restricted, one to each side of the triangle. The order of the integrations required

distinguished. It is true that in a few cases differences of opinion have arisen as to the principles, and discordant results have been arrived at, as in the now celebrated *three-point* problem, by Mr. WOOLHOUSE, and the *four-point* problem of Professor SYLVESTER; but all feel that this arises, not from any inherent ambiguity in the subject matter, but from the weakness of the instrument employed; our undisciplined conceptions of a novel subject requiring to be repeatedly and patiently reviewed, tested, and corrected by the light of experience and comparison, before they are purged from all latent error.

The object of the present paper is, principally, the application of the Theory of Probability to straight lines drawn at random in a plane; a branch of the subject which has not yet been investigated. It will be necessary to begin by some remarks on the general principles of Local Probability. Some portion of what follows I have already given elsewhere*.

2. The expression "*at random*" has in common language a very clear and definite meaning; one which cannot be better conveyed than by Mr. WILSON'S expression "*according to no law.*" It is thus of very wide application, being often used in cases altogether beyond the province of mathematical measurement or calculation.

In Mathematical Probability, which consists essentially in arithmetical calculation, when we speak of a thing of any kind taken at random, there is always a direct reference to the *assemblage of things* to which it belongs and from which it is taken, at random,—which here comes to the same thing as saying that any one is as likely to be taken as any other. When we have a clear conception of what the assemblage is, from which we take, and not till then, we can proceed to sum up the favourable cases.

In many problems on probability there is no difficulty in forming a clear conception of the total number of cases. Thus if balls are drawn from an urn, the number of cases is the number of balls, or of certain combinations of them; and if the number of balls be supposed infinite, no obscurity arises from this. But there are several classes of questions in which the totality of cases is not merely infinite, but of an inconceivable nature. Thus if we try to imagine how to determine completely by experiment the probability of a hemisphere thrown into the air falling on its base, we may suppose an infinite number of persons to make one trial each; afterwards we may suppose each person to make two, three, or an infinite number of trials; again, we may suppose for every trial that has taken place an infinite number of others, varying, for instance, in the substance, size, &c. of the body employed; and so on. We can thus continually suppose variations of the experiment, each variation giving a new infinity of cases. Now problems of this nature are treated by means of the following principle:—

In any question of probability regarding an infinite number of cases, all equally pro-

---

is thus reduced by *three*. The same method applies to any polygon, and also to the points taken in space within a tetrahedron. It is to be hoped that Professor SYLVESTER will give these remarkable results to the public in a detailed form: a general account of them was given to the British Association at Birmingham in 1865.

* Educational Times, May 1867.

bable, the result will be unaltered if we take, instead of these cases, *any lesser infinity of cases, chosen at random from among them**.

3. The case of a point or straight line taken at random in a plane or in space is a problem of the above description. Thus, if a point be taken at random in a plane, the total number of cases is of an inconceivable nature, inasmuch as a plane cannot be *filled* with mathematical points, any infinitesimal element of the plane containing an unlimited number of points. We see, however, by means of the above principle, that we may consider the assemblage we are dealing with, as *an infinity of points all taken at random in the plane.*

Let us examine the nature of this assemblage. As the points continue to be scattered at random over the plane, their density tends to become uniform. It is evident, in fact, that a random point is as likely to be in any element dS of the surface, as in any equal element dS'; and therefore by continuing to multiply points, the number in dS will be equal (or *subequal*, to use a term of Professor De Morgan's) to that in dS'. Of course, though the density tends to become uniform, the disposition of the points does not tend to become symmetrical; those within any element dS will be dispersed in the most irregular manner over that element†. However, it is important to remark that, *for all purposes of calculation*, the ultimate disposition may be supposed symmetrical; for as the position of any point is determined by that of the element dS, within which it falls, it matters not what arbitrary arrangement we assume for the points within the element.

* This proposition, of which, in a somewhat different form, a mathematical demonstration is given by Laplace (Théorie Analytique des Probabilités, chap. 3), may be regarded as almost axiomatic. Thus, suppose an urn to contain an infinite number of black and white balls, in the proportion of 2 to 3; if any lesser infinite number of balls be drawn from it, the black ones among them will be to the white as 2 to 3. For, imagine all the balls ranged in a row ACB, the black from A to C, the white from C to B; if we now select an infinite number at random from among them, it appears self-evident that, if the line be divided into five equal parts, the numbers of balls taken from each part will be the same, or rather, will *tend* to equality on being increased indefinitely. Hence the black balls selected will be to the white as AC to CB, or as 2 to 3. When the numbers are *large*, but not infinite, this principle is approximately true, and forms, as is well known, the basis of most of the practical applications of Probability. Thus the chance of an infant living to the age of twenty is as truly found from, say, 1,000,000 of observed cases, as it would be from the total number.

In its strict mathematical form, the proposition may be thus stated:—In any unlimited number of cases, divided into favourable and unfavourable, if p be the ratio of the favourable to the whole number of cases, and if we select any infinite number of cases at random from among them, *the probability is infinitely small, that the same ratio, as determined from the selected cases, shall differ from* p *by a finite quantity.*

† Order thus results from disorder, the uniform density of the aggregate being unaffected by the disorder and irregularity of arrangement of its ultimate constituents; much as a nebula of uniform brightness is related to the stars which compose it. This remarkable law is to be traced, under one form or another, in most of the applications of the Theory of Probability.

"Elle mérite l'attention des philosophes, en faisant voir comment la regularité finit par s'établir dans les choses même qui nous paraissent entièrement livrées au hasard."—*Laplace.*

A familiar illustration of the tendency to uniform density in the random points may be derived by observing the drops of rain on a pavement at the commencement of a shower: as the drops multiply, it will be evident to the eye that their density tends more and more to uniformity.

Hence we may, if we please, assume that, when a point is taken at random in a plane, those from which it is taken are an infinite number symmetrically disposed over the plane.

Likewise, points taken at random in a line may be supposed equidistant. And if random values be taken for any *quantity*, they may be supposed to form an arithmetical series, with an infinitesimal difference.

Let us now consider the case of a straight line drawn at random in an infinite plane: the assemblage from which we select it is, as before, *an infinity of lines drawn at random in the plane.* What is the nature of this aggregate? First, since any direction is as likely as any other, as many of the lines are parallel to any given direction as to any other. Consider one of these systems of parallels; let them be cut by any infinite perpendicular. As this infinite system of parallels is drawn at random, they are as thickly disposed along any part of the perpendicular as along any other; the intersections being in fact random points on the perpendicular. Hence it is easily seen that, for all purposes of calculation, the assemblage of lines may be thus conceived. Divide the angular space round any point into a number of equal angles $\delta\theta$, and for every direction let the plane be ruled with an infinity of equidistant parallel lines, the common infinitesimal distance being the same for every set of parallels. Or we may suppose one such system of parallels drawn, and then turned through an angle $\delta\theta$, then through another equal angle, and so on, till they have returned to their former direction.

If we take any fixed axes in the plane, a random line may be represented by the equation

$$x\cos\theta+y\sin\theta=p,$$

where p and θ are constants taken at random.

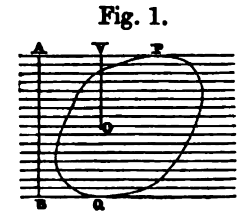
There is no difficulty in extending now our conceptions to points, straight lines, and planes, taken at random in *space.*

4. We may take any plane area as the *measure* of the number of random points within it: in the case of random lines, I proceed to prove the following important principle:—

The measure of the number of random lines which meet a given closed convex plane boundary, is the length of the boundary.

Draw any system of parallels meeting the boundary, their common infinitesimal distance being δp. *If we take this distance as unity*, the number of these parallels is AB, a line cutting them at right angles. Let $\text{AB}=\mathfrak{C}$, and let θ be its inclination to any fixed direction in the plane; conceive now a consecutive system of parallels inclined to the former at an angle $\delta\theta$, then a third, and so on, till the parallels return to the direction in the figure; then the total number of lines will be

Fig. 1.



$$\frac{1}{\delta\theta}\int_0^{\pi}\mathfrak{C}d\theta;$$

or, if O be any fixed pole inside the boundary, and $\text{OV}=p$, the perpendicular on the

tangent to the boundary, θ its inclination to a fixed axis, the measure of the number of lines* is

$$N=\int_0^{2\pi} p\,d\theta.$$

Now the integral $\int p\,d\theta$ extended through four right angles gives the *whole length of the boundary*, whatever be its nature, provided it be convex†.

Hence if L be the length of the boundary,

$$N=L.$$

This result may be obtained also as follows. It may be shown very simply by the above principles that the measure of the number of random lines which meet any finite straight line of length a, is $2a$ (it may indeed be assumed as self-evident that the number is proportional to a). Conceiving now the boundary L as consisting of straight elements, the number of lines meeting any element ds, is $2ds$; so that the whole number which meet the boundary would be 2L; but as each line cuts the boundary in *two* points, we should thus count each line twice over; hence the true number is L.

Hence if L be the length of any convex boundary, and l that of another, lying wholly inside the former, the probability that a line drawn at random across L shall also intersect l, is

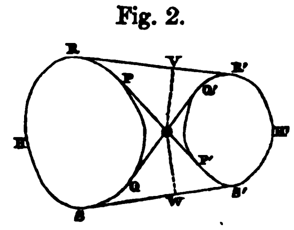
$$p=\frac{l}{L}.$$

It is important to observe that the measure of the number of lines which meet any *non-convex* boundary is *the length of a string drawn tightly round it*; as is obvious on consideration. The same is true for a boundary which is not closed.

5. Let there be any two boundaries external to each other: let X be the length of an endless band passing round both, and crossing between them, and Y the length of another endless band also enveloping both, but not crossing; then *the measure of the number of random lines which meet both boundaries is* X−Y.

It will be easily found from the principles explained above, that the number required will be the integral $\int p\,d\theta$ (referred to O as pole), taken for the left-hand curve from the position RR′ of its tangent, to the position PO; then for the right-hand one from the position P′O of its tangent, to the position S′S; then for the left-hand one, from SS′ to QO; then for the right-hand one, from Q′O to R′R. Now the values of these integrals are, drawing the perpendiculars OV, OW to RR′, SS′,

Fig. 2.



* It will be well to remember that this *measure* of the number of lines, N, means *the actual number multiplied by the constant factor* $\delta\theta$. Our notation is thus simplified, and no confusion need arise from sometimes saying "the number of lines," for shortness, instead of "the measure of the number of lines." As $\delta\theta$ remains constant throughout our investigations, henceforth we will denote it by δ.

† As $L=\int_0^{\pi} C\,d\theta$, we see that *the mean breadth of any convex area is equal to the diameter of a circle whose circumference equals the length of the boundary*. By *breadth* is meant the distance between two parallel tangents, whose direction is supposed to alter by uniform increments.

1. the mixed line RPO − RV,
2. „ „ S′P′O − S′W,
3. „ „ SQO − SW,
4. „ „ R′Q′O − R′V,

and the sum of these is evidently equal to X−Y.

I will add a different proof of this proposition, deduced from art. 4, as it is interesting to see our results verified.

For shortness, I will use the symbol N(S) for "the number of random lines meeting the space S;" and N(S, S′) for the number meeting both S and S′.

The number of lines meeting both boundaries is evidently identical with the number meeting both the mixtilinear figures OPHQ, OP′H′Q′. These two figures together form the mixtilinear reentrant figure HPP′H′Q′Q, and by art. 4, N(HPP′H′Q′Q)=Y.

Now N(OPHQ)+N(OP′H′Q′)=N(HPP′H′Q′Q)+N(OPHQ, OP′H′Q′). But OPHQ, OP′H′Q′ being convex figures, the number of lines meeting each is represented by its length; therefore

$$X=Y+N(HPQ,\ H'P'Q').$$

The probability that a line drawn at random across a given convex boundary of length L shall also meet a given *external* boundary is therefore

$$p=\frac{X-Y}{L}.$$

6. If two convex boundaries L, L′ intersect each other, in two or more points, it may be proved in a similar manner that the number of random lines which meet both is represented by L+L′−Y, where Y is the length of an endless band passing round both. Hence the probability that a line which meets L shall also meet L′, is

$$p=\frac{L+L'-Y}{L}.$$

7. It may easily be proved that the measure of *the number of random lines which pass between two given convex boundaries is*

$$N=PP'+QQ'-\text{arc } PQ-\text{arc } P'Q',$$

where PP′, QQ′ are the two common tangents which cross each other.

Thus the number of random lines which pass between the two branches of an hyperbola is represented by Δ, the difference between the whole length of the hyperbola and that of its asymptotes. This difference, as is known, is given by the definite integral

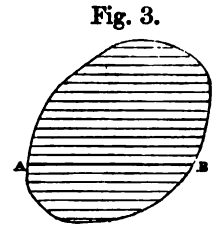
$$\Delta=4a\int_0^{\alpha}\sqrt{1-e^2\sin^2\theta}\,.\,d\theta,$$

where $\sin\alpha=\frac{1}{e}$.

8. Two lines are drawn at random across a given convex area: to find the probability of their intersection lying within the area.

Let AB be the internal portion of any random line crossing the area: the number of its intersections with all the random lines in the area is the number of those lines which meet it. Now this number is $\frac{2AB}{\delta}$ (art. 4); hence the number of intersections of the system of parallels to AB with all the random lines in the area, is twice the sum of the lengths of all these parallel chords divided by δ. But this sum is the area of the figure (we have taken the common distance δp of the chords as unity).

Fig. 3.



Let Ω be the area, L the length of the boundary. As, then, $\frac{2\Omega}{\delta}$ is the number of intersections for any system of parallels, and the number of those systems is $\frac{\pi}{\delta}$, the total number of intersections is $\frac{2\pi\Omega}{\delta^2}$. But we have thus counted each intersection twice; so that the real number of intersections which fall inside the area Ω is $\frac{\pi\Omega}{\delta^2}$.

Hence the required probability is

$$p=\frac{2\pi\Omega}{L^2},$$

since the whole number of intersections is $\frac{1}{2}\left(\frac{L}{\delta}\right)^2$.

Thus it is an even chance that two random chords of a circle intersect within the circle; for any other figure the chance is less than $\frac{1}{2}$.

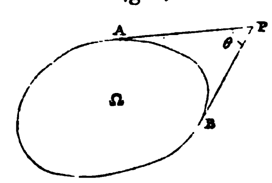
If an infinity of lines are drawn at random in an infinite plane, *the density of their intersections* (*i. e.* the measure of the number* of intersections in any given space, divided by the space) is uniform, and equal to π.

9. If an infinity of random lines meet a given area, *the density of their intersections, at any external point* P, is

$$\rho=\theta-\sin\theta,$$

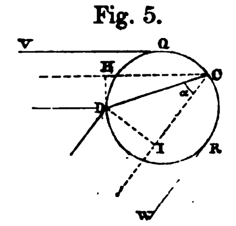
where θ is the apparent angular magnitude of the area from that point.

Fig. 4.



Conceive an infinitely small circle, or other figure (whose dimensions, however, infinitely exceed δp), at P, and let us calculate the number of the said intersections which fall inside this circle. Let the figure represent this circle, *magnified* as it were; QV, RW being the tangents PA, PB. Draw one of the random lines CD, which meet both the circle and the area Ω, the actual number of intersections which lie on CD will be $\frac{1}{\delta}N(\Omega, CD)$, which is found from art. 5 to be

Fig. 5.



$$\frac{1}{\delta}(2CD-CH-CI),$$

* We take for this measure the actual number multiplied by $\delta\theta^2$, or δ^2 (see note, art. 4).

or

$$\frac{\mathrm{CD}}{\delta}\Big(2-\cos\alpha-\cos(\theta-\alpha)\Big).$$

Hence the actual number of intersections on all the chords parallel to CD is

$$\frac{1}{\delta}(\text{area of circle})\Big(2-\cos\alpha-\cos(\theta-\alpha)\Big).$$

Therefore the measure* of the whole number of intersections lying within the circle is

$$\tfrac{1}{2}(\text{area})\int_0^\theta\Big(2-\cos\alpha-\cos(\theta-\alpha)\Big)d\alpha=(\text{area})(\theta-\sin\theta),$$

which proves the theorem.

10. The number of the intersections external to the given area is, then, measured by the integral

$$\iint(\theta-\sin\theta)d\mathrm{S}$$

extended over the whole plane outside Ω; $d\mathrm{S}$ being the element of the area. Now the number of internal intersections is $\pi\Omega$ (art. 8), and the sum of both is $\frac{1}{2}\mathrm{L}^2$. We obtain thus, in a singular manner, the following remarkable theorem in Definite Integrals:—

If θ be the angle between the tangents drawn from any external point (x, y) *to any given convex boundary, of length* L, *enclosing an area* Ω, *then*

$$\iint(\theta-\sin\theta)dxdy=\tfrac{1}{2}\mathrm{L}^2-\pi\Omega,$$

the integration extending over the whole space outside Ω.

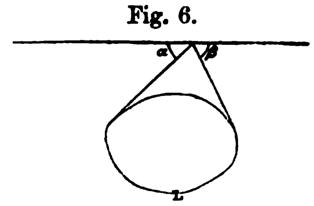
It does not seem easy to deduce this integral, in its generality, by any other method. It may be verified by direct integration for the cases of a circle, and of a finite straight line. It forms a striking example of what will doubtless be found, as the study of Local Probability advances, to be one of its most remarkable applications, viz. the evaluation of Definite Integrals. All who have studied the subject must have remarked the variety of ways in which almost every problem may be considered; now it often happens that a question in which we are baffled by the difficulties of the integration, when we attempt it in a particular way, may be solved with comparative ease by other considerations: we can then return to the integrals which we were unable to solve, and assign their values. I proceed to give some further applications of the above theory to Integration.

11. Given any infinite straight line outside a given convex boundary of length L, let dx be any element of this line; α, β the inclinations of dx to the two tangents drawn from it to the boundary, then

$$\int_{-\infty}^{\infty}(\cos\alpha+\cos\beta)dx=\mathrm{L}.$$

* We take for this measure the actual number multiplied by $\delta\theta^2$, or δ^2 (see note, art. 4).

Fig. 6.



It is easy to see from art. 5 that the number of random lines cutting L, which also meet dx, is $dx(\cos\alpha+\cos\beta)$; now the sum of all such elements gives the number of lines cutting both L and the given infinite straight line; that is, L (art. 4). This integral may be otherwise verified.

If the boundary L be enclosed within any outer convex boundary, let ds be the differential of the length of the latter, α, β the inclinations of ds to the tangents from it to L, then we find in the same manner,

$$\int(\cos\alpha+\cos\beta)ds=2\mathrm{L},$$

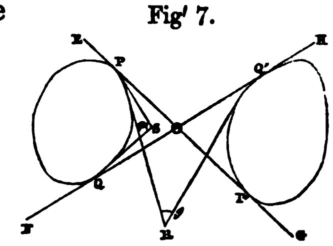
the integral extending all round the outer curve.

I mention this merely as an illustration; it is in fact easy to show independently that

$$\mathrm{L}=\int\cos\alpha ds=\int\cos\beta ds.$$

12. If an infinite number of random lines pass between two convex areas, the density of their intersections will be (as in art. 9) at any point R in the angle FOG, or in EOH,

Fig' 7.



$$\varrho=\theta-\sin\theta;$$

and at any point S in the spaces POQ, P'OQ',

$$\varrho=\pi-\varphi-\sin\varphi;$$

now the whole number of intersections is (art. 7) measured by

$$\tfrac{1}{2}(\mathrm{PP'}+\mathrm{QQ'}-\mathrm{PQ}-\mathrm{P'Q'})^2.$$

Hence

$$\iint(\theta-\sin\theta)d\mathrm{S}+\iint(\pi-\varphi-\sin\varphi)d\mathrm{S}=\tfrac{1}{2}(\mathrm{PP'}+\mathrm{QQ'}-\mathrm{PQ}-\mathrm{P'Q'})^2,$$

the first integral extending over the infinite spaces FOG, EOH, and the second over the spaces POQ, P'OQ'.

Thus if θ be the angle between the tangents drawn from any external point to an hyperbola,

$$\iint(\theta-\sin\theta)dxdy=\tfrac{1}{2}\Delta^2,$$

where Δ is the difference between the hyperbola and its asymptotes, and θ means the *external* angle of the tangents, in the cases where they touch the same branch of the curve, the integral extending over the whole space outside the hyperbola.

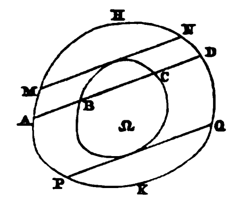
Received February 13, 1868.

13. If we consider a system of random lines disposed over the whole surface of an infinite plane, and a second system all of which meet a given convex area Ω within the plane, and then fix our attention on the infinite system of points in which the latter system cuts the former, it will be seen that *the density of these intersections, at any point* (x, y) *exterior to* Ω, *is equal to* 2θ, θ being the angle which Ω subtends at the point

(x, y); hence $2\iint\theta dxdy$ represents the number of these intersections which lie on any given portion of the plane outside Ω.

Take now an arbitrary convex boundary surrounding Ω; we will calculate in a different way the number of intersections which lie on the annular space between the two boundaries, and thus arrive at a value for the above definite integral, extended over the same annulus.

Fig. 8.



Let AD be a random line of the second system, meeting Ω; the number (within the annulus) of its intersections with the first system will be measured by (art. 4) $2AB+2CD$; and hence the total number of intersections of all parallels to AD (between the tangents MN, PQ), with the first system, will be measured by double the area, cut from the annulus, between MN and PQ. Hence if Θ represent the annulus, the actual number of intersections which lie on those random lines of the second system which are parallel to those in the figure, is

$$n_0=\frac{1}{\delta}(2\Theta-2 \text{ segment MHN}-2 \text{ segment PKQ}).$$

Making now the parallel tangents MN, PQ revolve by constant changes of inclination, δ, through two right angles, we have for the *measure* of the total number of intersections, if φ be the inclination of MN to a fixed line,

$$\mathrm{N}=\int_0^\pi(2\Theta-2\mathrm{MHN}-2\mathrm{PKQ})d\varphi.$$

But if we make the tangent MN revolve through 4 right angles instead of 2, it will occupy all the positions of PQ; denoting then the segment MHN by Σ, we have

$$\mathrm{N}=2\pi\Theta-2\int_0^{2\pi}\Sigma d\varphi;$$

therefore

$$\iint\theta dxdy=\pi\Theta-\int_0^{2\pi}\Sigma d\varphi.$$

The mean or average value of the segment Σ, as the tangent alters by uniform changes of inclination, is

$$\mathrm{A}=\frac{1}{2\pi}\int_0^{2\pi}\Sigma d\varphi;$$

we have, then, the following theorem:—

If θ be the angle subtended at any point (x, y) *by a given convex area Ω, then*

$$\iint\theta dxdy=\pi(\Theta-2\mathrm{A}),$$

the integration extending over the annulus between Ω and any given exterior convex boundary; Θ standing for the area of that annulus, and A *denoting the average area of the segments cut from the annulus by the tangents to the boundary of Ω.*

This theorem gives the value of the integral in those cases where we are able to calculate the value of A: if Σ is constant, we have the theorem:—

Let there be any two convex boundaries so related that a tangent to the inner cuts off a constant area from the outer. Let θ be the angle subtended by the inner boundary at any external point (x, y)*; and let Δ be the difference of the parts into which the annular space between the two is divided by any tangent to the inner, then*

$$\iint \theta dxdy = \pi\Delta,$$

the integration extending over the whole of the annulus.

For instance, we may apply the theorem to two similar coaxial ellipses. We may deduce thus the following definite integral,

$$\iint \tan^{-1}\left(\frac{2\sqrt{a^2y^2+b^2x^2-a^2b^2}}{x^2+y^2-a^2-b^2}\right)dxdy = \pi abk^2(\pi\sin^2\tfrac{1}{2}\alpha - \alpha + \sin\alpha),$$

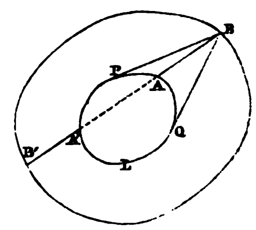
the limits being given by $1 < \frac{x^2}{a^2}+\frac{y^2}{b^2} < k^2$; putting $\cos\frac{1}{2}\alpha = \frac{1}{k}$.

In the case where $k^2=2$, the value of the integral is $2\pi ab$; that is, the area of the outer ellipse.

14. If we suppose an infinite plane covered with random lines, and then imagine these divided into two systems, the first comprising all those lines which meet a given convex boundary, the second all those which do not meet it, and if we now consider the assemblage of points in which the first system intersects the second, we shall find (as in art. 9) that *the density of these intersections, at any point outside the boundary*, is $2\sin\theta$, θ being, as before, the apparent angular magnitude of the boundary.

Hence the number of intersections which lie on any given space is represented by the integral $2\iint \sin\theta dS$.

Fig. 9.



If we now suppose an endless string (of length Y) passed round the given boundary (whose perimeter we call L), and if this string be kept stretched by a moving point which thus traces out a new contour enclosing the given one (as the outer of any two confocal ellipses may be generated from the inner), we may estimate in a different manner the number of intersections which lie on the intermediate annular space, and thus obtain the following value for the above integral extended over that space,

$$\iint \sin\theta dS = L(Y-L).$$

Let AB be a line of the first system meeting the two boundaries in A, B; the number of points in which AB is cut by a system of random lines *covering the whole plane* is (art. 4)

$$\frac{2AB}{\delta}.$$

If we subtract from this the number of intersections of AB with those lines which

meet the boundary L, the remainder will be the number of intersections of AB with the *second* system of lines above, viz. (art. 6)

$$\frac{2\mathrm{AB}}{\delta}-\frac{1}{\delta}(2\mathrm{AB}+\mathrm{L}-\mathrm{Y}),$$

that is,

$$\frac{1}{\delta}(\mathrm{Y}-\mathrm{L}).$$

This is constant for every position of AB; hence the number of intersections lying on the annulus will be the above constant, multiplied by the number of positions of AB; now this number is $\frac{2\mathrm{L}}{\delta}$ (art. 4) (remembering that for every line AB, there is also one A′B′, forming a portion of the same straight line). Hence the total number of intersections is

$$\frac{2}{\delta^2}\mathrm{L}(\mathrm{Y}-\mathrm{L}).$$

If, then, the integration extend over the annulus,

$$\iint \sin\theta\,.\,d\mathrm{S}=\mathrm{L}(\mathrm{Y}-\mathrm{L}).$$

This theorem will apply to an ellipse, the outer boundary being a confocal ellipse. A particular case, which admits of verification by using elliptic coordinates, will be:—

If θ be the angle which two fixed points F, F′ subtend at the element dS,

$$\iint \sin\theta d\mathrm{S}=8c(a-c);$$

the integration extending over an ellipse whose foci are F, F′, $2a$ being the axis of the ellipse, and $2c$=FF′.

The above method will also show that in this case the integral remains unchanged in value, if it extend over any Cartesian oval whose *internal* foci are FF′, and whose axis is $2a$. An instance of such a Cartesian is a circle from F as centre with a as radius, provided $a>2c$. The same will appear by means of elliptic coordinates*.

15. I will mention the following integral here, as, though strictly not derived from the theory which forms the subject of this paper, the principle used in obtaining it is, as in the cases which precede, the calculation of the number of intersections lying on a given space, of a given reticulation of straight lines.

Given a closed convex boundary without salient points; if we draw an infinity of tangents to it, each making an infinitesimal angle (δ) with the preceding, and consider the *intersections* of all these tangents with each other, it will not be difficult to show (as in art. 9) that the number of intersections lying on any element dS will be

$$\frac{1}{\delta^2}\cdot\frac{\sin\theta}{\mathrm{TT'}}d\mathrm{S},$$

* The general integral above admits also of being established by means of a certain generalization of elliptic coordinates, which defines the position of a point by the sum and difference of two strings, each of which is attached to a fixed point on a given oval curve; they are then wrapped round the curve in opposite directions, and leave it as two tangents, meeting and terminating at the proposed point.

where T, T′ *are the tangents from* dS *to the boundary, and* θ *their mutual inclination.*

Now the whole number of tangents is $\frac{2\pi}{\delta}$, and that of intersections $\frac{2\pi^2}{\delta^2}$. We infer therefore that

$$\iint \frac{\sin\theta}{TT'}\,dS = 2\pi^2;$$

the integral extending over the whole external surface.

If the integral extend over *the annulus between the given boundary and an outer line along which* θ *has a constant value* (α), then

$$\iint \frac{\sin\theta}{TT'}\,dS = 2\pi(\pi-\alpha).$$

If the same integral extend over *the space between the given boundary and two fixed tangents, including an angle* α, its value will be $\frac{1}{2}(\pi-\alpha)^2$. If it extend over *the infinite angle formed by those tangents produced*, its value will be $\frac{1}{2}\alpha^2$.

If the given boundary contain salient points, then for every such point, where the bounding line changes direction abruptly through an angle A, a number of the tangents, equal to $\frac{A}{\delta}$, meet at that point; hence a number $\left(\frac{1}{2}\frac{A^2}{\delta^2}\right)$ of intersections coincide there, and consequently we must subtract $\frac{1}{2}A^2$ from each of the above integrals. Hence if there are any number of salient points $A\ A'\ A''$, &c. in the boundary, the first integral becomes

$$\iint \frac{\sin\theta}{TT'}\,dS = 2\pi^2 - \tfrac{1}{2}\Sigma A^2,$$

and likewise for the second.

Thus for a regular polygon of (n) sides, the value is

$$2\pi^2\left(1-\frac{1}{n}\right).$$

If instead of drawing tangents to the given boundary at uniform angular intervals, we draw a system of tangents whose points of contact are distant from each other by a common infinitesimal interval, we shall find that the density of the intersections in this case varies as

$$\frac{\rho\rho'}{TT'}\sin\theta,$$

where $\rho\rho'$ *are the radii of curvature of the boundary at the points of contact of* TT′: this gives us the integral

$$\iint \frac{\rho\rho'}{TT'}\sin\theta\,dS = \tfrac{1}{2}L^2,$$

L being the whole perimeter of the boundary, the integral extending over the whole plane.

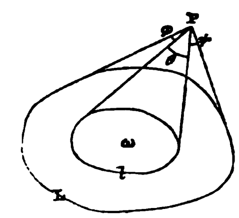
Many analytical definite integrals may be deduced by expressing the general theorems now given, in the language of different systems of coordinates, for various particular

cases. Thus the first theorem in this article, applied to the ellipse, gives

$$\iint \frac{a^2y^2+b^2x^2}{\{(x^2+y^2+c^2)^2-4c^2x^2\}\sqrt{a^2y^2+b^2x^2-a^2b^2}}\,dxdy=\pi^2;$$

the equation of limits being $\frac{x^2}{a^2}+\frac{y^2}{b^2}>1$.

Fig. 10.



16. Let there be a closed convex area ω, length of boundary l, enclosed within another of length L; let θ be the apparent magnitude of ω at any external point; by considering two systems of random lines, one crossing the boundary L, and the other l, and examining the law of the density of the intersections of the former with the latter, we arrive at the theorem:—if we put for shortness

$$\alpha-\sin\alpha=u_\alpha,$$

$$\iint(u_{\theta+\phi}+u_{\theta+\psi}-u_\phi-u_\psi)dS+2\iint\theta dS=Ll-2\pi\omega;$$

the first integral extending over the whole space outside L, the second over the space between L and l.

17. But few problems on random straight lines admit of such simple results and of such generality as those we have been discussing. In general they can only be solved for particular forms of the boundaries. However, the above principles, applied to each particular question, generally suffice to reduce it at least to a problem of the Integral Calculus. I will give one or two examples.

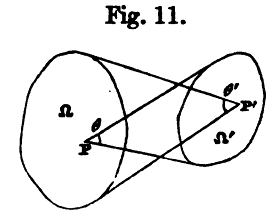
If two random lines cross a given convex area, the chance of their intersection falling on any *internal* portion of the area ω, is evidently (art. 8)

$$p=\frac{2\pi\omega}{L^2}.$$

But the chance of the intersection falling on any *external* area is less easy to find; it depends on the integral $\iint(\theta-\sin\theta)dS$ extended over that area. Could we succeed in finding the required probability by any different method, we could give the value of this integral for any external area.

A line is drawn at random across each of two given convex areas Ω, Ω', external to each other, lengths of boundaries L, L'; to find the chance of their intersection being outside both areas.

Fig. 11.



The density of the intersections of the system of random lines crossing Ω with those crossing Ω', at any point P within Ω, is 2θ, where θ means the apparent magnitude of Ω' at P. Within Ω', the density is $2\theta'$. Hence it is easy to see that, as the whole number of intersections is LL', the required probability is

$$p=1-\frac{2}{LL'}(\iint\theta dS+\iint\theta' dS'),$$

the integrals extending over Ω and Ω' respectively. It is evident that these integrals, however, can only be evaluated for particular forms of the areas.

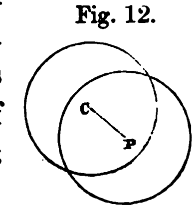
18. The following problems relate to a circular boundary:—

1. A random point falls within a given circle and a random straight line is drawn across the circle; to find the chance of the line passing within a given distance of the point.

As the general solution is somewhat complicated, I will take the particular case where the given distance is the radius of the circle, which will serve equally well as an example of the application of the foregoing principles.

Let C be the centre of the given circle, P any position of the random point, r the radius of the given circle; draw an equal circle with P as centre; then the number of random lines meeting the given circle and passing within a distance r of the point P, is the same as the number of random lines cutting *both* circles; this number is measured (art. 6) by the excess of the two circumferences over an endless band wrapped round them; that is, putting $CP=\varrho$,

Fig. 12.



$$2\pi r-2\varrho.$$

If dS be an element of the surface at P, the sum of the favourable cases will be

$$F=\iint(2\pi r-2\varrho)dS=2\int_0^r(\pi r-\varrho).2\pi\varrho d\varrho\,;$$

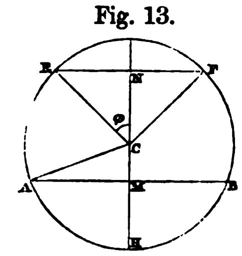
$$\therefore F=(\pi-\tfrac{2}{3})2\pi r^3.$$

But the whole number of cases is $2\pi r\times\pi r^2$; hence the required chance is

$$p=1-\frac{2}{3\pi}.$$

I will give another solution of this problem:—Let AB be a position of the random line; take $MN=r$, then all the favourable positions of the random point are within the segment EHF; the number of favourable points is therefore

Fig. 13.



$$r^2(\pi-\phi+\sin\phi\cos\phi).$$

We have to multiply this by the differential of CM, and integrate from $CM=r$ to $CM=0$, which will give the favourable combinations for all random lines parallel to AB, passing between C and H; doubling this, we have the result for *all* lines parallel to AB; that is,

$$F_0=2r^2\int_r^0(\pi-\phi+\sin\phi\cos\phi)d.r\cos\phi$$

$$=2r^3\int_0^{\frac{\pi}{2}}(\pi-\phi+\sin\phi\cos\phi)\sin\phi\,d\phi\,;$$

$$\therefore F_0=2r^3(\pi-\tfrac{2}{3}).$$

Now if the system of lines parallel to AB revolve through two right angles, we have for the *measure* of the whole number of favourable cases

$$F=2\pi r^3(\pi-\tfrac{2}{3})\,;$$

hence, as before,

$$p=1-\frac{2}{3\pi}.$$

The problem in its general form can be solved without any great difficulty by the same methods. The result may be expressed in this form:—Let D be the given maximum distance; draw a circle of radius D with its centre on the given circumference; let Y be a band enveloping both circles, and 2θ the inclination of the two straight portions of this band; then the probability of the line passing within a distance D of the point will be

$$p=\frac{2\pi r+2\pi D-Y}{2\pi r}+\frac{\cos^3\theta}{3\pi};$$

or, if p_0 be the probability when the point is taken anywhere on the *circumference* of the given circle, then the general value of the probability is

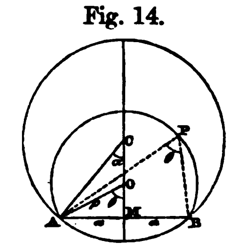
$$p=p_0+\frac{\cos^3\theta}{3\pi}.$$

If a random point and a random straight line be taken within any convex boundary of length L, the chance that the line shall pass within a distance D of the point, D being small, is approximately,

$$p=\frac{2\pi D}{L}.$$

2. If three lines are drawn at random across a given circle, to determine the probability that their three intersections shall lie within the circle.

Fig. 14.



Let AB be one of the random lines. The total number of favourable triads of random lines, each triad of which includes AB, is the same as the *number of intersections, which fall within the circle*, of all random lines which cross AB. For every such intersection which lies within the circle, gives a pair of lines meeting AB, forming a triad whose intersections all lie within the circle. Now if θ be the angle which AB subtends at any internal point P, the number of these intersections will be measured by (art. 9)

$$N=\iint(\theta-\sin\theta)dS,$$

extended over the whole circle.

To integrate this, conceive the circle divided into an infinite number of elementary crescents, by segments of circles on AB; let O be the centre of the segment APB, ϱ its radius; then the area of the segment APB is, putting $AB=2a$,

$$\text{segment }=(\pi-\theta)\varrho^2+a\varrho\cos\theta,\text{ or as }\varrho\sin\theta=a$$

$$=a^2\left(\frac{\pi-\theta}{\sin^2\theta}+\cot\theta\right).$$

Differentiating this for θ, we obtain for the area of the crescent between APB and the consecutive arc on AB,

$$\text{crescent} = \frac{2a^2}{\sin^2\theta}\big(1+(\pi-\theta)\cot\theta\big)d\theta.$$

Hence the number of intersections *above* AB will be

$$N=2a^2\int_\alpha^\pi \frac{\theta-\sin\theta}{\sin^2\theta}\big(1+(\pi-\theta)\cot\theta\big)d\theta;$$

$$\therefore\quad \frac{N}{2a^2}=\int_\alpha^\pi\left\{\frac{\theta d\theta}{\sin^2\theta}+\pi\frac{\theta\cos\theta d\theta}{\sin^3\theta}-\frac{\theta^2\cos\theta}{\sin^3\theta}d\theta-\frac{d\theta}{\sin\theta}-\pi\frac{\cos\theta d\theta}{\sin^2\theta}+\frac{\theta\cos\theta}{\sin^2\theta}d\theta\right\}.$$

All these are elementary integrals, and give (reducing the indeterminate forms by the usual methods)

$$\frac{N}{2a^2}=\frac{3}{2}-\frac{\alpha^2}{\sin^2\alpha}-\frac{\pi-\alpha}{\sin\alpha}+\frac{\pi}{2}\cot\alpha+\frac{\pi\alpha}{2\sin^2\alpha}.$$

To find the number of intersections *below* AB, change α into $\pi-\alpha$; this gives for the whole number of favourable triads (including AB),

$$N=2a^2\left(3-\frac{\pi}{\sin\alpha}+\frac{\alpha\pi-\alpha^2}{\sin^2\alpha}\right);$$

or if c be the radius of the given circle, $a=c\sin\alpha$;

$$\therefore\quad N=2c^2(3\sin^2\alpha-\pi\sin\alpha+\alpha\pi-\alpha^2).$$

Multiply this by the differential of CM, and integrate from c to $-c$, and we have the sum of all favourable triads, each of which includes *any one* of the random lines parallel to AB,

$$F=2c^3\int_0^\pi(3\sin^2\alpha-\pi\sin\alpha+\alpha\pi-\alpha^2)\sin\alpha d\alpha$$

$$=2c^3\left(8-\frac{\pi^2}{2}\right).$$

Multiply this by π, and we have the measure* of the total number of favourable triads: however, this must be divided by 3, as it is clear we should thus count each triad thrice; hence total value of

$$F=\frac{\pi c^3}{3}(16-\pi^2);$$

and the whole number of cases being $\frac{1}{6}(2\pi c)^3$, we find for the probability sought,

$$P=\frac{4}{\pi^3}-\frac{1}{4}\dagger.$$

19. An interesting inquiry, though of a more difficult nature than that which has occupied us in this Paper, would be the extension of the foregoing principles to straight lines and planes drawn at random in space. It involves several intricate and curious points relating to the general theory of surfaces. With regard to the *measure of the number of random straight lines which meet a given closed convex surface*, it is easy to show that this measure is *the surface itself*.

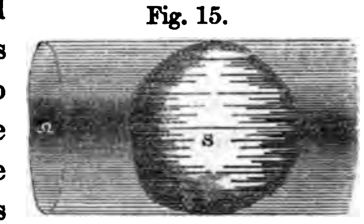
* *i. e.* the actual number multiplied by δ^3 (as art. 8).

† It is not unlikely that this result may be obtained in some simpler manner.

It may be assumed as self-evident that if space be filled with an infinity of random straight lines, and they be cut by any infinite plane, the points in which it cuts them are distributed with uniform density over the plane; and this density will be the same for any other plane. Hence the number of the random lines which meet any plane area is proportional to that area Hence the number meeting any plane element dS of the surface is proportional to dS; the same is true for every other element; and each random line cuts two elements and only two; hence the whole number of lines is proportional to S.

We might view the question as follows. The entire body of random lines may be considered (as in art. 3) as a system of parallels disposed uniformly and symmetrically in space, which is afterwards turned round by infinitely small angular displacements, into every possible position. Let the figure represent one of these systems of parallels meeting the surface S, and of course bounded by the cylinder, enveloping S, whose generatrix is parallel to these lines. Let Ω be the area of the perpendicular section of this cylinder, then Ω is the measure of the number of these parallels. Let θ, ϕ be the angular coordinates of the direction of these parallels, and let them now pass into every possible angular position; the whole number of lines which meet S will be proportional to

Fig. 15.



$$\iint \Omega \sin \theta d\theta d\phi,$$

extended through half the solid angular space round a point. We infer from this that

$$\int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \Omega \sin \theta d\theta d\phi = k\mathrm{S}.$$

To determine the constant k, we may apply the theorem to any particular case, as a sphere; this gives $k=\frac{\pi}{2}$. We may accept this manner of viewing a system of random lines, then, as a proof of the theorem in surfaces:—

If Ω be the area of the section of a cylinder enveloping a convex surface S; θ, ϕ the angular coordinates of the generatrix of the cylinder,

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \Omega \sin \theta d\theta d\phi = \frac{\pi}{2}\mathrm{S}.$$

The measure of *the number of random planes* which meet a given surface is easily seen to be (as in art. 4)

$$\mathrm{N} = \int_0^{2\pi}\int_0^{\pi} p \sin \theta d\theta d\phi,$$

where p is the perpendicular from any internal point on the tangent plane, and θ, ϕ the angular coordinates of p. I am not aware that this integral has ever been considered. It is probable that it admits of some simple geometrical representation, which possibly will be found to be the length of some closed curve, traced upon the given surface, and bearing some remarkable relation to the general curvature of the surface.

IX. *On the Proof of the Law of Errors of Observations.*

By MORGAN W. CROFTON, *F.R.S.*

Received March 24,—Read April 22, 1869.

1. So much has been published upon the Theory of Errors, that some apology seems to be required from a new writer who does not profess to have arrived at any results which were unknown to his predecessors. Nevertheless, so great, as is well known, are the difficulties of the theory, whether we seek to form a correct estimate of the principles on which it rests, or to follow the subtle mathematical analysis which has been found indispensable in reasoning upon them, that any contribution which tends to simplify the processes, without weakening their logical exactness, will probably be considered of some value. My object in this paper is to give the mathematical proof, in its most general form, of the law of single errors of observations, on the hypothesis that an error in practice arises from the joint operation of a large number of independent sources of error, each of which, did it exist alone, would produce errors of extremely small amount as compared generally with those arising from all the other sources combined. Now this proof is contained in a process given for a different object, namely, POISSON'S generalization of LAPLACE'S investigation of the law of the mean results of a large number of observations, to be found in his 'Recherches sur la Probabilité des jugements,' and which is reproduced in Mr. TODHUNTER'S valuable 'History of the Theory of Probability.'

It is obvious that we should altogether restrict the generality of the proof, confining it merely to a few artificial and conventional cases, if we were to suppose each source of error to give positive and negative errors with equal facility, or to assume the law of error (even supposing it unknown) to be the same for all the sources. None of the processes, therefore, contained in the 4th chapter of the 'Théorie Analytique des Probabilités' are of sufficient generality for our purpose, though some writers have so employed them; nor will the method apply here which LESLIE ELLIS has given in his memoir "On the Method of Least Squares" (Camb. Phil. Trans. 1844), based upon FOURIER'S theorem, on account of the assumption of equal facility for positive and negative errors. The proof which follows will be found, I think, of full generality,—the only cases excluded being incompatible with the existence of the exponential law (see art. 7), and at the same time greatly simpler than POISSON'S, dispensing with his refined and difficult analysis*.

2. It is remarkable that the well-known exponential function which is now pretty

* The *length* of this communication may seem at variance with the statement that the proof here given is a simpler one than those of former writers. Still I think it will be found to be so on examination; the length of the paper arises from fuller explanations being given than is usually the case. I am persuaded that the doubts and misconceptions which have prevailed so extensively with relation to this subject have been in great part occasioned by the extreme brevity and scanty explanation of the great writers who have treated it.

generally received among mathematicians as expressing the law of frequency of single errors of observation, does not seem to have been distinctly given by any one of the three great philosophers LAPLACE, GAUSS, and POISSON (who may be called the founders of the Theory of Errors) as being, in their opinion, the expression of that law. It has been erroneously supposed, as LESLIE ELLIS points out, that GAUSS's and LAPLACE's proofs of the method of Least Squares depend upon that assumption. It is true that GAUSS's first method, in the 'Theoria Motûs,' does require it; but he does not present that method as other than tentative and hypothetical: and later, in the 'Theoria Combinationis Observationum,' he says, speaking of the law of single errors, "plerumque incognita est."

As, however, this law of error seems in our day to have been adopted by general consent, some inquiry into the grounds on which its validity rests will be appropriate here. And first I would remark that it can scarcely be maintained that any attempt hitherto made to establish this law independently of the hypothesis I have named in art. 1 has been successful. We may pass by GAUSS's proof in the 'Theoria Motûs,' which shows that the law must hold *if we take as an axiom* that the arithmetical mean of several observations is the most probable result. Now this really is not an axiom, but only a convenient rule which is generally near the truth: this we see by considering any case in which we are certain that the errors do not follow the exponential law; does the mind see here *à priori* that the rule does not give the most probable result? It seems certain that we should have just the same confidence in it here as in any case; yet GAUSS's proof shows that it does not give the most probable result*. It should indeed be stated that GAUSS himself (as might have been expected from that acute and accurate mind) is very far from asserting the above assumption to be an axiom; consequently he does not give his proof as more than hypothetical. He only states that the rule is generally accepted—"axiomatis loco haberi solet hypothesis." A method of remarkable simplicity was given by Sir J. HERSCHEL in a very interesting review of QUETELET's 'Letters on Probability†,' which conducts to the same law of error by means of one or two bold assumptions; but striking as the coincidence is, it can hardly be seriously viewed as a *demonstration*; nor is it formally so presented by its distinguished author. However, the methods both of GAUSS and Sir J. HERSCHEL are of great interest to the natural philosopher, as showing that certain *à priori* mathematical assumptions of a very simple kind lead to the same law of error which reasoning based on a study of the facts which surround us also points out as expressing, at least approximately, what generally does occur *in rerum naturâ*: though we can see no necessity that the facts

* See ELLIS, *loc. cit.* p. 207.

† Edinburgh Review, July 1850. See a criticism by LESLIE ELLIS in the Philosophical Magazine, vol. xxxvii. Also BOOLE (Edinb. Trans. vol. xxi.) and THOMSON and TAIT (Natural Philosophy), who speak more favourably. M. QUETELET's 'Lettres' will amply repay a perusal; in connexion with our present inquiry, he points out that not only errors of observations, but the variations of many other fluctuating magnitudes, such as the stature of men, the temperature of the weather, &c. from their mean values, seem to follow the same law. If this be so, the inference seems legitimate that these divergences from the mean types, or *errors of Nature* herself, as they may be called, are produced in each case, not by one or two, but by a vast number of hidden coexisting causes.

should be so, it being quite easy to *conceive* a different economy of nature in which no such accordance would subsist*.

It is possible *à priori* to conceive that the law of single errors of observation might be of any form whatever, varying with each kind of observation: how far it is true that in practice one general law will be found to prevail, is essentially a question of facts—an inquiry, not into what *might be*, but what *is*. Now the hypothesis above mentioned,—namely, that errors *in rerum naturâ* result from the superposition of a large number of minuter errors arising from a number of independent sources,—when submitted to mathematical analysis, leads to the law which is generally received; as far therefore as this hypothesis is in accordance with fact, so far is the law practically true. Fully to decide how far this hypothesis does agree with facts is an extremely subtle question in philosophy, which would embrace not only an extended inquiry into the laws of the material universe, but an examination of the senses and faculties of man, which form an important element in the generation of error. Still, without pretending to enter on a demonstration of the truth of this hypothesis, a few reflections upon the facts, especially in the case of Astronomy (which is *par excellence* the science of observation, and where accordingly the lessons of experience are the clearest and most complete), will, I think, at least convince us of its *reasonableness* in certain large classes of errors of observations. Now if we attend to what has taken place in the history of astronomical observation, we find that the gross errors of the earlier observers proceeded mainly from three or four principal causes—for instance, refraction, imperfect measurement of time, and the use of the naked eye in pointing to objects. When these few capital occasions of error were removed (at least approximately), refraction being discovered and allowed for, and the pendulum and telescopic sights introduced, it was found that observations at once attained a high order of accuracy, showing that the principal sources of error had been eliminated. It would seem, in fact, that in coarse and rude observations the errors proceed from *a very few* principal causes; and in this case, consequently, our hypothesis will probably represent the facts only imperfectly, and the frequency of the errors will only approximate roughly and vaguely to the law which follows from it†. But when

* The extreme simplicity of the exponential relation itself, whether considered as expressing the law of single errors, or that of the mean results of a large number of observations, as contrasted with the long and difficult methods by which it was established, has naturally led to several attempts to dispense with or simplify the latter; in some the hypothesis we here adopt is taken as a basis; but, so far as the present writer is aware, every process given, except POISSON's, fails in generality. In a recent Memoir on the Law of Frequency of Error by Professor TAIT (Edin. Trans. vol. xxiv.) (where, it should be stated, the learned author speaks with some hesitation, and only gives his method as an attempt), it is assumed that each of the elementary errors which are combined can be assimilated to the deviation from its most probable value of the number of white balls among a given large number of balls drawn from an urn, which contains white and black in a given proportion. It is then shown (as indeed is done in LAPLACE's 3rd chapter) that this error follows the exponential law. Thus the proof only applies to the combination of a number of elementary errors, each of which follows that law. But it is quite certain that many simple errors do not follow that law; hence the method is altogether deficient in generality.

† We cannot, however, assert this positively, if there is reason to believe that the error which arises from each principal cause is itself a composite error, which certainly is often the case. The "error in time," for

astronomers, not content with the degree of accuracy they had reached, prosecuted their researches into the remaining sources of error, they found that, not three or four, but a *great number* of minor sources of error, of nearly coordinate importance, began to reveal themselves, having been till then masked and overshadowed by the graver errors which had been now approximately removed*. It was as if a small number of forest trees had been cut down, leaving an innumerable growth of shrubs and brushwood at their feet, remaining to be cleared. There were errors of graduation, and many others, in the construction of instruments; other errors of their adjustments; errors (technically so called) *of observation*; errors from changes of temperature, of weather, from slight irregular motions and vibrations; in short, the thousand minute disturbing influences with which modern astronomers are familiar, and which it is superfluous to recapitulate here. Many of these are known and allowed for, or eliminated, at least approximately, in practical astronomy; still we seem to be justified in considering the error which remains as the result of a great number of yet minuter errors, each inconsiderable in itself. Thus a cursory view of the nature of astronomical errors, and the light which this throws on various cognate classes of observations, seem to lead to the conclusion that the above hypothesis will be found to hold, generally, in the case of refined and delicate observations. No doubt much more would be necessary to justify us in asserting

instance, is certainly not a simple error, but one resulting from the joint action of several causes, one or more of which we can conceive detected and allowed for, leaving the others in operation. An error may thus arise from the superposition of only three or four component errors, which at first sight are of simple origin, but in reality represent each a group of minor errors; and the hypothesis would then hold. It is questionable whether, among the causes which in practice vitiate any observation, any *simple* error ever does enter, of considerable magnitude and importance as compared with the others combined; such, for instance, as would be the error produced in the time (or through the time on some astronomical magnitude) by the pendulum being $\frac{1}{10}$ of an inch too long or too short, every thing else being pretty accurate. If it be said that ignorance or negligence might produce such a result, we may answer that such negligence or ignorance would make itself felt in other ways also: one such error would not stand alone. Isolated acts of neglect by a careful observer would come under the head of *occasional* errors, as explained further on.

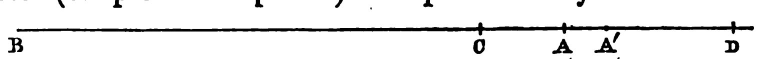
It seems very difficult to discern, *à priori*, the nature of the errors incurred in *estimating* magnitudes by the eye, or of errors arising from the imperfection of our senses, such as those incurred in pointing to a star with the naked eye. It is quite possible that such errors may arise each from *several* sources, though their nature be hidden from our view.

* A similar law to that mentioned above seems to prevail in many kindred cases. Thus in the successive improvements in artillery, machinery, &c., in proportion as the greater sources of imperfection and inaccuracy are understood and remedied, the *number* of minor disturbing influences which are thus rendered perceptible, and still vitiate the results, though to a less extent, increases rapidly. We may even trace a sort of analogy here in various phenomena both of the moral and material universe, which apparently have no bearing on the point we are considering. Thus the principal wants of human nature, the necessaries of life in fact, are very few; and so long as these are supplied with difficulty, minor wants are scarcely felt, as we see in uncivilized communities: but when the greater wants are satisfied, the number and variety of the *secondary* requirements of our nature are visible in the multitudinous productions of civilized life. The diseases which mainly operate in shortening human existence are very few in number; but could they be extirpated, the number of minor causes, of nearly coordinate importance, which still would influence the rate of mortality would be very large. The statistics of crime, and many other phenomena, would give rise to remarks of a similar nature.

this absolutely; thus it is not enough for our purpose to show, could we do so conclusively, that each error in practice is compounded of a large number of smaller errors; we must also show that they are *independent*, at least for the most part. Thus we may conceive one of the minute errors affecting an astronomical magnitude to be an error in the refraction proceeding from a rise in the general temperature, and another affecting the same observation to be an error of time arising from the expansion of the pendulum through the same cause; now these two minute errors are not independent, and would have to be mathematically combined in quite a different way from two that were independent; and, indeed, such a change of temperature would influence the actual error of the observation in other ways also. However, we may at least safely conclude that the hypothesis in question is not a mere arbitrary assumption, but a reasonable and probable account of what does in fact take place in the case of careful and refined observations.

3. In proceeding to submit this hypothesis to mathematical analysis, the minute simple errors which go to form the observed compound error will be assumed to follow each its own unknown law, expressed by different unknown functions of the utmost generality *: positive and negative values of each error will not be assumed equally possible; on the contrary, the cases will be included, as obviously ought to be done, of minute disturbing influences which always cause the observed magnitude to err in excess, and of others which cause it to err only in defect. I will exclude all mention of the term *probability*, and will consider solely the *frequency* or *density* of the error, viewed as a function of its magnitude.

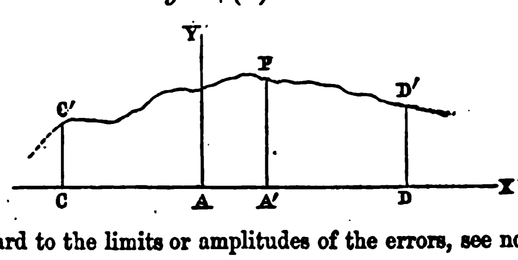
Let any magnitude which has to be determined by observations affected with some one cause of error (simple or compound) be represented by the line BA;



let a large number of such observations be made, and let the observed values be represented by a number of lengths BA′, measured from B: it will be found in general that in the neighbourhood of A the line will be dotted over with a multitude of points A′, the distance AA′ being the error in each case. These dots will begin at some point C, and end at some point D, which generally are on opposite sides of A, but may both be at the same side. Between C and D the dots will be distributed over CD with a variable density: this density, at any point A′, will represent the *frequency* or *density* of errors of magnitude AA′.

If at every point A′ we erect an ordinate A′P representing the density at A′, we shall thus trace out a locus or curve C′D′, whose equation we may call, taking A as origin,

$$y=\phi(x). \qquad (1)$$



* With regard to the limits or amplitudes of the errors, see note on art. 7.

This we may call the *curve* or *function* of Error*. It is of course generally discontinuous, as it is only to include values of x between the points C, D. The function $\phi(x)$ strictly speaking should vanish for all values of x beyond C and D; however, we shall not require any consideration of the analytical methods of expressing such functions. If N be the number of observations taken, and if we put $AD=a$, $AC=b$, then as ydx denotes the number of errors lying between x and $x+dx$,

$$N=\int_{-b}^{a}\phi(x)dx. \quad \ldots \ldots \ldots \quad (2)$$

It is well to notice that, if C be any constant, the equation

$$y=C\phi(x)$$

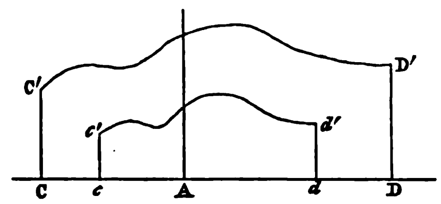
really is the same function of Error as (1), the number of observations only being altered.

4. In order not unduly to limit the generality of the investigation, it is necessary further to study the nature of the possible ways in which the dots we have spoken of as representing the observations may be scattered along the line CD, in the case of various unknown *simple* causes of error; noting also what becomes of the *function* $\phi(x)$ and the curve C′D′ in each case. And first, in many cases the dots will be distributed *continuously* along CD, thus giving a *curve* without gaps or intervals. It is by no means necessary that this curve should descend towards CD at its two extremities more than in the middle; in other words, the extreme values of a simple Error are not always less probable than the intermediate ones. There may be cases where the extreme values are the most probable; for instance, the Error occasioned by supposing a point fixed, which is in reality performing extremely minute and slow oscillations about its mean position. But besides the cases of continuous distribution, there are others, not only conceivable, but which we may be sure do actually occur, in which a function or curve does not assist our conceptions, and we shall do better merely to consider the points or dots themselves. There may be what is called a *constant* Error; that is, some cause which gives the observation always too great (or too small) by the same fixed minute amount: the distribution here is simply a group of N coincident points somewhere on CD. Or a certain cause may only admit of two or more definite values for the error; the distribution will be two or more groups of coincident points, the numbers in each group being equal or unequal. Again, an important class of Errors are those which may be called *occasional Errors*, that is, produced by intermittent causes not always in operation. In such a case, if N observations be made, a certain number of them (say n) are unaffected by the Error; the remaining $N-n$, made when the cause is in operation, we may suppose represented by dots continuously or discontinuously distributed; we have then a group of n coincident points at A, besides a number $N-n$ distributed in some way over CD. Errors of mistake or forgetfulness, and many others also, are of this description.

* The word "error" is sometimes used for shortness to express a source of error. To avoid confusion we may write it with a capital E, when used in this sense. Thus "an Error" will mean a source of error, or the assemblage of actual errors (or the curve or function symbolizing them) which that source produces in a large number of trials, and which form a visible manifestation or representation of it: "an error" will mean a particular magnitude.

5. If we alter the ordinate and abscissa of every point in the curve C′D′ in a given ratio, changing the limits a, $-b$ of the Error in the same ratio, we find the curve $c'd'$ represented by

$$\frac{y}{i}=\phi\left(\frac{x}{i}\right), \quad \ldots \ldots \ldots \ldots \quad (3)$$



which may be called a *similar Error* to (1) or C′D′. The number of observations will be different in the two cases, being represented by the areas of the two figures. We may find it convenient to suppose the number of observations the same; if so

$$y=\frac{1}{i}\phi\left(\frac{x}{i}\right) \quad \ldots \ldots \ldots \ldots \quad (4)$$

will be a *similar function of Error* to $y=\phi(x)$, the number of observations being the same for both, the limits of the error in (3) and (4) being ia, $-ib$.

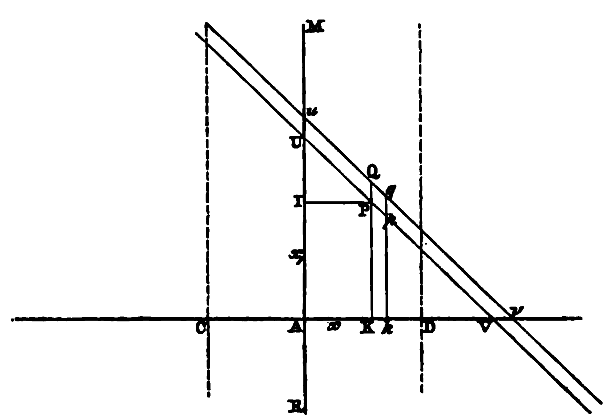
6. *To find the function of Error resulting from the combination of a given Error whose equation is*

$$y_1=f(x_1) \quad \ldots \ldots \ldots \ldots \quad (5)$$

(*the limits being* $\pm\infty$) *with another independent Error*

$$y=\phi(x), \quad \ldots \ldots \ldots \ldots \quad (6)$$

whose limits are a, $-b$.



We shall do this most clearly by help of a geometrical construction. Let the (N) values of the first Error be measured from A according to their signs along the indefinite line MR; likewise measure the (n) values of the second Error along CD, where AD$=a$, AC$=b$. Take any two values, AI$=x_1$ of the first, and AK$=x$ of the second; they give a value $x+x_1$ of the compound Error, to which will correspond a point P of the plane, whose *coordinates* are x, x_1. The number of such points contained within the element

$dx\,dx_1$, each point corresponding to a compound error, will be

$$yy_1\,dx\,dx_1 \text{ or } f(x_1)\phi(x)d\mathrm{S},$$

dS being the element of the area. Draw through P a line UV equally inclined to the axes, then $x+x_1$ is constant along this line; put $\xi=x+x_1=\mathrm{AV}$, take $\mathrm{V}v=d\xi$, and draw uv parallel to UV; take $\mathrm{K}k=dx$, then the number of points within the elementary parallelogram PQpq will be

$$f(\xi-x)\phi(x)d\xi\,dx.$$

Hence the whole number of points between the parallels UV and uv (that is, *the number of compound errors whose magnitudes lie between* ξ and $\xi+d\xi$) will be

$$d\xi\int_{-b}^{a} f(\xi-x)\phi(x)dx.$$

The total number of compound errors thus obtained will be Nn; however, for uniformity, we will suppose the number of observations taken, affected with the compound Error, to be N, the same as for (5). This will oblige us to divide by

$$n=\int_{-b}^{a}\phi(x)dx.$$

Thus if we represent the compound Error by a curve whose coordinates are (ξ, η), it will be

$$\eta=\frac{\int_{-b}^{a} f(\xi-x)\phi(x)dx}{\int_{-b}^{a}\phi(x)dx} \quad \ldots\ldots\ldots\ldots \quad (7)$$

Thus if we wish to find the Error resulting from the combination of the two Errors whose equations are

$$y=\frac{\mathrm{N}}{\theta\sqrt{\pi}}e^{-\frac{(x-\alpha)^2}{\theta^2}},\quad y=\frac{\mathrm{N}'}{\phi\sqrt{\pi}}e^{-\frac{(x-\beta)^2}{\phi^2}},$$

we have from formula (7) (N, N′ denoting the numbers of observations),

$$\eta=\frac{\mathrm{N}}{\pi\theta\phi}\int_{-\infty}^{\infty} e^{-\frac{(\xi-x-\alpha)^2}{\theta^2}}\,e^{-\frac{(x-\beta)^2}{\phi^2}}\,dx,$$

whence

$$\eta=\frac{\mathrm{N}}{\sqrt{(\theta^2+\phi^2)\pi}}e^{-\frac{(\xi-\alpha-\beta)^2}{\theta^2+\phi^2}}.$$

Hence it is easy to see that if any number of Errors of the forms

$$y=\frac{\mathrm{N}}{\theta\sqrt{\pi}}e^{-\frac{(x-\alpha)^2}{\theta^2}},\quad y=\frac{\mathrm{N}'}{\phi\sqrt{\pi}}e^{-\frac{(x-\beta)^2}{\phi^2}},\quad y=\frac{\mathrm{N}''}{\psi\sqrt{\pi}}e^{-\frac{(x-\gamma)^2}{\psi^2}},\ \&\text{c.}$$

be combined, the resultant Error will be

$$y=\frac{\mathrm{N}}{\sqrt{\pi(\theta^2+\phi^2+\psi^2+\ldots)}}e^{-\frac{(x-\alpha-\beta-\gamma-\ldots)^2}{\theta^2+\phi^2+\psi^2+\ldots}} \quad \ldots\ldots\ldots \quad (8)$$

Expanding $f(\xi-x)$ in formula (7), it becomes

$$\eta=f(\xi)-\alpha f'(\xi)+\frac{\lambda}{2}f''(\xi)-\frac{\sigma}{2.3}f'''(\xi)+\&\text{c.},$$

where

$$\alpha=\frac{\int_{-b}^{a}x\phi(x)dx}{\int_{-b}^{a}\phi(x)dx},\ \lambda=\frac{\int_{-b}^{a}x^2\phi(x)dx}{\int_{-b}^{a}\phi(x)dx},\ \sigma=\frac{\int_{-b}^{a}x^3\phi(x)dx}{\int_{-b}^{a}\phi(x)dx},\ \&\text{c.},$$

α being the mean value of the Error $y=\phi(x)$, λ its mean square, σ its mean cube, &c.

7. In the problem of finding the law of error resulting from the superposition of a great number of Errors, each of very small importance by itself, we will consider each component Error as the *diminutive* of some Error of finite importance* (see art. 5). Thus if $y=\mathrm{F}(x)$ be some possible finite Error, and we reduce its dimensions in the ratio i, where i is infinitesimal, the diminished Error will be $\frac{y}{i}=\mathrm{F}\left(\frac{x}{i}\right)$; and if the mean value, mean square, mean cube, &c. of the former be called

$$\mathrm{E}_1,\ \mathrm{E}_2,\ \mathrm{E}_3,\ \ldots\ldots,$$

it is easy to see that the same means, for the reduced Error, will be

$$i\mathrm{E}_1,\ i^2\mathrm{E}_2,\ i^3\mathrm{E}_3,\ \ldots\ldots$$

Now adopting the usual axiom that *no function can represent a finite Error unless* E_1, E_2, E_3, ... *are finite*, it follows that the mean cube, mean 4th power, &c. of the

* Thus all conceivable cases of Errors whose extreme limits, or amplitude, are very small, are contained in the above method of proof; also those small Errors which, though their extreme amplitude be not very small, are merely possible finite Errors (of great or infinite amplitude) on a reduced scale. It is necessary, however, to observe, in examining the nature of all the minute simple Errors which our hypothesis in its generality comprises, that there are cases quite conceivable, and involving no absurdity, of simple Errors of trivial or infinitesimal importance which come under neither of these categories, and to which the method in the text will not apply. To give a simple instance, imagine an *occasional* source of Error, which rarely operates, but which, when it does, gives a fixed finite error k (thus we may conceive an observer to mistake, once in a thousand times, the succeeding division of his instrument for the true one). Let this happen on an average once for n times that the cause is not in operation (n being supposed very great); then the mean value of the Error is $\frac{k}{n+1}$, its mean square is $\frac{k^2}{n+1}$, &c. It is therefore of infinitesimal importance whether, with LAPLACE, we estimate the *importance* of an Error by its mean value (irrespective of sign), or, with GAUSS, by its mean square; but as its mean cube &c. cannot be rejected in comparison with the mean square, the above analysis cannot be applied to it. Minute *simple* Errors of such a description must then be excepted from those which are supposed to enter into the composition of the actual errors of observations. If an appreciable number of them did enter, the received exponential law could not hold for the compound Error. Thus were we to combine a large number of small Errors of the nature of the simple instance just cited, the resultant Error would be of a discontinuous nature, represented by groups of coincident points, with finite intervals between them.

Though it is necessary clearly to understand that the full generality of the hypothesis is restricted by the exceptions explained in this note, yet there seems every reason to suppose that such cases are too rare in practice to cause any sensible deviation from the exponential law of error, the great majority of the minute component Errors which jointly affect any observation *in rerum naturâ* having each, it is natural to suppose, a very minute range or amplitude.

diminutive Error may be rejected in comparison with its mean square*. We infer, therefore,

If $y=f(x)$ represent any Error of indefinite amplitude, and if a new Error, $y=\phi(x)$, of indefinitely small importance as compared with it, be superposed, the resulting compound Error will be represented by the equation

$$y=f(x)-\alpha\frac{d}{dx}f(x)+\frac{\lambda}{2}\frac{d^2}{dx^2}f(x), \quad \dots \dots \dots \quad (9)$$

where α, λ are infinitesimal constants, viz. the mean value of the new Error and the mean value of its square†, the number of observations being supposed the same for the Error (9) as for the former, $y=f(x)$.

If we now conceive $y=\phi(x)$ in the above to be one of a large number of independent infinitesimal Errors, and $y=f(x)$ to be the compound finite Error resulting from the combination of all the others, we infer from (9) that each elementary Error $y=\phi(x)$ affects the law of the combined Errors in a manner which only involves (α) the mean value of the elementary Error, and (λ) its mean square. But if this be so, we may, for our present purpose, substitute for $y=\phi(x)$ any other Error whatever which has the same mean value and mean square (provided of course its mean cube &c. may be neglected in comparison with its mean square). We may therefore *for our purpose* replace $y=\phi(x)$ by‡

$$y=\frac{n}{\sqrt{2\pi(\lambda-\alpha^2)}}e^{-\frac{(x-\alpha)^2}{2(\lambda-\alpha^2)}}, \quad \dots \dots \dots \quad (10)$$

which fulfils these conditions.

Likewise, if there be another elementary Error whose mean value is β and mean square μ, we may replace it by

$$y=\frac{n'}{\sqrt{2\pi(\mu-\beta^2)}}e^{-\frac{(x-\beta)^2}{2(\mu-\beta^2)}},$$

and so on, for all the elementary Errors. Hence (see equation 8) the Error compounded of any number of them will be

$$y=\frac{\mathrm{N}}{\sqrt{2\pi(\lambda+\mu+\nu+\dots-\alpha^2-\beta^2-\gamma^2-\dots)}}e^{-\frac{(x-\alpha-\beta-\gamma-\dots)^2}{2(\lambda+\mu+\dots-\alpha^2-\beta^2-\dots)}}.$$

* We cannot neglect the mean square as compared with the mean 1st power, as the latter is the algebraical sum of a number of positive and negative elements, which sum may be of any amount, however small, and may sometimes vanish altogether; whereas the former is the sum of a number of *positive* elements, and therefore cannot vanish.

† If the new Error be such as to give any discontinuous distribution of points (see art. 4), it is easy to satisfy ourselves, by the method of art. 6, that the above proposition still holds good. In fact, if the n values of the new Error be x_1, x_2, x_3, ..., we shall have, instead of the formula (7),

$$\eta=\frac{1}{n}\{f(\xi-x_1)+f(\xi-x_2)+f(\xi-x_3)+\&c.\},$$

which is true in all cases, whether the distribution be continuous or discontinuous, or a mixture of both; and hence the formula (9) will follow.

‡ This suggestion is due to Professor J. C. ADAMS, one of the Referees charged by the Royal Society with the duty of reporting upon the present Paper. The remainder of the proof, which was of a different nature in the Paper as originally presented, is much simplified thereby.

We conclude therefore that *if a great number of minute independent Errors be combined, and if we write*

$$\left.\begin{aligned} m&=\alpha+\beta+\gamma+\ldots=\text{sum of mean Errors,}\\ h&=\lambda+\mu+\nu+\ldots=\text{sum of mean squares of Errors,}\\ i&=\alpha^2+\beta^2+\gamma^2+\ldots=\text{sum of squares of mean Errors,}\end{aligned}\right\}^{*} \quad . \quad . \quad . \quad . \quad (11)$$

the resulting function of Error will be

$$y=\frac{\mathrm{N}}{\sqrt{2\pi(h-i)}}e^{-\frac{(x-m)^2}{2(h-i)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The Probability of an error being found to lie between x and $x+dx$ is of course

$$\frac{1}{\sqrt{2\pi(h-i)}}e^{-\frac{(x-m)^2}{2(h-i)}}\,dx\dagger.$$

If positive and negative errors in the observation are equally probable, as generally can be secured in practice, at least approximately, then $m=0$; that is, the sum of the mean values of the elementary component Errors vanishes, and the Probability is expressed by the usual value

$$\frac{1}{c\sqrt{\pi}}e^{-\frac{x^2}{c^2}}\,dx.$$

If we calculate by integration from equation (12) the mean value of the composite Error (or, as GAUSS calls it, *the constant part* of the Error) and the mean value of its square, we shall find

$$\begin{aligned}&\text{Mean Error}=m=\text{sum of mean values of component Errors,}\\ &\text{Mean Square of Error}=h+m^2-i.\end{aligned}$$

We have thus a verification of the correctness of our analysis, as the same results may be found from independent algebraical computation‡.

8. Considering the celebrity of the question, it may not be superfluous to show how the result might have been obtained without any antecedent knowledge of the peculiar property of combination of the Errors in equation (8).

* We may observe that $h-i$ *is always positive*; for if we take any set of numbers, positive or negative, the mean of their squares is always greater than the square of the mean (see TODHUNTER's 'Algebra,' p. 407). Therefore

$$\lambda>\alpha^2,\ \text{also}\ \mu>\beta^2,\ \nu>\gamma^2,\ \&\text{c.}$$

Consequently $h>i$.

† This expression will be found to agree with POISSON's final result in the memoir already cited.

‡ If

$$\mathrm{U}=a+b+c+d+\&\text{c.},$$

where each of the quantities a, b, c, d, &c. may take any number (different for each quantity) of different independent values, adopting for shortness the symbol M(K) for "the mean value of K," it is not difficult to prove, by elementary algebra, that

$$\mathrm{M}(\mathrm{U})=\mathrm{M}(a)+\mathrm{M}(b)+\mathrm{M}(c)+\&\text{c.}=\Sigma\mathrm{M}(a),$$

$$\mathrm{M}(\mathrm{U}^2)=\mathrm{M}(a^2)+\mathrm{M}(b^2)+\mathrm{M}(c^2)+\ldots+2\Sigma\{\mathrm{M}(a)\mathrm{M}(b)\},$$

or

$$\mathrm{M}(\mathrm{U}^2)=\Sigma\mathrm{M}(a^2)+\{\Sigma\mathrm{M}(a)\}^2-\Sigma\{\mathrm{M}(a)\}^2.$$

Let us suppose all the infinitesimal simple Errors which it is proposed to combine to be successively superposed upon some assumed function of Error $y=f(x)$; then by equation (9) the new function arising from the first of them will be, putting $D=\frac{d}{dx}$,

$$y=\left(1-\alpha D+\frac{\lambda}{2}D^2\right)f(x).$$

If another be now superposed upon this, we shall have

$$y=\left(1-\beta D+\frac{\mu}{2}D^2\right)\left(1-\alpha D+\frac{\lambda}{2}D^2\right)f(x),$$

and finally the function arising from the superposition of all the given Errors upon the assumed Error $y=f(x)$ will be

$$y=\left(1-\alpha D+\frac{\lambda}{2}D^2\right)\left(1-\beta D+\frac{\mu}{2}D^2\right)\left(1-\gamma D+\frac{\nu}{2}D^2\right)\ldots.f(x). \quad . \quad . \quad (13)$$

But as α, λ are infinitesimals, we have, retaining the square of α,

$$1-\alpha D+\frac{\lambda}{2}D^2=e^{-\alpha D+\frac{1}{2}(\lambda-\alpha^2)D^2}.$$

Thus (13) will become

$$y=e^{-(\alpha+\beta+\gamma+\ldots.)D+\frac{1}{2}(\lambda-\alpha^2+\mu-\beta^2+\ldots.)D^2}f(x),$$

or, adopting the notation (11),

$$y=e^{\frac{1}{2}(h-i)D^2}e^{-mD}f(x)=e^{\frac{1}{2}(h-i)D^2}f(x-m). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

9. Let us now take as the assumed function of Error

$$y=f(x)=\frac{N}{\theta\sqrt{\pi}}e^{-\frac{x^2}{\theta^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

(where N is the number of observations), and imagine the whole given system of small Errors superposed upon it; the resulting function is

$$y=\frac{N}{\theta\sqrt{\pi}}e^{\frac{1}{2}(h-i)D^2}e^{-\frac{(x-m)^2}{\theta^2}}.$$

Now by a theorem in the Differential Calculus*,

$$e^{aD^2}e^{-kx^2}=\frac{1}{\sqrt{1+4ak}}e^{-\frac{kx^2}{1+4ak}};$$

* This theorem, which is new to the present writer, may be proved in various ways. Thus if we put

$$u=e^{aD^2}e^{-kx^2},$$

and differentiate with regard to a, we have

$$\frac{du}{da}=e^{aD^2}D^2e^{-kx^2}=e^{aD^2}(4k^2x^2-2k)e^{-kx^2}:$$

again,

$$\frac{du}{dk}=e^{aD^2}(-x^2e^{-kx^2});$$

we thus obtain the partial differential equation

$$\frac{du}{da}+4k^2\frac{du}{dk}+2ku=0,$$

hence

$$y=\frac{N}{\sqrt{\pi}\sqrt{2(h-i)+\theta^2}}e^{-\frac{(x-m)^2}{2(h-i)+\theta^2}}.$$

Now we may here assume θ as small as we please*,—that is, we may assume the Error (15) upon which the given system was superposed, to be of as small importance as we please. We conclude, then, rejecting this Error altogether, that a system of very small Errors, when combined, give for the resulting function of Error

$$y=\frac{N}{\sqrt{2\pi(h-i)}}e^{-\frac{(x-m)^2}{2(h-i)}}$$

as before.

---

the integral of which is

$$u=k^{-\frac{1}{2}}\phi\left(4a+\frac{1}{k}\right).$$

To determine the arbitrary function ϕ, we remark that if $a=0$, $u=e^{-kx^2}$,

$$\therefore \phi\left(\frac{1}{k}\right)=k^{\frac{1}{2}}e^{-kx^2},$$

hence

$$u=k^{-\frac{1}{2}}\left(4a+\frac{1}{k}\right)^{-\frac{1}{2}}e^{-x^2\left(4a+\frac{1}{k}\right)^{-1}}=(1+4ak)^{-\frac{1}{2}}e^{-\frac{kx^2}{1+4ak}}.$$

Another proof may be obtained by employing Poisson's ingenious transformation (Traité de Mécanique, tom. ii. p. 356), which gives

$$e^{aD^2}\phi(x)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-\omega^2}\phi(x+2\omega\sqrt{a})d\omega.$$

* In order that we may retain the three first terms only in the expansion

$$y=f(x)-\alpha f'(x)+\frac{\lambda}{2}f''x-\&c.,$$

it is necessary to show that $f'''(x)$ and the succeeding differential coefficients are not infinite. Now they generally will be infinite in the case where $y=f(x)$ is an infinitesimal Error, as $f(x)$ will be of the form $K\phi\left(\frac{x}{\epsilon}\right)$, where ϵ is infinitesimal; but in the case where

$$y=f(x)=\frac{N}{\theta\sqrt{\pi}}e^{-\frac{x^2}{\theta^2}},$$

we may take θ as small as we please, and yet retain only the three first terms above, because the differential coefficients of y do not here become infinite; in fact it is easy to see that any differential coefficient $\frac{d^ny}{dx^n}$ will consist of a series of terms of the form

$$C\frac{x^p}{\theta^r}e^{-\frac{x^2}{\theta^2}}:$$

now by the rules in the Differential Calculus for evaluating indeterminate forms, this quantity tends to zero as θ diminishes.

- 10 -

SUR

L'INTÉGRATION DES DIFFÉRENTIELLES

QUI CONTIENNENT

UNE RACINE CARRÉE D'UN POLYNÔME DU TROISIÈME OU DU QUATRIÈME DEGRÉ.

PAR

P. TCHÉBYCHEV.

Lu le 20 janvier 1854.

§ 1.

Dans le Mémoire *Sur l'intégration des différentielles irrationnelles*, publié, en 1853, dans le *Journal de Mathématiques pures et appliquées* de M. Liouville, nous avons donné une méthode pour trouver la partie algébrique dans l'expression de l'intégrale $\int \frac{f_0 x}{F_0 x} \frac{dx}{\sqrt{\theta x}}$, en tant qu'elle est possible sous forme finie, et déterminer séparément tous les termes logarithmiques à l'aide de certaines conditions qu'ils doivent vérifier. A présent, nous allons montrer comment on peut trouver, d'après ces conditions, les termes logarithmiques, dans le cas le plus simple et le plus intéressant, savoir: celui où la différentielle contient une racine carrée d'un polynôme du troisième ou du quatrième degré. Faute de méthode générale, on ne connaît que des cas très particuliers, où une pareille différentielle s'intègre sous forme finie; dans plusieurs autres cas, pour lesquels cette intégration a aussi lieu, on n'y parvient qu'en essayant différentes transformations, et, le plus souvent, on renonce à l'idée de chercher l'intégrale après avoir fait beaucoup de tentatives sans succès. Or, d'après nos recherches, citées plus haut, les méthodes particulières et les essais de différentes transformations qu'on emploie dans cette intégration, seront remplacés par une méthode générale et directe dès qu'on sera parvenu à définir les termes logarithmiques dans la valeur de $\int \frac{f_0 x}{F_0 x} \frac{dx}{\sqrt{\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \lambda}}$, d'après les conditions que nous avons trouvées pour leur détermination. C'est ce que nous allons faire ici, en donnant la méthode, d'après laquelle la recherche de ces termes se réduit toujours à cette question résolue par Abel:

«Trouver toutes les différentielles de la forme $\frac{\rho\, dx}{\sqrt{R}}$, où ρ et R sont des fonctions entières «de x, dont les intégrales puissent s'exprimer par une formule de la forme $\log \frac{p + q\sqrt{R}}{p - q\sqrt{R}}$. (*Oeuvres compl. T. I, pag. 33.*)

Cette intégration sera donc due à Abel et par le principe fondamental, d'où nous sommes partis dans nos recherches sur l'intégration des différentielles irrationnelles, et par la méthode de résoudre la question citée, à laquelle se réduit finalement la détermination des termes logarithmiques dans la valeur de $\int \frac{f_0 x}{F_0 x} \frac{dx}{\sqrt{\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \lambda}}$. Ainsi, nos recherches, comme nous nous plaisons à le croire, rempliront, sous un certain rapport, une lacune qui restait entre les

Mémoires de ce grand Géomètre, où il donne la forme générale des intégrales des différentielles algébriques, en tant qu'elles sont possibles sous forme finie, et ceux où il cherche leur valeur, en faisant une hypothèse particulière.

La réduction de nos équations, dont nous venons de parler, est indispensable aussi pour simplifier l'intégration des différentielles plus compliquées. Quant aux différentielles qui ne contiennent sous le signe du radical carré qu'une fonction du premier ou du second degré, cette réduction conduit immédiatement à trouver la partie logarithmique de leurs intégrales. Outre cela, cette réduction est remarquable par différents résultats relatifs à la nature des intégrales qu'on peut en tirer, et cela nous fournit un rapprochement très intéressant de la construction des valeurs irrationnelles avec la règle et le compas, et l'intégration des différentielles sous forme finie. Ainsi on verra que, la somme des nombres n^0, n', n'', ... étant impaire, l'intégrale

$$\left(n^0x+\frac{n'\Delta(a')}{x-a'}+\frac{n''\Delta(a'')}{x-a''}+\ldots+C\right)\frac{dx}{\Delta(x)},$$

où nous avons fait pour abréger $\Delta(x)=\sqrt{x^4+\beta x^3+\gamma x^2+\delta x+\lambda}$, ne peut être exprimée sous forme finie, si, d'après les quantités

$$a',\ a'',\ldots.\ \beta,\ \gamma,\ \delta,\ \lambda,$$

et à l'aide de la règle et du compas, on ne peut construire aucune des racines de l'équation

$$x^4+\beta x^3+\gamma x^2+\delta x+\lambda=0.$$

Par exemple, on reconnaît que les intégrales

$$\int\frac{x+C}{\sqrt{x^4+2x^2-8x+9}}dx,\quad \int\frac{n^0x+\dfrac{3(2n'-n^0-1)}{x}+C}{\sqrt{x^4+2x^2-8x+9}}dx,\quad \int\frac{n^0x+\dfrac{2n'}{x-1}+\dfrac{3(2n''-n^0-n'-1)}{x}+C}{\sqrt{x^4+2x^2-8x+9}}dx,$$

etc., etc., etc......

sont impossibles sous forme finie, parce que, à l'aide de la règle et du compas, on ne peut pas inscrire dans le cercle un polygone régulier de 7 côtés, ce qui est nécessaire pour la construction des racines de l'équation $x^4+2x^2-8x+9=0$.

Il y a d'autres questions de l'Analyse transcendante, où la même méthode de réduction peut être avantageusement employée, savoir, quand on cherche à exprimer la somme des intégrales

$$\int^{x_{\prime}}\frac{f_0x}{\alpha_{\prime}x+\beta_{\prime}}\frac{dx}{\sqrt{\theta x}}+\int^{x_{\prime\prime}}\frac{f_0x}{\alpha_{\prime}x+\beta_{\prime}}\frac{dx}{\sqrt{\theta x}}+\ldots..$$

par une somme d'un nombre déterminé d'intégrales semblables, en y ajoutant une certaine onction algébrique et logarithmique.

Enfin, cette même méthode, appliquée aux nombres, nous donne un procédé à l'aide duquel on trouvera la représentation d'un nombre donné par la forme x^2-ny^2, toutes les fois que ce nombre peut être mis sous cette forme et qu'on connaît la valeur de x, pour laquelle la

forme $x^2 - n$ est divisible par ce nombre. Dans le cas de $n = -1$, cela se réduit à la méthode ingénieuse que M. Hermite a employée pour démontrer que tous les nombres premiers de la forme $4k+1$ sont toujours décomposables en une somme de deux carrés, et pour effectuer en même temps cette décomposition.

§ 2.

Si dans les formules de notre Mémoire, cité plus haut, on fait

$$m = 2,\ \Delta = \sqrt[m]{\theta x} = \sqrt{\theta x},$$

on trouve que l'équation

$$x^m - 1 = 0,$$

dont l'une des racines primitives nous a servi pour composer des nombres complexes, se réduit à $x^2 - 1 = 0$, et comme la racine primitive de cette équation est égale à -1, les nombres complexes que nous avons désignés par

$$\overset{\circ}{M}_i,\ \ M'_i,\ \ M''_i,\ \ldots\ldots$$

deviennent réels et rationnels. De plus, la forme générale des termes logarithmiques

$$A \log [\varphi(\Delta) . \varphi^{\alpha}(\alpha\Delta) . \varphi^{\alpha^2}(\alpha^2\Delta) \ldots . \varphi^{\alpha^{m-1}}(\alpha^{m-1}\Delta)],$$

à cause de $m = 2$, $\Delta = \sqrt{\theta x}$, devient

$$A \log [\varphi(\sqrt{\theta x}) . \varphi^{-1}(-\sqrt{\theta x})] = A \log \frac{\varphi(\sqrt{\theta x})}{\varphi(-\sqrt{\theta x})},$$

et comme φ est une fonction entière, on aura

$$\varphi(\sqrt{\theta x}) = X_0 + X\sqrt{\theta x},\ \ \varphi(-\sqrt{\theta x}) = X_0 - X\sqrt{\theta x},$$

où X_0, X sont des fonctions entières.

Donc, les termes logarithmiques, dans la valeur de l'intégrale $\int \frac{Fx}{fx} \frac{dx}{\sqrt{\theta x}}$, s'écriront ainsi:

$$A \log \frac{X_0 + X\sqrt{\theta x}}{X_0 - X\sqrt{\theta x}}.$$

En cherchant à déterminer ces termes, nous avons trouvé que le coefficient A sera égal à une valeur connue, divisée par un nombre entier inconnu, et si l'on désigne ce nombre par n_i, le degré de la fonction $\frac{X_0 + X\sqrt{\theta x}}{X_0 - X\sqrt{\theta x}}$ sera exprimé par le produit $n_i . \overset{\circ}{M}_i$, où $\overset{\circ}{M}_i$ est une valeur connue. De plus, cette fonction, pour toutes les valeurs finies de x, sera en rapport fini avec la puissance n_i^{me} de la fonction

$$(x - x')^{M'_i} . (x - x'')^{M''_i} . (x - x''')^{M'''_i} \ldots . (x - x^{(\lambda-1)})^{M_i^{(\lambda-1)}} (x - x^{(\lambda-1)});$$

où

$$M'_i,\ M''_i,\ M'''_i,\ \ldots\ldots M_i^{(\lambda-1)},$$

dans le cas que nous examinons, sont réels et rationnels. En passant à la détermination des inconnus n_i, X_0, X, nous remarquons que n_i doit être susceptible de réduire les produits

$$n_i \overset{\circ}{M}_i,\ n_i \overset{\prime}{M}_i,\ n_i \overset{\prime\prime}{M}_i,\ n_i \overset{\prime\prime\prime}{M}_i,\ n_i M_i^{\lambda-1}$$

à des nombres entiers: car le produit $n_i M_i^\circ$ désigne le degré de $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$, qui ne peut être fractionnaire, X_0, X étant des fonctions entières; la même chose a lieu relativement aux produits

$$n_i \overset{\prime}{M}_i,\ n_i \overset{\prime\prime}{M}_i,\ n_i \overset{\prime\prime\prime}{M}_i \ldots\ldots n_i M_i^{\lambda-1},$$

qui sont égaux aux exposants de $x-x'$, $x-x''$, $x-x'''$, ... $x-x^{(\lambda-1)}$ dans les premiers termes du développement de $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$ suivant les puissances croissantes de $x-x'$, $x-x''$, $x-x'''$... ...$x-x^{(\lambda-1)}$. Donc, n_i doit être divisible par le plus petit dénominateur, auquel les quantités

$$\overset{\circ}{M}_i,\ \overset{\prime}{M}_i,\ \overset{\prime\prime}{M}_i \ldots\ldots M_i^{\lambda-1}$$

peuvent être réduites, et par conséquent, si l'on désigne ce dénominateur par σ, et le quotient $n_i:\sigma$ par $+\rho$ ou $-\rho$, on aura

$$n_i = \pm\rho\sigma,$$

où nous prendrons celui des deux signes qui appartient à la valeur de M_i°. D'après cela, $n_i M_i^\circ$, le degré de $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$, sera exprimé par $\pm\sigma M_i^\circ\rho$, où $\pm\sigma M_i^\circ$ se réduira à un nombre entier et positif. En dénotant ce nombre par π, et désignant d'après la notation d'Abel, le degré de $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$ par $\delta\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$, nous aurons, relativement à ρ, cette équation

$$\delta\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}} = \pi\rho.$$

Quant à la fonction qui, pour toutes les valeurs finies de x, reste dans un rapport fini avec la fonction $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$, en vertu de $n = \pm\rho\sigma$, elle se réduit à

$$\left[(x-x')^{\overset{\prime}{M}_i}.(x-x'')^{\overset{\prime\prime}{M}_i}.(x-x''')^{\overset{\prime\prime\prime}{M}_i}\ldots.(x-x^{(\lambda-1)})^{M_i^{\lambda-1}}.\ (x-x^{(\lambda+1)})\right]^{\pm\rho\sigma},$$

et comme les produits $\sigma\overset{\prime}{M}_i$, $\sigma\overset{\prime\prime}{M}_i$, ...$\sigma M_i^{(\lambda-1)}$, d'après la propriété du nombre σ, se réduisent à des nombres entiers, la fonction

$$\left[(x-x')^{\overset{\prime}{M}_i}.(x-x'')^{\overset{\prime\prime}{M}_i}.(x-x''')^{\overset{\prime\prime\prime}{M}_i}\ldots.(x-x^{(\lambda-1)})^{M_i^{\lambda-1}}.\ (x-x^{(\lambda+1)})\right]^{\pm\sigma}$$

ne peut être que rationnelle. Donc, si nous faisons, pour abréger,

$$\left[(x-x')^{\overset{\prime}{M}_i}.(x-x'')^{\overset{\prime\prime}{M}_i}.(x-x''')^{\overset{\prime\prime\prime}{M}_i}\ldots..(x-x^{(\lambda-1)})^{M_i^{\lambda-1}}.\ (x-x^{(\lambda+1)})\right]^{\pm\sigma} = \frac{u}{v};$$

où u, v sont des fonctions entières, et que nous convenions de désigner par la lettre T toutes les

fonctions qui restent finies, tant que x n'est pas infini, la propriété de la fonction $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$ en question, sera exprimée par cette équation

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=T\left(\frac{u}{v}\right)^\rho.$$

C'est d'après cette équation, combinée avec la suivante:

$$\delta\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=\pi\rho,$$

que nous devons chercher le nombre ρ et les fonctions X_0 et X.

Ces équations seront le plus souvent très compliquées à cause du degré élevé des fonctions u et v, et de la valeur considérable de π. Or, nous allons montrer qu'on peut les réduire à la forme, où le degré de uv, plus le nombre π, sera au-dessous du degré de $\sqrt{\theta x}$.

§ 3.

Il n'est pas difficile de s'assurer, que θ_1, θ_2 étant deux fonctions entières dont le produit est égal à θx, et p et q des fonctions entières quelconques, on peut mettre la fonction cherchée $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$ sous la forme

$$\left(\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}\right)^\rho \frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}},$$

en choisissant convenablement les fonctions entières P_0 et Q_0. En effet, le quotient

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}:\left(\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}\right)^\rho$$

se réduit à

$$\frac{(X_0+X\sqrt{\theta x})(p\sqrt{\theta_1}-q\sqrt{\theta_2})^\rho}{(X_0-X\sqrt{\theta x})(p\sqrt{\theta_1}+q\sqrt{\theta_2})^\rho}=\frac{(X_0+X\sqrt{\theta x})(p\theta_1-q\sqrt{\theta x})^\rho}{(X_0-X\sqrt{\theta x})(p\theta_1+q\sqrt{\theta x})^\rho},$$

expression qu'on peut mettre sous la forme $\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}$, en dénotant par P_0 la partie rationnelle du produit $(X_0+X\sqrt{\theta x})(p\theta_1-q\sqrt{\theta x})^\rho$, et par $Q_0\sqrt{\theta x}$ celle qui a pour facteur $\sqrt{\theta x}$.

Mais, si l'on substitue dans les équations

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=T\left(\frac{u}{v}\right)^\rho,\quad \delta\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=\pi\rho \;\ldots\ldots\ldots\; \text{(I)}$$

le produit $\left(\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}\right)^\rho\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}$ à la place de $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$, elles se réduisent à celles-ci

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=T\left(\frac{u}{v}\cdot\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{p\sqrt{\theta_1}+q\sqrt{\theta_2}}\right)^\rho,$$

$$\delta\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=\left(\pi-\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}\right)\rho,$$

et si les fonctions p et q sont choisies de manière à ce qu'elles vérifient les équations

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}=T\,\frac{u}{v}\cdot\frac{v'}{u'};\quad \delta\,\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}+q\sqrt{\theta_2}}=\pi_1,$$

les équations qui déterminent les nouvelles inconnues P_0 et Q_0 deviennent

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=T\left(\frac{u'}{v'}\right)^{\rho},\quad \delta\,\frac{P_0+Q_0\sqrt{\theta x}}{P_0+Q_0\sqrt{\theta x}}=(\pi-\pi_1)\rho. \;.\;.\;.\;.\;.\;.\; (2)$$

Ces équations seront plus ou moins simples selon les valeurs de p et q, qu'on emploiera dans la réduction dont nous venons de parler. Or, nous allons montrer que, dans les équations réduites (2), la somme du degré de $u'v'$ et de la valeur numérique de $\pi-\pi_1$ sera au-dessous du degré de $\sqrt{\theta x}$, si l'on prend pour p et q des fonctions qu'on trouve de la manière suivante:

1) On cherche une fonction entière S, pour laquelle les fractions $\frac{S\sqrt{\theta_1}+\sqrt{\theta_2}}{u}$, $\frac{S\sqrt{\theta_1}-\sqrt{\theta_2}}{v}$ ne deviennent pas infinies, tant que x reste fini.

2) On développe $\frac{S-\frac{\sqrt{\theta_2}}{\sqrt{\theta_1}}}{uv}$ en fraction continue, et parmi les fractions réduites on trouve une fraction dont le dénominateur est d'un degré moins élevé que $\sqrt{\frac{uv\theta_1.x^{\pi}}{\sqrt{\theta x}}}$; mais qui est suivie d'une fraction dont le dénominateur est d'un degré plus élevé que $\sqrt{\frac{uv\theta_1.x^{\pi}}{\sqrt{\theta x}}}$.

3) En dénotant cette fraction par $\frac{M}{N}$, on prend

$$q=N,\quad p=SN-Muv. \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (3)$$

En effet, d'après les équations (3) et la propriété de la fonction S, on voit que les expressions

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u},\quad \frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}$$

restent finies pour toutes les valeurs finies de x. Donc, si l'on dénote par

$$\alpha,\ \alpha_1,\ \alpha_2,\ \ldots\ldots,\ \beta,\ \beta_1,\ \beta_2,\ \ldots\ldots,$$

les valeurs de x qui rendent ces expressions égales à zéro, et par

$$f,\ f_1,\ f_2,\ \ldots\ldots,\ g,\ g_1,\ g_2,\ \ldots\ldots,$$

les exposants de

$$x-\alpha,\ x-\alpha_1,\ x-\alpha_2,\ \ldots\ldots,$$

$$x-\beta,\ x-\beta_1,\ x-\beta_2,\ \ldots\ldots$$

dans leur développement suivant les puissances croissantes de ces différences, on aura

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}=T_1(x-\alpha)^f(x-\alpha_1)^{f_1}(x-\alpha_2)^{f_2}\ldots\ldots;$$

$$\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}=T_2(x-\beta)^g(x-\beta_1)^{g_1}(x-\beta_2)^{g_2}\ldots\ldots;$$

où T_1, T_2 désignent des fonctions qui restent finies pour toutes les valeurs finies de x.

Mais, comme les exposants de $x-\alpha$, $x-\alpha_1$, $x-\alpha_2$, ... $x-\beta$, $x-\beta_1$, $x-\beta_2$, ... dans le développement de $\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}$, $\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}$ ne peuvent contenir d'autres fractions que $\frac{1}{2}$, ces équations se réduiront à cette forme

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}=T_1v'\sqrt{w},\quad \frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}=T_2u'\sqrt{w'},\quad \ldots\ldots (4)$$

où u', v', w, w', sont des fonctions entières, dont les deux dernières ne contiennent que des facteurs simples. Par la multiplication de ces équations nous trouvons

$$\frac{p^2\theta_1-q^2\theta_2}{uv}=T_1T_2u'v'\sqrt{ww'},$$

et par conséquent,

$$\frac{p^2\theta_1-q^2\theta_2}{uvu'v'\sqrt{ww'}}=T_1T_2.$$

Cette équation prouve, évidemment, que T_1T_2 est une constante; car, d'après la propriété des fonctions T_1, T_2, leur produit ne devient ni zéro ni infini pour x fini, tandis que cette équation montre que le carré de T_1T_2 est une fraction rationnelle $\frac{(p^2\theta_1-q^2\theta_2)^2}{(uvu'v')^2ww'}$ qui ne peut rester finie pour toutes les valeurs finies de x, à moins qu'elle ne se réduise à une constante. Donc

$$T_1T_2=C,$$

et par conséquent, l'équation précédente devient

$$\frac{p^2\theta_1-q^2\theta_2}{uvu'v'\sqrt{ww'}}=C. \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Or cette égalité suppose que ww' est un carré parfait, et comme les fonctions w, w' n'ont que des facteurs simples, cela ne peut avoir lieu à moins qu'on n'ait

$$w=w'. \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

D'après cela, en divisant les équations (4) l'une par l'autre, on trouve

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}=T\frac{u}{v}\cdot\frac{v'}{u'},$$

en mettant, pour abréger, T à la place de $\frac{T_1}{T_2}$.

Il nous reste maintenant à prouver que si l'on fait

$$\pi_1=\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}},$$

la somme de $\delta(u'v')$ avec la valeur numérique de $\pi-\pi_1$ sera au-dessous de $\delta\sqrt{\theta x}$. Or, selon que $\pi-\pi_1$ est positif ou négatif, cette somme sera égale à $\delta(u'v')+\pi-\pi_1$ ou à $\delta(u'v')-\pi+\pi_1$. Nous allons montrer que ces deux quantités sont effectivement plus petites que $\delta\sqrt{\theta x}$, tant que p et q sont déterminés comme nous l'avons dit.

Pour s'en assurer, nous remarquons que d'après la substitution de δx^{π} et $\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}+q\sqrt{\theta_2}}$ à la place de π et π_1, ces quantités deviennent

$$\delta(u'v')+\delta\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{p\sqrt{\theta_1}+q\sqrt{\theta_2}}x^{\pi}, \quad \delta(u'v')+\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}x^{-\pi}.$$

Mais, d'après l'équation (5), nous trouvons

$$\delta(u'v') \overline{\overline{<}}\, \delta\frac{p^2\theta_1-q^2\theta_2}{uv}.$$

Donc, les quantités précédentes sont égales ou inférieures à celles-ci

$$\delta\frac{p^2\theta_1-q^2\theta_2}{uv}+\delta\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{p\sqrt{\theta_1}+q\sqrt{\theta_2}}x^{\pi}=2\delta\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{\sqrt{uv}}x^{\frac{\pi}{2}},$$

$$\delta\frac{p^2\theta_1-q^2\theta_2}{uv}+\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}x^{-\pi}=2\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}.$$

Mais la première de ces quantités, par la substitution des valeurs de p et q d'après (3), devient

$$2\delta\frac{(SN-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}}{\sqrt{uv}}x^{\frac{\pi}{2}}=\delta\sqrt{\theta x}+2\delta\left(\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}-\frac{M}{N}\right)N\sqrt{\frac{uv\theta_1 . x^{\pi}}{\sqrt{\theta x}}},$$

quantité qui est au-dessous de $\delta\sqrt{\theta x}$, tant que $\frac{M}{N}$, dans la série des fractions réduites de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$, est suivie par une fraction dont le dénominateur est d'un degré plus élevé que $\sqrt{\frac{uv\theta_1 . x^{\pi}}{\sqrt{\theta x}}}$. Quant à la seconde quantité, nous remarquons qu'elle peut être mise sous cette forme

$$2\delta\left[\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}+\frac{2q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}\right],$$

et par conséquent, qu'elle ne surpasse pas au moins l'une de ces deux valeurs

$$2\delta\left[\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}\right]=2\delta\left[\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{\sqrt{uv}}x^{\frac{\pi}{2}}\right]-2\pi,$$

$$2\delta\frac{q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}=\delta\sqrt{\theta x}-2\delta\cdot\frac{1}{q}\sqrt{\frac{uv\theta_1 . x^{\pi}}{\sqrt{\theta x}}}.$$

Mais, comme nous venons de trouver que $2\delta\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{\sqrt{uv}}x^{\frac{\pi}{2}}$ est plus petit que $\delta\sqrt{\theta x}$, et que nous avons pris $q=N$, d'un degré moins élevé que $\sqrt{\frac{uv\theta_1 . x^{\pi}}{\sqrt{\theta x}}}$, il s'en suit que ces deux

quantités sont au-dessous de $\delta\sqrt{\theta x}$. Ainsi l'on parvient à s'assurer que les valeurs de p et q, déterminées d'après la méthode énoncée, sont effectivement susceptibles par la substitution

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=\left(\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}\right)^\rho \frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}},$$

de réduire les équations

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=T\left(\frac{u}{v}\right)^\rho, \quad \delta\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=\pi\rho,$$

à ces autres

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=T\left(\frac{u'}{v'}\right)^\rho; \quad \delta\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=(\pi-\pi_1)\rho;$$

où la somme du degré de $u'v'$, plus la valeur numérique de $\pi-\pi_1$, est au-dessous du degré de $\sqrt{\theta x}$.

Nous montrerons maintenant, que cette réduction sera toujours possible, tant que les équations primitives elles mêmes ne remplissent pas la condition

$$\delta(uv)+\pi<\delta\sqrt{\theta x}.$$

Il est facile de remarquer que la détermination de p et q, dont nous venons de parler, ne suppose que l'existence de deux fractions réduites de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$ telles que l'une ait pour dénominateur une fonction d'un degré moins élevé que $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$, tandis que la suivante a le dénominateur d'un degré plus élevé que $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$.

Or nous verrons, que cela aura toujours lieu, tant que la condition $\delta(uv)+\pi<\delta\sqrt{\theta x}$ n'est pas remplie, et que l'on décompose convenablement la fonction θx en deux facteurs $\theta_1 . \theta_2$; savoir: de manière que $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$ soit d'un degré fractionnaire. En effet, dans ces suppositions, le degré de $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$ est au-dessus de zéro, et par conséquent, si l'on commence la série des fractions réduites de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$ par $\frac{0}{1}$, où le dénominateur est du degré zéro, on est sûr de trouver parmi elles au moins une fraction dont le dénominateur soit d'un degré moins élevé que $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$.

Mais alors, dans la série infinie des fractions réduites de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{\sqrt{uv}}$, on trouvera nécessairement deux fractions consécutives telles que l'une a pour dénominateur une fonction d'un degré moins élevé que $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$, tandis que le dénominateur de l'autre est d'un degré plus élevé que $\sqrt{\frac{uv\theta_2 . x^\pi}{\sqrt{\theta x}}}$, si toutefois aucune des fractions réduites de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$ n'a son dénominateur du

même degré que $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$. Or cela n'aura pas lieu, tant que cette fonction est d'un degré fractionnaire; car, pour θx de degré pair, toutes les fractions réduites de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}=\frac{s-\frac{\theta_2}{\sqrt{\theta x}}}{uv}$ ne contiennent que les puissances entières de x, et pour θx de degré impair, le degré de $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$ a la forme $k \pm \frac{1}{4}$, tandis que les degrés fractionnaires de x, dans la fonction $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$, sont de la forme $k+\frac{1}{2}$.

Nous remarquerons encore que, dans la série des fractions réduites de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}=\frac{s-\frac{\theta_2}{\sqrt{\theta x}}}{uv}$, on ne rencontrera des puissances fractionnaires de x qu'après la fraction $\frac{M}{N}$, qui sert pour trouver les fonctions p et q. En effet, les puissances fractionnaires de x ne peuvent y entrer que dans le cas où θx est de degré impair. Mais alors toutes les fonctions de la forme $\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}$ sont évidemment du degré 0, et par conséquent, $\pi=0$. Or, π étant égal à zéro, d'après ce que nous venons de dire sur la détermination de p et q, le dénominateur N sera d'un degré moins élevé que $\sqrt{\frac{uv\theta_1}{\sqrt{\theta x}}}$, et avec un tel dénominateur la fraction réduite ne donne, en général, la fonction, d'où elle résulte par le développement en fraction continue, qu'avec une exactitude jusqu'aux quantités de l'ordre plus élevé que $\frac{1}{\left(\sqrt{\frac{uv\theta_1}{\sqrt{\theta x}}}\right)^2}=\frac{\sqrt{\theta x}}{\theta_1}\frac{1}{uv}=\frac{1}{uv}\sqrt{\frac{\theta_2}{\theta_1}}$.

Mais la partie irrationnelle de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$ est justement de cet ordre.

Donc, dans ce cas, cette partie n'a aucune influence sur la fraction $\frac{M}{N}$, de manière qu'on peut la supprimer dans la formule $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$, et chercher $\frac{M}{N}$ par le développement seulement de $\frac{s}{uv}$ en fraction continue.

§ 4.

Nous allons montrer maintenant le parti que l'on peut tirer de la réduction, qui vient d'être exposée, pour la solution des équations

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=T\left(\frac{u}{v}\right)^\rho, \quad \delta\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=\pi\rho,$$

dans le cas, où θx est du 3^me ou du 4^me degré. Après avoir trouvé les fonctions p et q, comme nous l'avons dit, et si l'on fait

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=\left(\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}\right)^{\rho}\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}},$$

on parvient à ces équations

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=T\left(\frac{u'}{v'}\right)^{\rho}, \quad \delta\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=(\pi-\pi_1)\rho.$$

Pour trouver la fonction $\frac{u'}{v'}$, on divisera $p^2\theta_1-q^2\theta_2$ par uv. D'après la méthode qui nous a servi pour trouver les fonctions p et q, il est clair que le quotient de cette division sera d'un degré moins élevé que $\sqrt{\theta x}$, et par conséquent, dans le cas de θx du 3^{me} ou du 4^{me} degré, ce quotient sera, en général, représenté par $ax+b$. Mais, d'après les équations (5, 6), ce quotient, à un facteur constant près, est égal à $u'v'w$. Donc, l'une des trois fonctions

$$u',\ v',\ w$$

sera égale à $ax+b$, et les autres se réduiront à des constantes, et par conséquent, l'on sera conduit à l'un de ces trois cas

$$\frac{u'}{v'}=\frac{1}{ax+b}, \quad \frac{u'}{v'}=ax+b, \quad \frac{u'}{v'}=\text{à une constante.}$$

Mais en faisant $x=-\frac{b}{a}$ dans les équations (4), où d'après (6) $w'=w$, on voit que le premier cas aura lieu, si cette valeur de x rend

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}=0,$$

le second, si l'on a

$$\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}=0,$$

et enfin le troisième, si, pour $x=-\frac{b}{a}$, on trouve en même temps

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}=0, \quad \frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}=0.$$

Donc, si nous convenons de désigner par ε une valeur qui se réduit à

$$+1, \quad -1, \quad 0,$$

selon que, pour $x=-\frac{b}{a}$, on trouve

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}=0,$$

$$\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}=0,$$

ou, en même temps,

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}=0, \quad \frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}=0,$$

la valeur de $\frac{u'}{v'}$ sera donnée par cette équation

$$\frac{u'}{v'}=\frac{1}{(ax+b)^{\varepsilon}}.$$

D'après cela, les équations qui déterminent P_0, Q_0 et ρ deviennent

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=\frac{T}{(ax+b)^{\varepsilon\rho}}, \quad \delta\,\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=(\pi-\pi_1)\rho. \quad \ldots\ldots (7)$$

Dans le cas, où a ne se réduit pas à zéro, on peut mettre ces équations sous une forme plus simple, en introduisant à la place de x une nouvelle variable z d'après l'équation

$$ax+b=\frac{a}{z}.$$

En effet, si l'on traite la valeur de $\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}$ comme fonction de cette nouvelle variable, on parvient facilement à reconnaître, que, d'après les équations précédentes, la fonction $\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}$ en z, sera déterminée par ces propriétés:

1) Elle reste finie, tant que z est fini et diffère de 0; car ces valeurs de z correspondent à celles de x différentes de $-\frac{b}{a}$ et finies.

2) Pour $z=0$, la limite du rapport $\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}:z^{(\pi-\pi_1)\rho}$ reste finie; car ce rapport n'est lui même que la limite de la valeur de $\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}:x^{(\pi-\pi_1)\rho}$ pour $x=\infty$.

3) Pour $z=\infty$, le rapport $\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}:z^{\varepsilon\rho}$ reste fini, car ce rapport est égal à $\frac{T}{a^{\varepsilon\rho}}$, quand on fait $x=-\frac{b}{a}$.

Donc, en faisant $ax+b=\frac{a}{z}$, on peut remplacer les équations (7) par celles-ci

$$\frac{P_0+Q_0\sqrt{\theta\left(\frac{a-bz}{az}\right)}}{P_0-Q_0\sqrt{\theta\left(\frac{a-bz}{az}\right)}}=Tz^{(\pi-\pi_1)\rho}, \quad \delta\,\frac{P_0+Q_0\sqrt{\theta\left(\frac{a-bz}{az}\right)}}{P_0-Q_0\sqrt{\theta\left(\frac{a-bz}{az}\right)}}=\varepsilon\rho.$$

Mais il n'est pas difficile de s'assurer, que a étant différent de zéro, on aura

$$\pi=\pi_1.$$

En effet, comme $\pi_1=\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}$, on peut mettre les différences $\pi-\pi_1$, $\pi_1-\pi$ sous ces formes

$$\delta\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{p\sqrt{\theta_1}+q\sqrt{\theta_2}}x^{\pi}=2\delta\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{\sqrt{uv}}x^{\frac{\pi}{2}}-\delta\frac{p^2\theta_1-q^2\theta_2}{uv},$$

$$\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{p\sqrt{\theta_1}-q\sqrt{\theta_2}}x^{-\pi}=2\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}-\delta\frac{p^2\theta_1-q^2\theta_2}{uv}.$$

Donc, si le coefficient a dans la valeur de $\frac{p^2\theta_1-q^2\theta_2}{uv}=ax+b$ n'est pas égal à zéro, les différences

$$\pi-\pi_1,\quad \pi_1-\pi$$

sont respectivement égales à

$$2\delta\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{\sqrt{uv}}x^{\frac{\pi}{2}}-1,\quad 2\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}-1.$$

Mais, d'après le § 3, on a

$$2\delta\frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{\sqrt{uv}}x^{\frac{\pi}{2}}<\delta\sqrt{\theta x},\quad 2\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}<\delta\sqrt{\theta x},$$

et comme θx n'est que du 4^{me} ou du 3^{me} degré, cela prouve que les différences $\pi-\pi_1$, $\pi_1-\pi$ sont au-dessous de 1, ce qui ne peut être à moins qu'on n'ait $\pi=\pi_1$. D'après cela, les équations qui déterminent P_0 et Q_0, en fonctions de z, deviennent

$$\frac{P_0+Q_0\sqrt{\theta\left(\frac{a-bz}{az}\right)}}{P_0-Q_0\sqrt{\theta\left(\frac{a-bz}{az}\right)}}=T,\quad \delta\frac{P_0+Q_0\sqrt{\theta\left(\frac{a-bz}{az}\right)}}{P_0-Q_0\sqrt{\theta\left(\frac{a-bz}{az}\right)}}=\varepsilon\varrho,$$

formules que nous mettrons sous la forme

$$\frac{P_0z^2+Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{P_0z^2-Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}=T,\quad \delta\frac{P_0z^2+Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{P_0z^2-Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}=\varepsilon\varrho,\quad \ldots \quad (8)$$

pour délivrer la fonction radiçale des puissances négatives de z.

Or, la première de ces équations ne diffère que par la forme de celle, qu'Abel a traitée dans son Mémoire «*Sur l'intégration de la formule différentielle $\frac{\varrho dx}{\sqrt{R}}$, R et ϱ étant des fonctions entières*», et d'après les recherches ingénieuses de ce grand Géomètre nous savons que cette équation est impossible, sauf le cas de $P_0=0$, ou $Q_0=0$, si la fraction continue résultante de $\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}$ n'est pas périodique, et dans le cas contraire, si l'on a

$$\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)} = r + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots + \cfrac{1}{r_2 + \cfrac{1}{r_1 + \cfrac{1}{2r + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots}}}}}}}$$

on vérifiera cette équation en prenant

$$\frac{P_0 z^2}{Q_0} = r + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots + \cfrac{1}{r_2 + \cfrac{1}{r_1}}}}$$

Quant à l'équation

$$\delta\, \frac{P_0 z^2 + Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{P_0 z^2 - Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}} = \varepsilon\rho,$$

on la vérifiera, en choisissant convenablement ρ, savoir en prenant

$$\rho = \frac{1}{\varepsilon}\delta\, \frac{P_0 z^2 + Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{P_0 z^2 - Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}} = \frac{1}{\varepsilon}\delta\, \frac{\frac{P_0 z^2}{Q_0} + \sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{\frac{P_0 z^2}{Q_0} - \sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}.$$

Donc, si l'on fait, pour abréger,

$$\varphi(z) = r + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots + \cfrac{1}{r_2 + \cfrac{1}{r_1}}}}$$

la valeur cherchée de $\frac{P_0 + Q_0\sqrt{\theta x}}{P_0 - Q_0\sqrt{\theta x}}$, en fonction de z, sera

$$\frac{\varphi(z) + \sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{\varphi(z) - \sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}},$$

et d'après l'équation $ax+b=\frac{a}{z}$, nous aurons, en fonction de x,

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=\frac{\left(x+\frac{b}{a}\right)^2\varphi\left(\frac{a}{ax+b}\right)+\sqrt{\theta x}}{\left(x+\frac{b}{a}\right)^2\varphi\left(\frac{a}{ax+b}\right)-\sqrt{\theta x}}. \quad \ldots\ldots\ldots\ldots (9)$$

Quant au nombre ρ, on le trouvera d'après l'équation

$$\rho=\frac{1}{\varepsilon}\delta\,\frac{\varphi(z)+\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{\varphi(z)-\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}.$$

Cette valeur de ρ nous montre que la solution des équations (8), que nous venons de trouver, ne peut être employée que dans le cas, où ε ne se réduit pas à zéro; car, pour $\varepsilon=0$, cette valeur de ρ devient infinie, tandis que ρ désigne chez nous un nombre fini. Mais dans ce cas on vérifiera, évidemment, nos équations par une valeur finie de ρ, en prenant une des solutions de l'équation

$$\frac{P_0z^2+Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{P_0z^2-Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}=T,$$

que nous avons exclues, savoir: $Q_0=0$ ou $P_0=0$, ce qui donne

$$\frac{P_0z^2+Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{P_0z^2-Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}=\pm1, \qquad \delta\,\frac{P_0z^2+Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{P_0z^2-Q_0\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}=\varepsilon\rho=0.$$

Dans ces solutions, pour $\varepsilon=0$, le nombre ρ reste arbitraire, et l'on pourra prendre $\rho=1$. Remarquons que ces solutions qu'on pouvait aussi tirer de la formule (9), en prenant $\varphi(z)$ égale à 0 ou ∞, ne pourront être employées, à leur tour, que dans le cas de $\varepsilon=0$, car, autrement, ρ serait égal à 0, tandis que ce nombre doit être différent de zéro.

Ainsi l'on trouve la fonction $\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}$ et le nombre ρ, si le quotient de la division de $p^2\theta_1-q^2\theta_2$ par uv se réduit à $ax+b$, et que a ne soit pas égal à zéro. Mais s'il arrive que $a=0$, les fonctions u', v', d'après ce que nous venons de dire relativement à leur détermination, se réduisent à des constantes, et par conséquent, les équations qui déterminent P_0, Q_0 et ρ deviennent

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=T, \qquad \delta\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=(\pi-\pi_1)\rho.$$

Or, comme ces équations sont de même nature que les équations (8), et que seulement

ici P_0, Q_0, $\sqrt{\theta x}$, $\pi-\pi_1$ remplacent $P_0 z^2$, Q_0, $\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}$, ε, nous concluons, d'après les formules précédentes, que la solution de ces équations sera donnée par ces formules

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0+Q_0\sqrt{\theta x}}=\frac{\varphi_0(x)+\sqrt{\theta x}}{\varphi_0(x)-\sqrt{\theta x}}, \qquad \rho=\frac{1}{\pi-\pi_1}\delta\frac{\varphi_0(x)+\sqrt{\theta x}}{\varphi_0(x)-\sqrt{\theta x}},$$

où l'on prendra pour $\varphi_0(x)$ zéro ou l'infini, si

$$\pi-\pi'=0,$$

et dans le cas contraire, on développera $\sqrt{\theta x}$ en fraction continue

$$r+\cfrac{1}{r_1+\cfrac{1}{r_2+\cdots+\cfrac{1}{r_2+\cfrac{1}{r_1+\cfrac{1}{2r+\cfrac{1}{r_1+\cdots}}}}}}$$

et l'on prendra

$$\varphi_0(x)=r+\cfrac{1}{r_1+\cfrac{1}{r_2+\cdots+\cfrac{1}{r_2+\cfrac{1}{r_1}}}}$$

Nous remarquerons encore que si les équations primitives

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=T\left(\frac{u}{v}\right)^{\rho}, \qquad \delta\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=\pi\rho$$

remplissent elles mêmes la condition

$$\delta(uv)+\pi<\sqrt{\theta x},$$

on trouvera leur solution au moyen des formules que nous venons de donner pour résoudre les équations réduites

$$\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=T\left(\frac{u'}{v'}\right)^{\rho}, \qquad \delta\frac{P_0+Q_0\sqrt{\theta x}}{P_0-Q_0\sqrt{\theta x}}=(\pi-\pi_1)\rho.$$

Dans ce cas, on prendra π au lieu de $\pi-\pi_1$, et l'on trouvera a, b, ε, en égalant

$$\frac{u}{v}=\frac{1}{(ax+b)^{\varepsilon}}.$$

§ 5.

D'après ce que nous venons de donner sur la solution des équations

$$\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=T\left(\frac{u}{v}\right)^\rho, \quad \delta\frac{X_0+X\sqrt{\theta x}}{X_0-X\sqrt{\theta x}}=\pi\rho, \ldots \ldots \ldots (10)$$

on peut prouver qu'elles sont impossibles, si, θx étant du quatième degré et $\frac{uvx^\pi}{\sqrt{\theta x}}$ de degré impair, l'équation $\theta x=0$ n'est pas vérifiée, en prenant pour x une valeur composée des racines de l'équation $uv=0$ et des coefficients de θx, à l'aide des seuls radicaux carrés, et que, par conséquent, on ne peut pas exprimer en termes finis toutes les intégrales, dont la détermination se réduit aux équations (10) de cette catégorie.

Pour le démontrer, nous remarquerons d'abord que, dans le cas où $\frac{uvw^\pi}{\sqrt{\theta x}}$ est de degré impair, on peut exécuter la réduction des équations (10), d'après le § 3, en prenant cette décomposition de θx en deux facteurs θ_1, θ_2:

$$\theta_1=1, \ \theta_2=\theta x,$$

et si, avec ces valeurs de θ_1, θ_2, et en supposant connues les racines de l'équation $uv=0$, on fait la réduction des équations (10), et qu'on cherche leur solution, on ne rencontre que l'extraction des racines carrées et les différentes opérations rationnelles. Donc, dans toute cette analyse, on n'aura que des quantités qui ne peuvent vérifier l'équation $\theta x=0$ dans le cas que nous examinons. Or nous allons prouver que tant que cela a lieu, on ne peut donner une solution des équations (10).

D'après le § précédent, dans la solution des équations (10) on ne peut se passer du développement de $\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}$ ou $\sqrt{\theta x}$ en fraction continue, que dans le cas, où l'on a $\varepsilon=0$, ou $\pi-\pi_1=0$, $a=0$.

Mais nous savons (voyez § 4) que la quantité ε ne se réduit à zéro, que dans le cas, où la valeur $x=-\frac{b}{a}$ vérifie ces deux équations

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}=0, \quad \frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v}=0,$$

et, comme le produit $\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}.\frac{p\sqrt{\theta_1}-p\sqrt{\theta_2}}{v}$ est égal à $\frac{p^2\theta_1-p^2\theta_1}{uv}=ax+b$, cela suppose que le développement de

$$\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{u}, \quad \frac{p\sqrt{\theta_1}-q\sqrt{\theta_2}}{v},$$

suivant les puissances de $x+\frac{b}{a}$, contient des exposants fractionnaires, ce qui ne peut avoir

lieu, à moins que θ_1 ou θ_2 ne contienne le facteur $x+\frac{b}{a}$, et par conséquent, cela suppose que la valeur $x=-\frac{b}{a}$ vérifie l'équation $\theta_1\theta_2=\theta x=0$, ce qui ne peut être admis, comme nous l'avons remarqué.

Le cas de $a=0$, $\pi-\pi_1=0$ ne peut avoir lieu, car nous avons trouvé, dans le § précédent,

$$\pi_1-\pi=2\delta\frac{p\sqrt{\theta_1}+q\sqrt{\theta_2}}{\sqrt{uv}}x^{-\frac{\pi}{2}}-\delta\frac{p^2\theta_1-q^2\theta_2}{uv},$$

et comme

$$\frac{p^2\theta_1-q^2\theta_2}{uv}=ax+b,$$

$$\theta_1=1, \qquad \theta_2=\theta x,$$

cela nous donne

$$\pi_1-\pi=2\delta\frac{p+q\sqrt{\theta x}}{\sqrt{uv}}x^{-\frac{\pi}{2}}-\delta(ax+b)=2\delta\frac{p+q\sqrt{\theta x}}{\sqrt[4]{\theta x}}-\delta\frac{uv.x^{\pi}}{\sqrt{\theta x}}-\delta(ax+b).$$

Mais dans le cas que nous examinons, la fonction $\frac{uvx^{\pi}}{\sqrt{\theta x}}$ est de degré impair et la fonction $\frac{p+q\sqrt{\theta x}}{\sqrt[4]{\theta x}}$ d'un degré entier; donc, si $a=0$, la différence $\pi_1-\pi$ est de la forme $2k+1$; et, par conséquent, ne peut se réduire à zéro.

Il nous reste maintenant à prouver qu'on ne parviendra pas non plus à la solution de nos équations par le développement de $\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}$ ou $\sqrt{\theta x}$ en fraction continue. Pour cela, nous allons montrer qu'en général, si aucune des racines de l'équation bicarrée $R=0$ ne peut être exprimée à l'aide des seuls radicaux carrés, la fraction continue, résultante de $\sqrt{R}$, ne peut être périodique, de la forme

$$r+\cfrac{1}{r_1+\cfrac{1}{r_2+\ddots+\cfrac{1}{r_2+\cfrac{1}{r_1+\cfrac{1}{2r+\cfrac{1}{r_1+\ddots}}}}}}$$

En effet, si cela avait lieu, nous savons, par les recherches d'Abel, qu'on parviendrait, par ce développement de $\sqrt{R}$, à la solution de l'équation

$$Y_0^2-Y^2R=C,$$

§ 6.

En terminant notre Mémoire, nous allons faire le résumé des procédés qui, d'après ce que nous venons d'exposer, constituent, avec nos recherches, citées plus haut, une méthode générale d'intégration des différentielles qui contiennent une racine carrée d'un polynome du 3^{me} ou du 4^{me} degré, en tant que cette intégration est possible sous forme finie.

Nous supposons que préalablement la partie rationnelle de ces différentielles a été séparée, et que le reste a été mis sous la forme $\frac{f_0 x}{F_0 x}\frac{dx}{\sqrt{\theta x}}$, où $f_0 x$, $F_0 x$ n'ont point de commun diviseur; nous supposons aussi que θx, qui n'est que du 3^{me} ou du 4^{me} degré, n'a pas de facteurs multiples; car, autrement, l'intégration de $\frac{f_0 x}{F_0 x}\frac{dx}{\sqrt{\theta x}}$ deviendrait très simple.

Pour trouver l'intégrale $\int\frac{f_0 x}{F_0 x}\frac{dx}{\sqrt{\theta x}}$ sous forme finie, en tant que cela est possible, on procédera de la manière suivante:

1) On cherchera le plus grand commun diviseur entre les fonctions $F_0 x\,\theta x$ et $\frac{d[F_0 x.\theta x]}{dx}$. Nous dénoterons ce diviseur par Q.

2) On déterminera les degrés des fonctions $\frac{Q.f_0 x}{F_0 x.\theta x}$, $\frac{Q}{x\sqrt{\theta x}}$. Si ces fonctions sont de degrés inférieurs à -1, le terme algébrique dans l'expression de l'intégrale cherchée est zéro. Dans le cas contraire, on prendra n égal au plus petit nombre entier supérieur aux degrés de ces fonctions, et on cherchera les coefficients du polynome

$$P = B_n x^n + B_{n-1}x^{n-1} + \ldots\ldots + B_2 x^2 + B_1 x + B_0$$

d'après cette condition:

«la fonction $f_0 x - \frac{F_0 x\,\theta x}{Q}\frac{dP}{dx} - \left(\frac{F_0 x\frac{d\theta x}{dx}}{2Q} - \frac{F_0 x.\theta x.\frac{dQ}{dx}}{Q^2}\right)P$, étant divisée par Q, donne zéro pour «reste et pour quotient une fonction d'un degré qui n'est pas plus élevé que celui de $\frac{F_0 x\sqrt{\theta x}}{xQ}$».

Si cette condition ne peut être remplie, on conclura, tout de suite, que l'intégrale cherchée est impossible sous forme finie. Dans le cas contraire, on trouvera les coefficients du polynome P, et l'on conclura que la partie algébrique dans l'intégrale cherchée a pour valeur $\frac{P}{Q}\sqrt{\theta x}$.

3) On mettra la fonction $\frac{f_0 x}{F_0 x} - \sqrt{\theta x}\,\frac{d\frac{P}{Q}\sqrt{\theta x}}{dx}$ sous la forme $\frac{fx}{Fx}$, où fx, Fx sont des fonctions entières qui n'ont point de commun diviseur; on cherchera les racines de l'équation $Fx = 0$, et l'on calculera des quantités K^0, K', K'', K''', ... $K^{(l)}$ d'après les équations

$$K^0 = \left[\frac{xfx}{Fx\sqrt{\theta x}}\right]_{x=\infty},\quad K' = \frac{f(x')}{F'(x')\sqrt{\theta(x')}},\quad K'' = \frac{f(x'')}{F'(x'')\sqrt{\theta(x'')}},\ \ldots . K^{(l)} = \frac{f(x^l)}{F'(x^l)\sqrt{\theta(x^l)}};$$

où x', x'', ... $x^{(l)}$ désignent les racines de l'équation $Fx = 0$ et $F'x = \frac{dFx}{dx}$.

4) On cherchera les nombres entiers M°, M', M'', ... $M^{(l)}$ qui rendent

$$M^\circ K^\circ + M'K' + M''K'' + \ldots\ldots + M^{(l)}K^{(l)} = 0.$$

Soient λ le nombre de toutes les équations de cette forme qui ne sont pas identiques entre elles par rapport à

$$K^\circ, \; K', \; K'', \; \ldots\ldots \; K^{(l)},$$

et

$$K^\circ = \sum_{i=0}^{i=l-\lambda} M_i^\circ K^{(\lambda+i)}, \quad K' = \sum_{i=0}^{i=l-\lambda} M_i' K^{(\lambda+i)}, \quad K'' = \sum_{i=0}^{i=l-\lambda} M_i'' K^{(\lambda+i)}, \quad \ldots\ldots \quad K^{(\lambda-1)} = \sum_{i=0}^{i=l-\lambda} M_i^{(\lambda-1)} K^{(\lambda+i)}$$

les valeurs de λ quantités de la série

$$K^\circ, \; K', \; K'', \; \ldots\ldots \; K^{(l)},$$

en fonctions des autres, qu'on tire de ces équations.

D'après cela on conclura que la partie logarithmique de l'intégrale cherchée est composée de ces $l-\lambda+1$ termes

$$\frac{K^{(\lambda)}}{n_0}\log W_0 + \frac{K^{(\lambda+1)}}{n_1}\log W_1 + \ldots\ldots + \frac{K^{(l)}}{n_{l-\lambda}}\log W_{l-\lambda},$$

où n_0, n_1, ... $n_{l-\lambda}$ désignent des nombres entiers, et W_0, W_1, ... $W_{l-\lambda}$ des fonctions de la forme $\frac{X_0 + X\sqrt{\theta x}}{X_0 - X\sqrt{\theta x}}$.

5) Pour trouver un terme quelconque $\frac{K^{(\lambda+i)}}{n_i}\log W_i$, on cherchera le plus petit dénominateur auquel les quantités $\overset{\circ}{M_i}$, $\overset{\prime}{M_i}$, $\overset{\prime\prime}{M_i}$, ... $M_i^{(\lambda-1)}$ peuvent être réduites. En dénotant ce dénominateur par σ, on fera

$$\pi = \pm M_i^\circ \sigma,$$

en prenant celui des deux signes $\pm$ qui appartient à M_i°, et l'on mettra l'expression

$$\left[(x-x')^{M_i'}(x-x'')^{M_i''}\ldots\ldots(x-x^{(\lambda-1)})^{M_i^{(\lambda-1)}}(x-x^{(\lambda+i)})\right]^{\pm\sigma}$$

sous la forme d'une fraction simple $\frac{u}{v}$.

6) On décomposera θx en deux facteurs $\theta_1 . \theta_2$, de manière que $\sqrt{\frac{uv\theta_1 . x^\pi}{\sqrt{\theta x}}}$ ne soit pas d'un degré entier, on trouvera une fonction entière S, pour laquelle les fractions

$$\frac{S\sqrt{\theta_1}+\sqrt{\theta_2}}{u}, \qquad \frac{S\sqrt{\theta_1}-\sqrt{\theta_2}}{v}$$

ne deviennent pas infinies, tant que x reste fini, et en développant l'expression $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$ en fraction continue, on cherchera parmi les fractions réduites de $\frac{s-\sqrt{\frac{\theta_2}{\theta_1}}}{uv}$ celle dont le dénominateur est du degré le plus proche de celui de $\sqrt{\frac{uv\theta_1 . x^{\pi}}{\sqrt{\theta x}}}$, mais moins élevé que celui de cette fonction.

En dénotant cette fraction par $\frac{M}{N}$, on cherchera le quotient de la division de

$$(NS-Muv)^2\theta_1-N^2\theta_2$$

par uv. Ce quotient sera toujours d'un degré au-dessous du second.

7) Si ce quotient est une fonction du premier degré $ax+b$, on prendra

$$\frac{K^{(\lambda+i)}}{n_i}\log W_i=\frac{K^{(\lambda+i)}}{\pm\sigma}\log\frac{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}},$$

dans le cas où les deux fonctions

$$\frac{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}{u}, \qquad \frac{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}}{v}$$

se réduisent à zéro pour $x=-\frac{b}{a}$.

Dans le cas contraire, on aura

$$\frac{K^{(\lambda+i)}}{n_i}\log W_i=\frac{\varepsilon K^{(\lambda+i)}}{\pm\rho\sigma}\log\left[\left(\frac{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}}\right)^{\frac{\rho}{\varepsilon}}\cdot\frac{\left(\frac{ax+b}{a}\right)^2\varphi\left(\frac{a}{ax+b}\right)+\sqrt{\theta x}}{\left(\frac{ax+b}{a}\right)^2\varphi\left(\frac{a}{ax+b}\right)-\sqrt{\theta x}}\right],$$

où $\varphi(z)$ est une fonction égale à

$$r+\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_2}+\frac{1}{r_1}$$

la fraction continue résultante de $\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}$ étant périodique et de la forme

$$r+\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_2}+\frac{1}{r_1}+\frac{1}{2r}+\frac{1}{r_1}+\frac{1}{r_2}+\cdots$$

ρ le degré de $\dfrac{\varphi(x)+\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}{\varphi(x)-\sqrt{z^4\theta\left(\frac{a-bz}{az}\right)}}$, $\varepsilon=+1$ ou -1, selon que $x=-\frac{b}{a}$ vérifie l'équation

$$\frac{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}{u}=0, \quad \text{ou} \quad \frac{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}}{v}=0.$$

8) Si ce quotient se réduit à une constante, on prendra

$$\frac{K^{(\lambda+i)}}{n_i}\log W_i=\frac{K^{(\lambda+i)}}{\pm\sigma}\log\frac{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}},$$

dans le cas où $\frac{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}}{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}x^{\pi}$ est du degré zéro. Dans le cas contraire, en désignant par ε le degré de $\frac{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}}{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}x^{\pi}$, on aura

$$\frac{K^{(\lambda+i)}}{n_i}\log W_i=\frac{\varepsilon K^{(\lambda+i)}}{\pm\rho\sigma}\log\left[\left(\frac{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}}\right)^{\frac{\rho}{\varepsilon}}\cdot\frac{\varphi_0(x)+\sqrt{\theta x}}{\varphi_0(x)-\sqrt{\theta x}}\right],$$

où

$$\varphi_0(x)=r+\cfrac{1}{r_1+\cfrac{1}{r_2+\ddots+\cfrac{1}{r_2+\cfrac{1}{r_1}}}}$$

la fraction continue résultante de $\sqrt{\theta x}$ étant périodique et de la forme

$$r+\cfrac{1}{r_1+\cfrac{1}{r_2+\ddots+\cfrac{1}{r_2+\cfrac{1}{r_1+\cfrac{1}{2r+\cfrac{1}{r_1+\cfrac{1}{r_2+\ddots}}}}}}}$$

ρ le degré de $\frac{\varphi_0 x+\sqrt{\theta x}}{\varphi_0 x-\sqrt{\theta x}}$.

9) Ce que nous venons de dire sur la détermination du terme $\frac{K^{(\lambda+i)}}{n_i}\log W_i$ ne sera applicable que dans le cas, où le degré de la fonction uvx^{π} surpasse 1. S'il arrive que le degré de uvx^{π} n'est pas au-dessus de 1, le terme $\frac{K^{(\lambda+i)}}{n_i}\log W_i$ sera aussi déterminé par les formules que nous venons d'exposer, seulement on fera $\frac{(NS-Muv)\sqrt{\theta_1}+N\sqrt{\theta_2}}{(NS-Muv)\sqrt{\theta_1}-N\sqrt{\theta_2}}=1$, on trouvera a, b, ε, en égalant $\frac{u}{v}=\frac{1}{(ax+b)^{\varepsilon}}$, et on prendra $\varepsilon=\pi$, dans le cas de $a=0$.

10) Après avoir trouvé tous les termes logarithmiques, on différentiera leur somme. Si cette différentielle ne se réduit pas à $\frac{fx}{Fx}\frac{dx}{\sqrt{\theta x}}$, l'intégrale cherchée est impossible sous forme finie; dans le cas contraire sa valeur sera précisément donnée par la somme

$$\frac{P}{Q}\sqrt{\theta x}+\frac{K^{(\lambda)}}{n_0}\log W_0+\frac{K^{(\lambda+1)}}{n_1}\log W_1+\ldots\ldots+\frac{K^{(l)}}{n_{l-\lambda}}\log W_{l-\lambda}.$$

S'il s'agit, par exemple, de trouver l'intégrale

$$\int\frac{2x^6+4x^5+7x^4-3x^3-x^2-8x-8}{(2x^2-1)^2\sqrt{x^4+4x^3+2x^2+1}}\,dx,$$

on cherchera le plus grand commun diviseur entre les fonctions

$$(2x^2-1)^2(x^4+4x^3+2x^2+1),\quad \frac{d[(2x^2-1)^2(x^4+4x^3+2x^2+1)]}{dx}.$$

Comme ce diviseur est $2x^2-1$, et que les fonctions

$$\frac{(2x^2-1)(2x^6+4x^5+7x^4-3x^3-2x^2-8x-8)}{(2x^2-1)^2(x^4+4x^3+2x^2+1)},\quad \frac{2x^2-1}{x\sqrt{x^4+4x^3+2x^2+1}}$$

sont de degrés au-dessous de 1, on cherchera les coefficients de la fonction du premier degré $P=B_1x+B_0$. Pour cela on divisera

$$2x^6+4x^5+7x^4-3x^3-x^2-8x-8-\frac{(2x^2-1)^2}{2x^2-1}(x^4+4x^3+2x^2+1)B_1$$
$$-\left[\frac{1}{2}\frac{(2x^2-1)^2(4x^3+12x^2+4x)}{2x^2-1}-\frac{(2x^2-1)^2(x^4+4x^3+2x^2+1).4x}{(2x^2-1)^2}\right](B_1x+B_0)$$

par $2x^2-1$. Comme cette division donne le reste

$$\left(4B_1+9B_0-\frac{17}{2}\right)x+\frac{9}{2}B_1+4B_0-\frac{13}{2},$$

et qu'on trouve, en outre, dans le quotient le terme

$$(1-B_1)x^4$$

qui est d'un degré plus élevé que $\frac{(2x^2-1)^2\sqrt{x^4+4x^3+2x^2+1}}{x(2x^2-1)}$, on égalera tout cela à zéro, ce qui donne les équations

$$4B_1+9B_0-\frac{17}{2}=0,\quad \frac{9}{2}B_1+4B_0-\frac{13}{2}=0,\quad 1-B_1=0,$$

et de là

$$B_1=1,\ B_0=\frac{1}{2},\ P=x+\frac{1}{2}.$$

Donc, le terme algébrique, dans l'expression de l'intégrale cherchée, a cette valeur

$$\frac{x+\frac{1}{2}}{2x^2-1}\sqrt{x^4+4x^3+2x^2+1}.$$

En réduisant l'expression

$$\frac{2x^6+4x^5+7x^4-3x^3-x^2-8x-8}{(2x^2-1)^2}-\sqrt{x^4+4x^3+2x^2+1}\,\frac{d\frac{x+\frac{1}{2}}{2x^2-1}\sqrt{x^4+4x^3+2x^2+1}}{dx}$$

à la forme la plus simple, on parvient à

$$\frac{6x^2+5x+7}{2x^2-1};$$

et comme les racines de l'équation $2x^2-1=0$ sont $x=-\frac{1}{\sqrt{2}}$, $x=\frac{1}{\sqrt{2}}$, on calculera les quantités K^0, K', K'' d'après les formules

$$K^0=\lim\left[\frac{x(6x^2+5x+7)}{(2x^2-1)\sqrt{x^4+4x^3+2x^2+1}}\right]_{x=\infty},\quad K'=\frac{6x'^2+5x'+7}{4x'\sqrt{x'^4+4x'^3+2x'^2+1}},$$

$$K''=\frac{6x''^2+5x''+7}{4x''\sqrt{x''^4+4x''^3+2x''^2+1}},$$

en prenant $x'=-\frac{1}{\sqrt{2}}$, $x''=\frac{1}{\sqrt{2}}$, ce qui donne

$$K^0=0,\quad K'=-\frac{5}{2},\quad K''=\frac{5}{2},$$

et, par conséquent, on aura

$$M^0K^0+M'K'+M''K''=0, \quad \ldots\ldots\ldots\ldots \quad (12)$$

si l'on prend

$$M^0=1,\quad M'=0,\quad M''=0,$$

ou

$$M^0=0,\quad M'=1,\quad M''=1.$$

Quant aux autres valeurs de M^0, M', M'' qui satisfont à l'équation (12), elles ne conduisent par rapport à K^0, K', K'', qu'à des équations identiques à celles qu'on trouve en prenant les valeurs mentionnées de M^0, M', M'', savoir:

$$1.K^0+0.K'+0.K''=0,\quad 0.K^0+1.K'+1.K''=0,$$

et ces équations nous donnent

$$K^0=0.K'',\quad K'=-K''.$$

D'après cela on conclut que la partie logarithmique de l'intégrale cherchée ne contient qu'un seul terme

$$\frac{\frac{5}{2}}{n_1}\log W_1.$$

Les coefficients de l'expression de K^0 et K', en fonction de K'', n'étant pas fractionnaires, et le coefficient de K'' dans la valeur de K^0 étant zéro, on prendra

$$\sigma = 1\,,\ \pi = 0\,;$$

on mettra le produit

$$\left(x+\frac{1}{\sqrt{2}}\right)^{-1}\cdot\left(x-\frac{1}{\sqrt{2}}\right)$$

sous la forme d'une fraction

$$\frac{x-\frac{1}{\sqrt{2}}}{x+\frac{1}{\sqrt{2}}},$$

et après avoir décomposé $x^4+4x^3+2x^2+1$ en deux facteurs $(x+1)(x^3+3x^2-x+1)$, on cherchera une fonction entière S pour laquelle les fractions

$$\frac{S\sqrt{x+1}+\sqrt{x^3+3x^2-x+1}}{x-\frac{1}{\sqrt{2}}},\qquad \frac{S\sqrt{x+1}-\sqrt{x^3+3x^2-x+1}}{x+\frac{1}{\sqrt{2}}}$$

ne deviennent pas infinies, tant que x reste fini; ou, ce qui revient au même, une fonction S qui, pour $x=\frac{1}{\sqrt{2}}$, $x=-\frac{1}{\sqrt{2}}$, se réduise respectivement à

$$\frac{-\sqrt{\left(\frac{1}{\sqrt{2}}\right)^3+3\left(\frac{1}{\sqrt{2}}\right)^2-\frac{1}{\sqrt{2}}+1}}{\sqrt{\frac{1}{\sqrt{2}}+1}}=-\frac{3-\sqrt{2}}{\sqrt{2}},$$

et à

$$\frac{\sqrt{\left(-\frac{1}{\sqrt{2}}\right)^3+3\left(-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{\sqrt{2}}+1}}{\sqrt{-\frac{1}{\sqrt{2}}+1}}=\frac{3+\sqrt{2}}{\sqrt{2}}.$$

D'après cela on trouve

$$S=1-3x.$$

En cherchant la fraction réduite de $\dfrac{1-3x-\sqrt{\frac{x^3+3x^2-x+1}{x+1}}}{\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)}$, dont le dénominateur est du degré le plus proche possible de celui de $\sqrt{\dfrac{\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)(x+1)x^0}{\sqrt{(x^4+4x^3+2x^2+1)}}}$, mais moins élevé que $\sqrt{\dfrac{\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)(x+1)x^0}{\sqrt{(x^4+4x^3+2x^2+1)}}}$, on prendra pour cette fraction $\frac{0}{1}$, et comme

$$\left[1.(1-3x)-0.\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)\right]^2(x+1)-1^2.(x^3+3x^2-x+1)=8x^3-4x,$$

divisé par $\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)=x^2-\frac{1}{2}$, donne pour quotient $8x$, et que $x=0$ rend nulle la dernière de deux expressions

$$\frac{(1-3x)\sqrt{x+1}+\sqrt{x^3+3x^2-x+1}}{x-\frac{1}{\sqrt{2}}}, \qquad \frac{(1-3x)\sqrt{x+1}-\sqrt{x^3+3x^2-x+1}}{x+\frac{1}{\sqrt{2}}},$$

le terme logarithmique, d'après le n° 7, sera donné par la formule

$$\frac{-1.\frac{5}{2}}{1.\rho}\log\left[\left(\frac{(1-3x)\sqrt{x+1}+\sqrt{x^3+3x^2-x+1}}{(1-3x)\sqrt{x+1}-\sqrt{x^3+3x^2-x+1}}\right)^{\frac{\rho}{1}}\cdot\frac{x^2\varphi\left(\frac{1}{x}\right)+\sqrt{x^4+4x^3+2x^2+1}}{x^2\varphi\left(\frac{1}{x}\right)-\sqrt{x^4+4x^3+2x^2+1}}\right];$$

où ρ désigne le degré de $\dfrac{\varphi(z)+\sqrt{z^4\left[\left(\frac{1}{z}\right)^4+4\left(\frac{1}{z}\right)^3+2\left(\frac{1}{z}\right)^2+1\right]}}{\varphi(z)-\sqrt{z^4\left[\left(\frac{1}{z}\right)^4+4\left(\frac{1}{z}\right)^3+2\left(\frac{1}{z}\right)^2+1\right]}}$.

Pour trouver la fonction $\varphi(z)$, on développera le radical

$$\sqrt{z^4\left[\left(\frac{1}{z}\right)^4+4\left(\frac{1}{z}\right)^3+2\left(\frac{1}{z}\right)^2+1\right]}=\sqrt{z^4+2z^2+4z+1}$$

en fraction continue. Comme on trouve

$$\sqrt{z^4+2z^2+4z+1}=z^2+1+\cfrac{1}{\frac{1}{2}z+\cfrac{1}{2z-2+\cfrac{1}{\frac{1}{2}z+\cfrac{1}{2(z^2+1)+\cfrac{1}{\frac{1}{2}z+\cfrac{1}{2z-2+\dotsb}}}}}}$$

on prendra

$$\varphi(z)=z^2+1+\cfrac{1}{\frac{1}{2}z+\cfrac{1}{2z-2+\cfrac{1}{\frac{1}{2}z}}}$$

ce qui donne $$\varphi(z)=\frac{z^5-z^4+3z^3+z^2+2}{z^3-z^2+2z},$$

et par conséquent ρ, degré de $\dfrac{\varphi(z)+\sqrt{z^4\left[\left(\frac{1}{z}\right)^4+4\left(\frac{1}{z}\right)^3+2\left(\frac{1}{z}\right)^2+1\right]}}{\varphi(z)-\sqrt{z^4\left[\left(\frac{1}{z}\right)^4+4\left(\frac{1}{z}\right)^3+2\left(\frac{1}{z}\right)^2+1\right]}}$, est égal à 10.

Ainsi, en faisant pour abréger,

$$\Delta=\sqrt{x^4+4x^3+2x^2+1},$$

le terme logarithmique cherché aura cette valeur

$$-\frac{5}{2.10}\log\left[\left(\frac{(1-3x)\sqrt{x+1}+\sqrt{x^3+3x^2-x+1}}{(1-3x)\sqrt{x+1}-\sqrt{x^3+3x^2-x+1}}\right)^{-10}\cdot\frac{x^2\left[\left(\frac{1}{x}\right)^5-\left(\frac{1}{x}\right)^4+3\left(\frac{1}{x}\right)^3+\left(\frac{1}{x}\right)^2+2\right]+\left[\left(\frac{1}{x}\right)^3-\left(\frac{1}{x}\right)^2+2\left(\frac{1}{x}\right)\right]\Delta}{x^2\left[\left(\frac{1}{x}\right)^5-\left(\frac{1}{x}\right)^4+3\left(\frac{1}{x}\right)^4+\left(\frac{4}{x}\right)^2+2\right]-\left[\left(\frac{1}{x}\right)^3-\left(\frac{1}{x}\right)^2+2\left(\frac{1}{x}\right)\right]\Delta}\right],$$

ou, ce qui revient au même,

$$\frac{1}{4}\log\left[\left(\frac{(1-3x)\sqrt{x+1}+\sqrt{x^3+3x^2-x+1}}{(1-3x)\sqrt{x+1}-\sqrt{x^3+3x^2-x+1}}\right)^{10}\cdot\frac{2x^5+x^3+3x^2-x+1-(2x^2-x+1)\Delta}{2x^5+x^3+3x^2-x+1+(2x^2-x+1)\Delta}\right].$$

Donc, si l'intégrale cherchée peut être exprimée sous forme finie, elle doit être égale à

$$\frac{x+\frac{1}{2}}{2x^2-1}\Delta+\frac{1}{4}\log\left[\left(\frac{(1-3x)\sqrt{x+1}+\sqrt{x^3+3x^2-x+1}}{(1-3x)\sqrt{x+1}-\sqrt{x^3+3x^2-x+1}}\right)^{10}\cdot\frac{2x^5+x^3+3x^2-x+1-(2x^2-x+1)\Delta}{2x^5+x^3+3x^2-x+1+(2x^2-x+1)\Delta}\right];$$

où

$$\Delta=\sqrt{x^4+4x^3+2x^2+1}.$$

Effectivement, on trouve par la différentiation, que c'est bien la valeur de l'intégrale

$$\int\frac{2x^6+4x^5+7x^4-3x^3-x^2-8x-8}{(2x^2-1)^2\sqrt{x^4+4x^3+2x^2+1}}\,dx$$

sur laquelle nous avons opéré.

-11-

MÉMOIRES

DE

L'ACADÉMIE IMPÉRIALE DES SCIENCES DE ST.-PÉTERSBOURG, VII^{e} SÉRIE.

Tome I, N° 5.

SUR L'INTERPOLATION

DANS LE CAS

D'UN GRAND NOMBRE DE DONNÉES

FOURNIES PAR LES OBSERVATIONS.

Par

P. Tchébychef,

Membre de l'Académie.

Lu le 29 octobre 1858.

St.-PÉTERSBOURG, 1859.

Commissionnaires de l'Académie Impériale des sciences:

MM. Eggers et Comp. à St.-Pétersbourg, M. Samuel Schmidt à Riga
et M. Léopold Voss à Leipzig.

Prix: 65 Kop. = 22 Ngr.

Imprimé par ordre de l'Académie.

Mai 1859.

C. Vessélofski, Secrétaire perpétuel.

Imprimerie de l'Académie Impériale des sciences.

SUR L'INTERPOLATION

DANS LE CAS

D'UN GRAND NOMBRE DE DONNÉES

FOURNIES PAR LES OBSERVATIONS.

Par **P. Tchébychef.**

Quand le nombre des valeurs données surpasse celui des termes que l'on conserve dans leur expression, l'interpolation peut être executée par diverses méthodes. Mais ces méthodes, dans chaque cas particulier, sont loin d'être également avantageuses; elles diffé-rent entre elles, soit par la prolixité plus ou moins grande des calculs, soit par la grandeur de l'erreur moyenne à craindre, tant qu'il s'agit d'interpolation de valeurs fournies par les observations, et conséquemment affectées d'erreurs. Comme on ne peut gagner au delà d'une certaine limite, sous un de ces rapports, sans perdre sous l'autre, il est impossible de donner une méthode d'interpolation qui soit en général préférable à toutes les autres; car, suivant les cas, on tient plus ou à la simplification des calculs, ou à la précision des résultats. Si l'on ne connait qu'un petit nombre de valeurs d'une fonction interpolée, il se présente peu de ressources pour atténuer l'influence de leurs erreurs sur le résultat cherché, et alors il est important de tirer des données d'interpolation tout le parti possible pour diminuer l'erreur moyenne à craindre, ce qu'on ne peut faire qu'à l'aide de *la méthode des moindres carrés*. Dans le cas contraire, le nombre considérable des données qu'on a à sa disposition, nous dispense de recourir à *la méthode des moindres carrés* qui exige des calculs trop longs. Alors, pour la simplification des opérations numériques, on peut bien sacrifier une partie plus ou moins considérable de ce que les valeurs données offrent pour apprécier le résultat cherché. Dans le Mémoire *Sur les fractions continues*, présenté à l'Académie en 1855, nous avons traité de l'interpolation parabolique d'après *la méthode des moindres carrés*, *) et nous sommes parvenu à une série qui fournit directement les résultats d'une telle interpolation, indispensable, comme nous venons de le voir, si le nombre des valeurs connues de la fonction interpolée est assez petit. Dans le présent Mémoire nous

*) La traduction française de ce Mémoire, dont je suis redevable à l'obligeance eclairée de M. Bienaymé, vient de paraître dans le Journal de M. Liouville, T. III, 2^{me} Série.

montrerons comment, d'après nos méthodes, on parvient à d'autres formules d'interpolation qui peuvent remplacer avec avantage celle dont nous venons de parler, en tant que son application, vu le grand nombre des valeurs données, cesse, d'une part, d'être importante, et de l'autre, devient peu praticable.

Des divers cas particuliers que peut présenter l'interpolation suivant le nombre plus ou moins grand des valeurs données, nous nous bornerons à considérer celui qui est la limite de tous les autres où le nombre des valeurs données est infini. Quoique, en réalité, ce nombre ne soit jamais infini, les formules qu'on trouve dans cette supposition peuvent être cependant d'une application utile; car elles présentent la limite vers laquelle convergent très rapidement les résultats d'interpolation, à mesure que ce nombre augmente, et il ne sera pas difficile de voir, dans chaque cas particulier, de quel degré d'approximation ces formules sont susceptibles d'après les valeurs données.

I.

§ 1. Nous commencerons par exposer la solution du problème qui servira de base à nos recherches.

Problème.

Etant donnée une suite de valeurs de $F(x) = A_0 + A_1x + \ldots + A_nx^n$ qui correspondent à des valeurs de x équidistantes et très rapprochées entr'elles, combiner les valeurs de $F(x)$, par la seule voie d'addition et de soustraction, de manière à ce que le résultat final ne contienne que le terme affecté du coefficient A_1, et que ce terme soit le plus grand possible.

Solution.

Soient

$$F(x_1),\ F(x_2), \ldots\ldots\ldots F(x_l)$$

les valeurs données de

$$F(x) = A_0 + A_1x + \ldots\ldots + A_nx^n.$$

En supposant que

$$F(x_1),\ F(x_2), \ldots\ldots\ldots\ldots\ldots F(x_\sigma),$$
$$F(x_{\sigma+\sigma'+1}),\ F(x_{\sigma+\sigma'+2}), \ldots\ldots F(x_{\sigma+\sigma'+\sigma''}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

soient les valeurs de $F(x)$ prises avec le signe $+$, et

$$F(x_{\sigma+1}),\ F(x_{\sigma+2}), \ldots\ldots\ldots\ldots\ldots F(x_{\sigma+\sigma'}),$$
$$F(x_{\sigma+\sigma'+\sigma''+1}),\ F(x_{\sigma+\sigma'+\sigma''+2}), \ldots\ldots F(x_{\sigma+\sigma'+\sigma''+\sigma'''}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

celles qui auront le signe —, on trouve que la combinaison cherchée des valeurs

$$F(x_1),\ F(x_2),\ldots\ldots F(x_i)$$

s'exprime par la formule

$$\sum_{\mu=1}^{\mu=\sigma} F(x_\mu) - \sum_{\mu=\sigma+1}^{\mu=\sigma+\sigma'} F(x_\mu) + \sum_{\mu=\sigma+\sigma'+1}^{\mu=\sigma+\sigma'+\sigma''} F(x_\mu) - \sum_{\mu=\sigma+\sigma'+\sigma''+1}^{\mu=\sigma+\sigma'+\sigma''+\sigma'''} F(x_\mu) + \ldots\ldots\ldots\ldots$$

Les valeurs

$$x_1,\ x_2,\ldots\ldots\ldots\ldots x_i$$

étant, par hypothèse, équidistantes, cette expression, à un facteur constant près et qui ne change en rien la solution cherchée, est égale à

$$\sum_{\mu=1}^{\mu=\sigma} F(x_\mu)(x_{\mu+1}-x_\mu) - \sum_{\mu=\sigma+1}^{\mu=\sigma+\sigma'} F(x_\mu)(x_{\mu+1}-x_\mu) + \sum_{\mu=\sigma+\sigma'+1}^{\mu=\sigma+\sigma'+\sigma''} F(x_\mu)(x_{\mu+1}-x_\mu) - \sum_{\mu=\sigma+\sigma'+\sigma''+1}^{\mu=\sigma+\sigma'+\sigma''+\sigma'''} F(x_\mu)(x_{\mu+1}-x_\mu) + \ldots,$$

ce qui se réduit à

$$\int_{x_1}^{x_\sigma} F(x)dx - \int_{x_\sigma}^{x_{\sigma+\sigma'}} F(x)dx + \int_{x_{\sigma+\sigma'}}^{x_{\sigma+\sigma'+\sigma''}} F(x)dx - \int_{x_{\sigma+\sigma'+\sigma''}}^{x_{\sigma+\sigma'+\sigma''+\sigma'''}} F(x)dx + \ldots\ldots\ldots,$$

les valeurs

$$x_1,\ x_2,\ldots\ldots\ldots\ldots x_i$$

étant très rapprochées entre elles.

Or, si l'on fait

$$x_\sigma = \eta_1,\ x_{\sigma+\sigma'} = \eta_2,\ x_{\sigma+\sigma'+\sigma''} = \eta_3,\ldots\ldots\ldots\ldots,$$

et que l'on désigne par a et b les valeurs extrèmes de x dans la suite

$$x_1,\ x_2,\ldots\ldots\ldots\ldots x_i,$$

et par ν le nombre des valeurs

$$x_\sigma,\ x_{\sigma+\sigma'},\ x_{\sigma+\sigma'+\sigma''},\ldots\ldots\ldots\ldots,$$

l'expression précédente peut être représentée ainsi:

$$\int_a^{\eta_1} F(x)dx - \int_{\eta_1}^{\eta_2} F(x)dx + \int_{\eta_2}^{\eta_3} F(x)dx - \ldots\ldots + (-1)^\nu \int_{\eta_\nu}^{b} F(x)dx.$$

A cause de quoi notre problème se réduit à la détermination des quantités

$$\eta_1,\ \eta_2,\ldots\ldots\ldots\ldots\eta_\nu,$$

sous la condition que, pour

$$F(x) = A_0 + A_1 x \ldots\ldots + A_n x^n,$$

on ait

$$(1) \ldots\ldots\ldots \int_a^{\eta_1} F(x)dx - \int_{\eta_1}^{\eta_2} F(x)dx + \int_{\eta^2}^{\eta_3} F(x)dx - \ldots\ldots + (-1)_\nu \int_{\eta_\nu}^{b} F(x)dx = sA_l,$$

et que le facteur s soit le plus grand possible, en supposant, bien entendu, que, conformément au sens du problème, les valeurs

$$a,\ \eta_1,\ \eta_2,\ \eta_3, \ldots\ldots\ldots \eta_\nu,\ b$$

présentent une série croissante.

§ 2. Pour tirer de ce qui précède les équations relatives à $\eta_1, \eta_2, \eta_3, \ldots\ldots \eta_\nu$ nous remarquerons que la formule (1), en prenant

$$F(x) = A_0 + A_1 x + \ldots\ldots\ldots\ldots + A_n x^n,$$

devient

$$\begin{aligned}
&A_0\left[-a + 2\eta_1 - 2\eta_2 + 2\eta_3 - \ldots\ldots\ldots\ldots\ldots\ldots - 2(-1)^\nu \eta_\nu + (-1)^\nu b\right] \\
&+ \tfrac{1}{2} A_1\left[-a^2 + 2\eta_1^2 - 2\eta_2^2 + 2\eta_3^2 - \ldots\ldots\ldots\ldots\ldots\ldots - 2(-1)^\nu \eta_\nu^2 + (-1)^\nu b^2\right] \\
&+ \tfrac{1}{3} A_2\left[-a^3 + 2\eta_1^3 - 2\eta_2^3 + 2\eta_3^3 - \ldots\ldots\ldots\ldots\ldots\ldots - 2(-1)^\nu \eta_\nu^3 + (-1)^\nu b^3\right] \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \tfrac{1}{l} A_{l-1}\left[-a^l + 2\eta_1^l - 2\eta_2^l + 2\eta_3^l - \ldots\ldots\ldots\ldots\ldots\ldots - 2(-1)^\nu \eta_\nu^l + (-1)^\nu b^l\right] \\
&+ \tfrac{1}{l+1} A_l\left[-a^{l+1} + 2\eta_1^{l+1} - 2\eta_2^{l+1} + 2\eta_3^{l+1} \ldots\ldots - 2(-1)^\nu \eta_\nu^{l+1} + (-1)^\nu b^{l+1}\right] \\
&+ \tfrac{1}{l+2} A_{l+1}\left[-a^{l+2} + 2\eta_1^{l+2} - 2\eta_2^{l+2} + 2\eta_3^{l+2} \ldots\ldots - 2(-1)^\nu \eta_\nu^{l+2} + (-1)^\nu b^{l+2}\right] \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ \tfrac{1}{n+1} A_n\left[-a^{n+1} + 2\eta_1^{n+1} - 2\eta_2^{n+1} + 2\eta_3^{n+1} - \ldots - 2(-1)^\nu \eta_\nu^{n+1} + (-1)^\nu b^{n+1}\right] \\
&= sA_l.
\end{aligned}$$

D'où résultent les équations

$$(2) \ldots\ldots \begin{cases}
a - 2\eta_1 + 2\eta_2 - \ldots\ldots\ldots\ldots\ldots\ldots + 2(-1)^\nu \eta_\nu - (-1)^\nu b = 0, \\
a^2 - 2\eta_1^2 + 2\eta_2^2 - \ldots\ldots\ldots\ldots\ldots\ldots + 2(-1)^\nu \eta_\nu^2 - (-1)^\nu b^2 = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a^l - 2\eta_1^l + 2\eta_2^l - \ldots\ldots\ldots\ldots\ldots\ldots + 2(-1)^\nu \eta_\nu^l - (-1)^\nu b^l = 0, \\
a^{l+2} - 2\eta_1^{l+2} + 2\eta_2^{l+2} - \ldots\ldots + 2(-1)^\nu \eta_\nu^{l+2} - (-1)^\nu b^{l+2} = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a^{n+1} - 2\eta_1^{n+1} + 2\eta_2^{n+1} - \ldots\ldots + 2(-1)^\nu \eta_\nu^{n+1} - (-1)^\nu b^{n+1} = 0,
\end{cases}$$

avec la condition que la quantité

$$(3) \ldots\ldots s = -\frac{1}{l+1}\left[a^{l+1} - 2\eta_1^{\,l+1} + 2\eta_2^{\,l+1} - \ldots\ldots\ldots + 2(-1)^\nu \eta_\nu^{\,l+1} - (-1)^\nu b^{l+1}\right]$$

ait la valeur la plus grande possible, et par suite la méthode usitée des *maxima* relatifs nous fournit ces équations:

$$\begin{aligned}
\eta_1^{\,l} &= \lambda_0 + \lambda_1\eta_1 + \lambda_2\eta_1^{\,2} + \ldots\ldots\ldots + \lambda_n\eta_1^{\,n},\\
\eta_2^{\,l} &= \lambda_0 + \lambda_1\eta_2 + \lambda_2\eta_2^{\,2} + \ldots\ldots\ldots + \lambda_n\eta_2^{\,n},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\eta_\nu^{\,l} &= \lambda_0 + \lambda_1\eta_\nu + \lambda_2\eta_\nu^{\,2} + \ldots\ldots\ldots + \lambda_n\eta_\nu^{\,n},
\end{aligned}$$

où $\lambda_0, \lambda_1, \lambda_2 \ldots\ldots \lambda_n$ sont des inconnues auxiliaires.

Les dernières équations nous montrent que les quantités

$$\eta_1,\ \eta_2, \ldots\ldots \eta_\nu$$

sont les racines de la même équation

$$\lambda_n\eta^n + \ldots\ldots + \lambda_2\eta^2 + \lambda_1\eta + \lambda_0 - \eta^l = 0.$$

Comme cette équation est tout au plus de degré n, et que les quantités

$$\eta_1,\ \eta_1, \ldots\ldots \eta_\nu$$

sont différentes entre elles, il en résulte que ν, nombre de ces quantités, ne peut surpasser n. Donc, relativement au nombre ν, il n'y a que ces $n+1$ hypothèses à faire:

$$\nu = n,\ \nu = n-1, \ldots\ldots \nu = 1,\ \nu = 0.$$

De plus, on reconnaît aisément que la première hypothèse comprend comme cas particulier toutes les autres, tant qu'on admet des solutions où quelques unes des quantités

$$\eta_n,\ \eta_{n-1},\ \eta_{n-2}, \ldots\ldots\ldots\ldots$$

sont égales à b. En effet, si l'on a

$$n = \nu,\ \eta_n = b,\ \eta_{n-1} = b, \ldots\ldots\ldots \eta_{n-\rho+1} = b,$$

dans nos formules fondamentales (1), (2), (3), ρ termes s'éliminent, et le reste devient identique à ce qu'on trouverait en prenant

$$\nu = n - \rho,$$

au lieu de

$$n = \nu.$$

C'est pourquoi, dans les recherches de $\eta_1, \eta_2, \ldots\ldots \eta_\nu$, nous nous bornerons à la première hypothèse sur le nombre ν, savoir: $\nu = n$.

Or, pour cette valeur de ν, les formules (2) et (3) nous donnent

$$(4)\ldots\ldots\begin{cases} a-2\eta_1+2\eta_2-\ldots\ldots\ldots\ldots\ldots\ldots+2(-1)^n\eta_n-(-1)^n b=0, \\ a^2-2\eta_1^2+2\eta_2^2-\ldots\ldots\ldots\ldots\ldots\ldots+2(-1)^n\eta_n^2-(-1)^n b^2=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^l-2\eta_1^l+2\eta_2^l-\ldots\ldots\ldots\ldots\ldots\ldots+2(-1)^n\eta_n^l-(-1)^n b^l=0, \\ a^{l+2}-2\eta_1^{l+2}+2\eta_2^{l+2}-\ldots\ldots+2(-1)^n\eta_n^{l+2}-(-1)^n b^{l+2}=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a^{n+1}-2\eta_1^{n+1}+2\eta_2^{n+1}-\ldots\ldots+2(-1)^n\eta_n^{n+1}-(-1)^n b^{n+1}=0, \end{cases}$$

$$(5)\ldots\ldots s=-\frac{1}{l+1}\left[a^{l+1}-2\eta_1^{l+1}+2\eta_2^{l+1}-\ldots\ldots\ldots+2(-1)^n\eta_n^{l+1}-(-1)^n b^{l+1}\right].$$

Les équations (4) sont en nombre suffisant pour déterminer toutes les quantités $\eta_1, \eta_2, \ldots\ldots \eta_n$. Ces équations pourront avoir plusieurs solutions, mais on distinguera facilement celle qui correspond à notre problème, en ayant égard à la valeur de

$$s=-\frac{1}{l+1}\left[a^{l+1}-2\eta_1^{l+1}+2\eta_2^{l+1}-\ldots\ldots\ldots\ldots+2(-1)^n\eta_n^{l+1}-(-1)^n b^{l+1}\right],$$

qu'on cherche à rendre aussi grande que possible. De plus, conformément à ce que nous avons vu, on rejetera toutes les solutions où les valeurs

$$a,\ \eta_1,\ \eta_2, \ldots\ldots\ldots \eta_n,\ b$$

ne présentent pas une série croissante.

§ 3. Il serait très difficile de résoudre les équations (4) par les méthodes ordinaires d'Algèbre; mais on y pervient très aisément à l'aide d'une méthode particulière, dont nous nous sommes servi dans le Mémoire cité plus haut; c'est ce que nous allons montrer.

En développant l'expression

$$\frac{1}{x-a}-\frac{2}{x-\eta_1}+\frac{2}{x-\eta_2}-\ldots\ldots\ldots+\frac{2(-1)^n}{x-\eta_n}-\frac{(-1)^n}{x-b}$$

suivant les puissances décroissantes de x, on a

$$\begin{aligned} &\frac{1}{x-a}-\frac{2}{x-\eta_1}+\frac{2}{x-\eta_2}+\ldots\ldots\ldots+\frac{2(-1)^n}{x-\eta_n}-\frac{(-1)^n}{x-b}= \\ &\frac{1}{x^2}\left[a-2\eta_1+2\eta_1-\ldots\ldots\ldots\ldots\ldots\ldots\ldots+2(-1)^n\eta_n-(-1)^n b\right] \\ &+\frac{1}{x^3}\left[a^2-2\eta_1^2+2\eta_2^2-\ldots\ldots\ldots\ldots\ldots\ldots+2(-1)^n\eta_n^2-(-1)^n b^2\right] \\ &+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+\frac{1}{x^{l+2}}\left[a^{l+1}-2\eta_1^{l+1}+2\eta_2^{l+1}-\ldots\ldots\ldots+2(-1)^n\eta_n^{l+1}-(-1)^n b^{l+1}\right] \\ &+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$+\frac{1}{x^{n+2}}\left[a^{n+1}-2\eta_1^{\,n+1}+2\eta_2^{\,n+1}-\ldots\ldots+2(-1)^n\eta_n^{\,n+1}-(-1)^n b^{n+1}\right]$$
$$+\frac{1}{x^{n+3}}\left[a^{n+2}-2\eta_1^{\,n+2}+2\eta_2^{\,n+2}-\ldots\ldots+2(-1)^n\eta_n^{\,n+2}-(-1)^n b^{n+2}\right]$$
$$+\frac{1}{x^{n+4}}\left[a^{n+3}-2\eta_1^{\,n+3}+2\eta_2^{\,n+3}-\ldots\ldots+2(-1)^n\eta_n^{\,n+3}-(-1)^n b^{n+3}\right]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

D'où, en vertu des équations (4), (5), et en faisant, pour abréger,

$$s'=a^{n+2}-2\eta_1^{\,n+2}+2\eta_2^{\,n+2}\ldots\ldots\ldots+2(-1)^n\eta_n^{\,n+2}-(-1)^n b^{n+2},$$
$$s''=a^{n+3}-2\eta_1^{\,n+3}+2\eta_2^{\,n+3}-\ldots\ldots+2(-1)^n\eta_n^{\,n+3}-(-1)^n b^{n+3},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

nous obtenons

$$\frac{1}{x-a}-\frac{2}{x-\eta_1}+\frac{2}{x-\eta_2}-\ldots\ldots+\frac{2(-1)^n}{x-\eta_n}-\frac{(-1)^n}{x-b}=-\frac{(l+1)s}{x^{l+2}}+\frac{s'}{x^{n+3}}+\frac{s''}{x^{n+4}}+\ldots\ldots\ldots$$

D'autre part, en posant

$$(6)\ldots\ldots\ldots\ldots\ldots\begin{cases}(x-\eta_1)(x-\eta_3)\ldots\ldots\ldots\ldots\ldots=\varphi(x),\\(x-\eta_2)(x-\eta_4)\ldots\ldots\ldots\ldots\ldots=\psi(x),\end{cases}$$

nous trouvons

$$\frac{1}{x-\eta_1}+\frac{1}{x-\eta_3}+\ldots\ldots\ldots\ldots=\frac{\varphi'(x)}{\varphi(x)},$$
$$\frac{1}{x-\eta_2}+\frac{1}{x-\eta_4}+\ldots\ldots\ldots\ldots=\frac{\psi'(x)}{\psi(x)},$$

et par là l'équation précédente donne

$$\frac{1}{x-a}-\frac{2\varphi'(x)}{\varphi(x)}+\frac{2\psi'(x)}{\psi(x)}-\frac{(-1)^n}{x-b}=-\frac{(l+1)s}{x^{l+2}}+\frac{s'}{x^{n+3}}+\frac{s''}{x^{n+4}}+\ldots\ldots\ldots$$

D'où, en intégrant,

$$\log(x-a)-2\log\varphi(x)+2\log\psi(x)-(-1)^n\log(x-b)=\log C+\frac{s}{x^{l+1}}-\frac{s'}{(n+2)x^{n+2}}-\frac{s''}{(n+3)x^{n+3}}-\ldots,$$

ou, ce qui revient au même,

$$\frac{(x-a)\psi^2(x)}{(x-b)^{(-1)^n}\varphi^2(x)}=Ce^{\frac{s}{x^{l+1}}-\frac{s'}{(n+2)x^{n+2}}-\frac{s''}{(n+3)x^{n+3}}-\ldots\ldots}$$

D'après la composition des fonctions $\varphi(x)$, $\psi(x)$, on voit que la plus haute puissance de x dans le développement de la fraction

$$\frac{(x-a)\psi^2(x)}{(x-b)^{(-1)^n}\varphi^2(x)}$$

aura pour coefficient 1, et comme le premier terme du développement de

$$Ce^{\frac{s}{x^{l+1}}-\frac{s'}{(n+2)x^{n+2}}-\frac{s''}{(n+3)x^{n+3}}-\ldots\ldots\ldots}$$

est C, l'équation précédente suppose

$$C=1,$$

et elle se réduit alors á celle-ci:

$$\frac{(x-a)\psi^2(x)}{(x-b)^{(-1)^n}\varphi^2(x)}=e^{\frac{s}{x^{l+1}}-\frac{s'}{(n+2)x^{n+2}}-\frac{s''}{(n+3)x^{n+3}}-\cdots\cdots}$$

ce qui nous donne

$$(7)\ldots\ldots\ldots\ldots\frac{\psi(x)}{\varphi(x)}=\sqrt{\frac{(x-b)^{(-1)^n}}{x-a}}\,e^{\frac{s}{2x^{l+1}}-\frac{s'}{2(n+2)x^{n+2}}-\frac{s''}{2(n+3)x^{n+3}}-\cdots\cdots}$$

C'est au moyen de cette formule que nous trouverons le coefficient s dans l'équation (1)

$$\int_a^{\eta_1}F(x)dx-\int_{\eta_1}^{\eta_2}F(x)dx+\int_{\eta_2}^{\eta_3}F(x)dx-\ldots\ldots+(-1)^\nu\int_{\eta_\nu}^{b}F(x)dx=sA_l$$

et les fonctions

$$\varphi(x),\ \psi(x)$$

qui, d'après (6), déterminent les quantités

$$\eta_1,\ \eta_2,\ \eta_3\ldots\ldots\ldots$$

Mais pour y parvenir nous devons examiner séparément le cas de n pair et celui de n impair.

Le nombre n est pair.

§ 4. Dans le cas de n pair, l'équation (7) devient

$$\frac{\psi(x)}{\varphi(x)}=\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}-\frac{s'}{2(n+2)x^{n+2}}-\frac{s''}{2(n+3)x^{n+3}}-\cdots\cdots\cdots}$$

D'où il résulte

$$\frac{\psi(x)}{\varphi(x)}-\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}=\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}\left(e^{-\frac{s'}{2(n+2)x^{n+2}}-\frac{s''}{2(n+3)x^{n+3}}-\cdots}-1\right),$$

et comme l'expression

$$e^{-\frac{s'}{2(n+2)x^{n+2}}-\frac{s''}{2(n+3)x^{n+3}}-\cdots\cdots}-1,$$

développée en série, ne contient x que dans les degrés inférieurs à $-(n+1)$, il s'en suit que la fraction

$$\frac{\psi(x)}{\varphi(x)}$$

est la valeur de $\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}$, exacte jusqu'à $\frac{1}{x^{n+2}}$. D'autre part, n étant un nombre pair, les fonctions $\varphi(x)$, $\psi(x)$, déterminées par les formules (6), sont du degré $\frac{n}{2}$, et dans ce cas la fraction $\frac{\psi(x)}{\varphi(x)}$ ne peut représenter la valeur de

$$\sqrt{\frac{x-b}{x-a}}\, e^{\frac{s}{2x^{l+1}}}$$

exactement jusqu'à $\frac{1}{x^{n+2}}$, à moins qu'elle ne soit égale à l'une des réduites de la fraction continue qui résulte du développement de $\sqrt{\frac{x-b}{x-a}}\, e^{\frac{s}{2x^{l+1}}}$. De plus, comme cette réduite doit représenter $\sqrt{\frac{x-b}{x-a}}\, e^{\frac{s}{2x^{l+1}}}$ exactement jusqu'à $\frac{1}{x^{n+2}}$, et que son dénominateur ne sera pas de degré plus élevé que $\varphi(x)$ ou $x^{\frac{n}{2}}$, la réduite qui vient après elle doit avoir un dénominateur de degré supérieur à $\frac{n}{2}+1$. D'où l'on voit que parmi les réduites de la fraction continue, résultant de $\sqrt{\frac{x-b}{x-a}}\, e^{\frac{s}{2x^{l+1}}}$, celle qui détermine la valeur de $\frac{\psi(x)}{\varphi(x)}$ est la dernière avec un dénominateur de degré inférieur à $\frac{n}{2}+1$. D'après cela, dès qu'on connaîtra la valeur de s, on trouvera la fraction $\frac{\psi(x)}{\varphi(x)}$, et par là, les fonctions $\psi(x)$, $\varphi(x)$, dépourvues de leur commun diviseur. Mais en ayant égard à la composition des formules (1), (4(, (5), (6), on voit que tous les facteurs communs des fonctions $\varphi(x)$, $\psi(x)$ ne donnent naissance qu'aux valeurs η_1, η_2, $\eta_3 \ldots$, égales deux à deux, et de telles valeurs de η_1, η_2, $\eta_3 \ldots$, dans les formules (1), (4), (5), ne produisent que des termes identiquement nuls.

Ainsi l'on s'assure qu'en dénotant par

$$\frac{P_{\frac{n}{2}}}{Q_{\frac{n}{2}}}$$

la dernière des réduites de $\sqrt{\frac{x-b}{x-a}}\, e^{\frac{s}{2x^{l+1}}}$ dont le dénominateur est de degré inférieur à $\frac{n}{2}+1$, et en faisant abstraction des facteurs communs des fonctions $\psi(x)$, $\varphi(x)$ qui n'ont aucune influence sur la composition de nos formules définitives, on aura

$$\psi(x) = C_0 P_{\frac{n}{2}}, \quad \varphi(x) = C_0 Q_{\frac{n}{2}},$$

où C_0 est une constante, et par là, en vertu de (6), les quantités

$$\eta_1, \eta_3, \ldots,$$
$$\eta_2, \eta_4, \ldots.$$

seront déterminées par la résolution des équations

$$Q_{\frac{n}{2}} = 0,$$
$$P_{\frac{n}{2}} = 0.$$

Comme les quantités

$$\eta_1,\ \eta_3, \ldots\ldots,$$
$$\eta_2,\ \eta_4, \ldots\ldots$$

présentent une série croissante, en disposant les racines des équations

$$Q_{\frac{n}{2}} = 0,\ P_{\frac{n}{2}} = 0$$

par ordre de grandeur, on trouvera deux suites de termes respectivement égaux à

$$\eta_1,\ \eta_3 \ldots\ldots,$$
$$\eta_2,\ \eta_4, \ldots\ldots$$

§ 5. D'après ce que nous avons montré, on trouvera facilement les quantités $\eta_1, \eta_2, \eta_3, \eta_4, \ldots\ldots$, dès qu'on connaîtra la valeur de la constante s, qui entre dans l'expression $\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}$. C'est la détermination de cette constante qui va nous occuper.

En dénotant par $\frac{P_{\frac{n}{2}+1}}{Q_{\frac{n}{2}+1}}$ celle des réduites de la fraction continue, résultant de $\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}$, qui vient immédiatement après $\frac{P_{\frac{n}{2}+1}}{Q_{\frac{n}{2}+1}} = \frac{\psi(x)}{\varphi(x)}$, nous trouvons que la différence

$$\frac{\psi(x)}{\varphi(x)} - \sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}} = \frac{P_{\frac{n}{2}}}{Q_{\frac{n}{2}}} - \sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}$$

est du même ordre que l'expression

$$\frac{1}{Q_{\frac{n}{2}} \cdot Q_{\frac{n}{2}+1}}.$$

Mais nous avons vu que la fraction $\frac{\psi(x)}{\varphi(x)}$ doit représenter la valeur de $\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}$ avec l'exactitude jusqu'à $\frac{1}{x^{n+2}}$; donc cette expression ne peut être de degré supérieur à $-(n+2)$, et par conséquent, la fonction $Q_{\frac{n}{2}+1}$, dénominateur de la réduite qui vient après $\frac{P_{\frac{n}{2}}}{Q_{\frac{n}{2}}} = \frac{\psi(x)}{\varphi(x)}$, devra être d'un degré supérieur à celui de $\frac{x^{n+1}}{Q_{\frac{n}{2}}}$. Or, d'après cela, on peut toujours trouver toutes les valeurs de s satisfaisant à nos équations. En effet, le dénominateur de la fraction

$$\frac{P_{\frac{n}{2}}}{Q_{\frac{n}{2}}} = \frac{\psi(x)}{\varphi(x)},$$

étant tout au plus du degré $\frac{n}{2}$, l'expression $\frac{x^{n+2}}{Q_{\frac{n}{2}}}$ ne peut être que de degré supérieur à $\frac{n}{2}+1$. Donc, la réduite

$$\frac{P_{\frac{n}{2}}}{Q_{\frac{n}{2}}}=\frac{\psi(x)}{\varphi(x)},$$

avec un dénominateur de degré inférieur à $\frac{n}{2}+1$, sera immédiatement suivie de la fraction $\frac{P_{\frac{n}{2}+1}}{Q_{\frac{n}{2}+1}}$, où le dénominateur est de degré supérieur à $\frac{n}{2}+1$. D'où l'on voit que la fraction continue, résultant du développement de $\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}$, n'aura pas de réduite avec un dénominateur du degré $\frac{n}{2}+1$. Mais si l'on trouve toutes les valeurs de s, avec lesquelles l'expression

$$\sqrt{\frac{x-b}{x-a}}\,e^{\frac{s}{2x^{l+1}}}$$

jouit de cette propriété*), en examinant chacune d'elles à part, on distinguera toutes celles qui, conformément à ce que nous avons vu sur la fraction $\frac{P_{\frac{n}{2}+1}}{Q_{\frac{n}{2}+1}}$, rendent le degré de $Q_{\frac{n}{2}+1}$ supérieur á celui de $\frac{x^{n+1}}{Q_{\frac{n}{2}}}$.

Ainsi on parviendra à déterminer les valeurs de s qui correspondent à toutes les solutions possibles de nos équations. Pour choisir parmi elles la valeur s qui résout notre problème, on exclura toutes celles qui conduisent à ses solutions impropres, c.-à-d., où les valeurs

$$-h,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ldots,+h,$$

contre le vrai sens du problème, ne sont pas toutes réelles ou bien ne présentent pas une série croissante. Après cela, la valeur de s, numériquement la plus grande parmi celles qui restent, correspondra, évidemment, à la solution cherchée de notre problème, où il s'agit de rendre la quantité

$$s=-\frac{1}{l+1}\left[a^{l+1}-2\eta_1^{\,l+1}+2\eta_2^{\,l+1}-\ldots-(-1)^{\nu}b^{l+1}\right]$$

aussi grande que possible.

---

*) Dans le Mémoire intitulé: ***Sur les questions de minima qui se rattachent à la représentation approximative des fonctions***, nous avons montré la marche à suivre pour trouver les valeurs d'une constante, déterminée par une condition de ce genre.

En suivant la marche indiquée, on finira toujours par trouver la valeur de s qui résout notre problème et qui détermine, comme nous l'avons vu, toutes les autres inconnues de la formule

$$\int_a^{\eta_1} F(x)dx - \int_{\eta_1}^{\eta_2} F(x)dx + \int_{\eta_2}^{\eta_3} F(x)dx - \ldots\ldots + (-1)^{\nu} \int_{\eta_\nu}^{b} F(x)dx = sA_l.$$

Mais dans plusieurs cas particuliers la détermination de s se simplifie notablement; car souvent la série des valeurs parmi lesquelles on cherchera celle qui résout nôtre problème, se réduira à un seul terme qui ne pourra être que la valeur cherchée de s. — Remarquons encore que dans toutes ces recherches on pourra faire abstraction des valeurs imaginaires de s qui ne sont pas conformes au sens du problème.

Le nombre n est impair.

§ 6. Dans ce cas la formule (7) devient

$$\frac{\psi(x)}{\varphi(x)} = \frac{1}{\sqrt{(x-a)(x-b)}}\, e^{\frac{s}{2x^{l+1}} - \frac{s'}{2(n+2)x^{n+2}} - \frac{s''}{2(n+3)x^{n+3}} - \cdots\cdots\cdots}$$

ce qui nous donne

$$\frac{\psi(x)}{\varphi(x)} - \frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}} = \frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}} \left(e^{-\frac{s'}{2(n+2)x^{n+2}} - \frac{s''}{2(n+3)x^{n+3}} - \cdots} - 1 \right).$$

Cette formule prouve que la fraction

$$\frac{\psi(x)}{\varphi(x)}$$

ne diffère de l'expression

$$\frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}}$$

que par des termes d'un ordre moins élevé que $\frac{1}{x^{n+3}}$, et comme d'après (6), pour n impair, on trouve que $\varphi(x)$, dénominateur de cette fraction, est du degré $\frac{n+1}{2}$, cela suppose qu'elle est égale à l'une des réduites de la fraction continue qu'on obtient par le développement de l'expression

$$\frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}}.$$

De plus, en dénotant par

$$\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}}$$

celle des réduites de la fraction continue, résultant de $\frac{e^{\frac{xs}{2xl+1}}}{\sqrt{(x-a)(x-b)}}$, qui est égale à $\frac{\psi(x)}{\varphi(x)}$, et par

$$\frac{P_{\frac{n+3}{2}}}{Q_{\frac{n+3}{2}}}$$

la réduite qui vient immédiatement après $\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}}$, on trouve que la différence

$$\frac{\psi(x)}{\varphi(x)}-\frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}}=\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}}-\frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}}$$

est du même degré que

$$\frac{1}{Q_{\frac{n+1}{2}}\,Q_{\frac{n+3}{2}}}.$$

D'où, suivant ce que nous avons remarqué relativement à la différence

$$\frac{\psi(x)}{\varphi(x)}-\frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}},$$

il résulte que la fonction $Q_{\frac{n+3}{2}}$ doit être de degré plus élevé que $\frac{x^{n+2}}{Q_{\frac{n+1}{2}}}$. Mais comme $\frac{P_{\frac{n+1}{1}}}{Q_{\frac{n+1}{2}}}$ est égale à la fraction $\frac{\psi(x)}{\varphi(x)}$, mise sous la forme la plus simple, et que $\varphi(x)$ n'est que du degré $\frac{n+1}{2}$, cela nous prouve que $Q_{\frac{n+3}{2}}$ sera de degré supérieur à $\frac{n+3}{2}$. D'où l'on voit que la réduite

$$\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}}=\frac{\psi(x)}{\varphi(x)},$$

dont le dénominateur n'est pas de degré supérieur à $\frac{n+1}{2}$, est suivie immédiatement de la réduite

$$\frac{P_{\frac{n+3}{2}}}{Q_{\frac{n+3}{2}}}$$

avec un dénominateur de degré plus élevé que $\frac{n+3}{2}$. Donc, parmi les réduites de la fraction continue, résultant du devéloppement de l'expression

$$\frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}},$$

la fraction

$$\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}}=\frac{\psi(x)}{\varphi(x)}$$

sera la dernière avec un dénominateur de degré inférieur ou égal à $\frac{n+1}{2}$.

Or, d'après ce que nous venons de montrer sur les réduites

$$\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}},\ \frac{P_{\frac{n+3}{2}}}{Q_{\frac{n+3}{2}}},$$

et en suivant la même marche que dans les §§ 4 et 5, on parvient, relativement à la détermination des quantités

$$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ldots\ldots,\ s,$$

dans le cas de n impair, à ces conclusions:

1) Les quantités

$$\eta_1,\ \eta_3,\ldots\ldots,$$

$$\eta_2,\ \eta_4,\ldots\ldots$$

sont les racines des équations

$$Q_{\frac{n+1}{2}}=0,\ P_{\frac{n+1}{2}}=0,$$

où $P_{\frac{n+1}{2}}$, $Q_{\frac{n+1}{2}}$ désignent les termes de la dernière réduite de la fraction continue, résultant du développement de

$$\frac{e^{\frac{s}{2x^{l+1}}}}{\sqrt{(x-a)(x-b)}},$$

dont le dénominateur $Q_{\frac{n+1}{2}}$ est tout au plus du degré $\frac{n+1}{2}$.

2) On cherchera la valeur de s parmi celles qui ne donnent pas à la fraction continue, résultant du développement de l'expression

$$\frac{e^{\frac{s}{2x^{l+2}}}}{\sqrt{(x-a)(x-b)}},$$

de réduite dont le dénominateur serait du degré $\frac{n+3}{2}$. — Dans la série des valeurs de s qui jouissent de cette propriété, on exclura, en premier lieu, toutes celles avec lesquelles la réduite $\frac{P_{\frac{n+3}{2}}}{Q_{\frac{n+3}{2}}}$ qui vient après $\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}} = \frac{\psi(x)}{\varphi(x)}$ a pour dénominateur une fonction moins élevée que $\frac{x^{n+3}}{Q_{\frac{n+1}{2}}}$, et puis, toutes celles qui, d'après le N° 1, donnent des valeurs

$$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4, \ldots\ldots$$

ne répondant pas à notre problème (Voyez le § 2). Parmi les valeurs restantes celle numériquement la plus grande sera égale à la valeur cherchée de s. Dans tout cela on fera abstraction des valeurs imaginaires de s.

II.

§ 7. Pour montrer l'usage des méthodes exposées, nous allons chercher les coefficients de la fonction

$$F(x) = A_0 + A_1 x + \ldots\ldots + A_n x^n$$

dans les cas de

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

Pour simplifier les calculs, nous supposerons que les valeurs données de $F(x)$ sont comprises entre $x = -h$ et $x = +h$, ce qui revient à prendre dans nos formules

$$a = -h,\ b = +h.$$

Pour ces valeurs de a et b, et en supposant n pair, nous remarquerons (§§ 4, 5) que la détermination du coefficient A_0 se rattache au développement de l'expression

$$\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x}}$$

en fraction continue. Or, au moyen de la méthode ordinaire, on trouve aisément que la fraction continue, résultant de cette expression, a la valeur suivante:

$$1 + \cfrac{s-2h}{2x+h-\frac{1}{2}s + \cfrac{(s-2h)^3+32h^3}{24(s-2h)x + \cfrac{s^6-12hs^5+60h^2s^4-720h^4s^2-2880h^5s+2880h^6}{10[(s-2h)^3+32h^3]x + etc.}}}$$

En examinant la composition de cette fraction continue, on voit que ses trois premiers quotients ne cessent d'être du premier degré, et conséquemment, donnent des réduites avec des dénominateurs respectivement des degrés 0, 1, 2, 3, tant que les quantités

$$s - 2h,$$
$$(s - 2h)^3 + 32h^3,$$
$$s^6 - 12hs^5 + 60h^2s^4 - 720h^4s^2 - 2880h^5s + 2880h^6$$

restent différentes de zéro.

Donc, pour que cela n'ait pas lieu, la quantité s doit vérifier au moins l'une de ces équations:

$$s - 2h = 0,$$
$$(s - 2h)^3 + 32h^3 = 0,$$
$$s^6 - 12hs^5 + 60h^2s^4 - 720h^4s^2 - 2880h^5s + 2880h^6 - 0.$$

D'autre part, en supposant consécutivement que la quantité s vérifie chacune de ces équations, on trouve que la fraction continue, résultant de $\sqrt{\frac{x-h}{x+h}}e^{\frac{s}{2x^{l+1}}}$, dans ces trois hypothèses sur s, devient respectivement

$$1 + \frac{s^2(s-6h)}{48x^3 + \text{etc.}},$$

$$1 + \cfrac{s-2h}{2x+h-\frac{s}{2}+\cfrac{3s^5-30hs^4+80h^2s^3-240h^4s+480h^5}{1920(s-2h)x^3+\text{etc.}}},$$

$$1 + \cfrac{s-2h}{2x+h-\frac{s}{2}+\cfrac{(s-2h)^3+32h^3}{24(s-2h)x+\cfrac{0}{x+\text{etc.}}}}$$

D'où, pour réduites de $\sqrt{\frac{x-h}{x+h}}e^{\frac{s}{2x}}$, on obtient

$$(8)\ldots\ldots\frac{1}{1},\ \frac{48x^3+s^2(s-6h)}{48x^3},\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$(9)\ldots\ldots\frac{1}{1},\ \frac{2x-h+\frac{1}{2}s}{2x+h-\frac{1}{2}s},\ \frac{3840(s-2h)x^4+\ldots\ldots\ldots}{3840(s-2h)x^4+\ldots\ldots\ldots},\ldots\ldots\ldots\ldots\ldots,$$

$$(10)\ldots\ldots\frac{1}{1},\ \frac{2x-h+\frac{1}{2}s}{2x+h-\frac{1}{2}s},\ \frac{48(s-2h)x^2+12(s-2h)^2x+(s-2h)^3+32h^3}{48(s-2h)x^2-12(s-2h)^2x+(s-2h)^3+32h^3},\ \frac{p_4}{q_4},\ \frac{p_5}{q_5},\ldots,$$

en désignant par $\frac{p_4}{q_4}, \frac{p_4}{q_5}, \ldots$ des réduites avec des dénominateurs de degrés supérieurs à 3.

Ainsi nous parvenons à trouver tous les cas, où la fraction continue, résultant de $\sqrt{\frac{x-h}{x+h}}e^{\frac{s}{2x}}$, n'a pas de réduites avec des dénominateurs des degrés 1, 2, 3. D'après cela,

en suivant la marche indiquée dans les §§ 4, 5, il est aisé de trouver la solution de notre problème pour $n = 0, 2, 4$. C'est ce dont nous allons nous occuper.

Cas de $n = 0$.

§ 8. Dans ce cas on doit chercher la valeur de s parmi celles, avec lesquelles la fraction continue, résultant du développement de $\sqrt{\frac{x-h}{x+h}}\,e^{\frac{s}{2x}}$, n'a pas de réduite dont le dénominateur soit du degré $\frac{0}{2} + 1 = 1$. Or, d'après ce que nous avons vu sur les réduites de $\sqrt{\frac{x-h}{x+h}}\,e^{\frac{s}{2x}}$, cela n'a lieu que dans le cas où

$$s - 2h = 0,$$

et comme cette équation ne donne qu'une valeur de s

$$s = 2h,$$

nous concluons sur le champ que c'est elle qui résout notre problème.

Pour trouver les quantités

$$\eta_1, \eta_2, \eta_3, \ldots\ldots,$$

nous chercherons parmi les réduites (8), obtenues dans l'hypothèse

$$s - 2h = 0,$$

celle qui est la dernière avec un dénominateur de degré inférieur à $\frac{0}{2} + 1 = 1$. Comme cette fraction est $\frac{1}{1}$, il s'en suit

$$P_{\frac{n}{2}} = 1,\ Q_{\frac{n}{2}} = 1,$$

et par là on reconnaît que le nombre des quantités

$$\eta_1, \eta_2, \eta_3, \ldots\ldots,$$

qui se déterminent par les équations

$$P_{\frac{n}{2}} = 0,\ Q_{\frac{n}{2}} = 0,$$

se réduit à 0.

Or, en portant dans la formule (1) la valeur trouvée de s et en réduisant la série des valeurs

$$a, \eta_1, \eta_2, \eta_3, \ldots\ldots, b$$

à

$$-h,\ +h,$$

on obtient, pour $n=0$ et $l=0$,

$$\int_{-h}^{+h} F(x)dx = 2hA_0,$$

équation qui se vérifie aisément, en remarquant que dans le cas de $n=0$, la fonction

$$F(x) = A_0 + A_1 x + \ldots\ldots\ldots + A_n x^n$$

devient égale à une constante.

Cas de $n=2$.

§ 9. Si $n=2$, on cherchera la valeur de s parmi celles avec lesquelles la fraction continue, résultant de l'expression

$$\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x}},$$

n'a pas de réduite dont le dénominateur soit du degré $\frac{2}{2}+1=2$. Or, comme nous l'avons vu (§ 7), cela ne peut avoir lieu que dans les cas oú l'une des équations

$$(11)\ldots\ldots\ldots\ldots s-2h=0,\ (s-2h)^3+32h^3=0$$

est satisfaite. Pour choisir parmi les racines de ces équations celle qui résout notre problème, remarquons que dans le cas de

$$s-2h=0,$$

d'après (8), les réduites de la fraction continue, résultant du développement de $\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x}}$, présentent cette série:

$$\frac{1}{1},\ \frac{48x^3+s^2(s-2h)}{48x^3},\ldots\ldots\ldots$$

Parmi ces fractions la dernière avec un dénominateur de degré inférieur à $\frac{2}{2}+1=2$ étant $\frac{1}{1}$, on aura, d'après notre notation, dans la supposition de $s-2h=0$,

$$Q_{\frac{n}{2}}=1,\ Q_{\frac{n}{2}+1}=48x^3,$$

Comme pour ces valeurs de $Q_{\frac{n}{2}}$, $Q_{\frac{n}{2}+1}$, le degré de $Q_{\frac{n}{2}+1}$ n'est pas supérieur à celui de $\frac{x^{2+1}}{Q_{\frac{n}{2}}}$, on conclut (§ 7) que l'équation

$$s-2h=0$$

ne donne pas la valeur de s qui résoudrait notre problème. D'après cela il ne reste qu'à chercher cette valeur parmi les racines de la dernière des équations (11), et comme cette équation n'a qu'une racine réelle

$$s = 2(1 - \sqrt[3]{4})h,$$

nous concluons sur le champ que c'est elle qui correspond à notre problème.

Pour trouver les quantités

$$\eta_1,\ \eta_2,\ \eta_3, \ldots\ldots,$$

remarquons que, dans le cas de

$$(s - 2h)^3 + 32h^3 = 0,$$

les réduites de la fraction continue, résultant du développement de

$$\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x}},$$

sont, comme nous l'avons vu (9),

$$\frac{1}{1},\ \frac{2x - h + \frac{1}{2}s}{2x + h - \frac{1}{2}s},\ \frac{3840(s-2h)x^4 + \ldots\ldots}{3840(s-2h)x^4 + \ldots\ldots}, \ldots\ldots$$

La fraction

$$\frac{2x - h + \frac{1}{2}s}{2x + h - \frac{1}{2}s}$$

étant la dernière parmi elles avec un dénominateur de degré au dessous de $\frac{2}{2} + 1 = 2$, nous concluons qu'on aura

$$P_{\frac{n}{2}} = 2x - h + \tfrac{1}{2}s,\quad Q_{\frac{n}{2}} = 2x + h - \tfrac{1}{2}s.$$

D'après cela, pour la détermination des quantités

$$\eta_1,\ \eta_3, \ldots\ldots,$$
$$\eta_2\ \eta_4, \ldots\ldots,$$

nous obtenons les équations

$$Q_{\frac{n}{2}} = 2x + h - \tfrac{1}{2}s = 0,$$
$$P_{\frac{n}{2}} = 2x - h + \tfrac{1}{2}s = 0.$$

D'où il résulte

$$\eta_1 = -\frac{h}{2} + \tfrac{1}{4}s,\quad \eta_2 = \frac{h}{2} - \tfrac{1}{4}s,$$

et en portant ici la valeur trouvée de s, on a définitivement

$$\eta_1 = -\sqrt[3]{\tfrac{1}{2}}\,h,\quad \eta_2 = +\sqrt[3]{\tfrac{1}{2}}\,h.$$

En vertu de ces valeurs de

$$s,\ \eta_1,\ \eta_2,$$

et en remarquant que dans le cas actuel

$$l = 0,\ a = -h,\ b = +h,$$

la formule (1) nous donne

$$\int_{-h}^{-\sqrt[3]{\frac{1}{2}}h} F(x)dx - \int_{-\sqrt[3]{\frac{1}{2}}h}^{+\sqrt[3]{\frac{1}{2}}h} F(x)dx + \int_{\sqrt[3]{\frac{1}{2}}h}^{h} F(x)dx = 2(1-\sqrt[3]{4})hA_0.$$

Cas de $n = 4$.

§ 10. Nous avons vu (§ 7) que la fraction continue, résultant du développement de l'expression

$$\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x}},$$

n'a pas de réduite avec un dénominateur du degré 3 seulement dans le cas où s remplit l'une des équations

$$(s-2h)^3 + 32h^3 = 0,$$

$$s^6 - 12hs^5 + 60h^2s^4 - 720h^4s^2 - 2880h^5s + 2880h^6 = 0.$$

D'après cela, comme le nombre $\frac{n}{2}+1$, pour $n=4$, devient 3, on cherchera, suivant le § 5, la valeur de s parmi les racines réelles de ces équations.

D'autre part, comme, dans la supposition

$$(s-2h)^3 + 32h^3 = 0,$$

nous avons trouvé que les fractions réduites sont

$$\frac{1}{1},\ \frac{2x-h+\frac{1}{2}s}{2x+h-\frac{1}{2}s},\ \frac{3840(s-2h)x^4+\ldots\ldots}{3840(s-2h)x^4+\ldots\ldots},\ \ldots\ldots\ldots,$$

et que la fraction

$$\frac{2x-h+\frac{1}{2}s}{2x+h-\frac{1}{2}s},$$

la dernière avec le dénominateur de degré inférieur à 3, est suivie de la fraction

$$\frac{3840(s-2h)x^4+\ldots\ldots}{3840(s-2h)x^4+\ldots\ldots},$$

dont le dénominateur n'est pas de degré supérieur à celui de

$$\frac{x^{4+1}}{2x+h-\frac{s}{2}},$$

nous concluons que l'équation

$$(s-2h)^3 + 52h^3 = 0$$

ne saurait donner la valeur cherchée de s, et par conséquent, qu'on doit la chercher parmi les racines réelles de l'équation

$$s^6 - 12hs^5 + 60h^2s^4 - 720h^4s^2 - 2880h^5s + 2880h^6 = 0.$$

Or, en cherchant les racines réelles de cette équation, on trouve que l'une d'elles est comprise entre $s = 6h$ et $s = 7h$, et l'autre entre $s = 0$ et $s = h$. Pour reconnaître parmi ces valeurs de s celle qui se rapporte à notre problème, nous passons aux valeurs de

$$\eta_1,\ \eta_2,\ \eta_3, \ldots\ldots$$

qui en résultent.

Comme dans le cas de

$$s^6 - 12hs^5 + 60h^2s^4 - 720h^4s^2 - 2880hs^5 + 2880h^6 = 0,$$

d'après (10), les réduites de la fraction continue qui résulte de

$$\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x}}$$

sont

$$\frac{1}{1},\ \frac{2x-h+\frac{1}{2}s}{2x+h-\frac{1}{2}s},\ \frac{48(s-2h)x^2+12(s-2h)^2x+(s-2h)^3+32h^3}{48(s-2h)x^2-12(s-2h)x+(s-2h)^3+32h^3},\ \frac{p_4}{q_4}, \ldots\ldots\ldots,$$

et que parmi elles la dernière avec le dénominateur de degré au dessous de $\frac{4}{2}+1=3$, est

$$\frac{48(s-2h)x^2+12(s-2h)x+(s-2h)^3+32h^3}{48(s-2h)x^2-12(s-2h)x+(s-2h)^3+32h^3},$$

nous trouvons, pour la détermination de

$$\eta_1,\ \eta_3, \ldots\ldots,$$

$$\eta_2,\ \eta_4, \ldots\ldots,$$

les équations

$$48(s-2h)x^2 - 12(s-2h)^2x + (s-2h)^3 + 32h^3 = 0,$$

$$48(s-2h)x^2 + 12(s-2h)^2x + (s-2h)^3 + 32h^3 = 0.$$

Or on reconnaît que ces équations n'ont point de solution réelle, si s surpasse $2h$. D'où nous concluons que la racine de l'équation

$$s^6 - 12hs^5 + 60h^2s^4 - 720h^4s^2 - 2880h^5s + 2880h^6 = 0,$$

comprise entre $s = 6h$ et $s = 7h$, ne donne pas de valeurs de

$$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4, \ldots\ldots,$$

propres à la solution de notre problème, et, par conséquent, que c'est son autre racine, comprise entre $s = 0$ et $s = h$, et dont la valeur approchée est $0{,}83446h$, qui correspondra à notre problème.

En portant cette valeur de s dans les équations que nous avons trouvées pour la détermination des quantités

$$\eta_1, \eta_3, \ldots\ldots,$$
$$\eta_2, \eta_4, \ldots\ldots,$$

on a

$$x^2 + 0{,}29138hx - 0{,}54362h^2 = 0,$$
$$x^2 - 0{,}29138hx - 5{,}54362h^2 = 0,$$

et comme les racines de ces équations, disposées par ordre de grandeur, sont

$$-0{,}89725h, \quad +0{,}60587h,$$
$$-0{,}60587h, \quad +0{,}89725h,$$

nous concluons qu'on aura

$$\eta_1 = -0{,}89725h, \quad \eta_3 = 0{,}60587h,$$
$$\eta_2 = -0{,}60587h, \quad \eta_4 = 0{,}89725h.$$

Ainsi nous trouvons les valeurs des quantités

$$s, \eta_1, \eta_2, \eta_3, \eta_4$$

pour $n = 4$ et en prenant

$$l = 0, \; a = -h, \; b = +h.$$

D'après cela la formule (1) nous donne

$$\int_{-h}^{-0,89725h} F(x)dx - \int_{-0,89725h}^{-0,60587h} F(x)dx + \int_{-0,60587h}^{0,60587h} F(x)dx - \int_{0,60587h}^{0,89725h} F(x)dx + \int_{0,89725h}^{h} F(x)dx = 0{,}83446hA_0.$$

Cas de $n = 1, 3, 5$.

§ 11. En cherchant pour ces valeurs de n la solution de notre problème, relatif à la détermination de A_0, et en prenant toujours

$$a = -h, \; b = +h,$$

on parvient définitivement aux formules identiques à celles que nous venons de trouver pour

$$n = 0, 2, 4,$$

respectivement; c'est ce qu'on pouvait prévoir, en remarquant que dans les formules

$$\int_{-h}^{h} F(x)dx = 2hA_0,$$

$$\int_{-h}^{-\sqrt[3]{\frac{1}{2}}h} F(x)dx - \int_{-\sqrt[3]{\frac{1}{2}}h}^{\sqrt[3]{\frac{1}{2}}h} F(x)dx + \int_{\sqrt[3]{\frac{1}{2}}h}^{h} F(x)dx = 2(1-\sqrt[3]{4})hA_0,$$

$$\int_{-h}^{-0{,}89725h} F(x)dx - \int_{-0{,}89725h}^{-0{,}60587h} F(x)dx + \int_{-0{,}60587h}^{0{,}60587h} F(x)dx - \int_{0{,}60587h}^{0{,}89725h} F(x)dx + \int_{0{,}89725h}^{h} F(x)dx = 0{,}83446hA_0,$$

tous les termes de la fonction

$$F(x) = A_0 + A_1x + \ldots\ldots + A_nx^n$$

avec les puissances impaires de x s'évanouissent.

§ 12. En cherchant de la même manière la solution de notre problème pour

$$l = 1,\ n = 1,\ 2,\ 3,\ 4,\ 5,$$

et en supposant toujours

$$a = -h,\ b = +h,$$

nous parvenons définitivement à ces formules:

Cas de $n = 1$ ou 2.

$$\int_{-h}^{0} F(x)dx - \int_{0}^{h} F(x)dx = -h^2A_1.$$

Cas de $n = 3$ ou 4.

$$\int_{-h}^{-\sqrt[4]{\frac{1}{2}}h} F(x)dx - \int_{-\sqrt[4]{\frac{1}{2}}h}^{0} F(x)dx + \int_{0}^{\sqrt[4]{\frac{1}{2}}h} F(x)dx - \int_{\sqrt[4]{\frac{1}{2}}h}^{h} F(x)dx = (\sqrt{2}-1)h^2A_1.$$

Cas de $n = 5$.

$$\int_{-h}^{-0{,}91682h} F(x)dx - \int_{-0{,}91682h}^{-0{,}67418h} F(x)dx + \int_{-0{,}67418h}^{0} F(x)dx - \int_{0}^{0{,}67418h} F(x)dx + \int_{0{,}67418h}^{0{,}91682h} F(x)dx - \int_{0{,}91682h}^{h} F(x)dx = -0{,}82277h^2A_1.$$

De même pour $l=2$ et en supposant successivement $n=2, 3, 4, 5$, nous trouvons:

Cas de $n=2$ **ou** 3.

$$\int_{-h}^{-\frac{1}{2}h} F(x)dx - \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} F(x)dx + \int_{\frac{1}{2}h}^{h} F(x)dx = \tfrac{1}{2}h^3 A_2.$$

Cas de $n=4$ **ou** 5.

$$\int_{-h}^{-0,87305h} F(x)dx - \int_{-0,87305h}^{-0,37305h} F(x)dx + \int_{-0,37305h}^{0,37305h} F(x)dx - \int_{0,37305h}^{0,87305h} F(x)dx + \int_{0,87305h}^{h} F(x)dx = -0,15139h^3 A_2.$$

En prenant $l=3$ et $n=3, 4, 5$, nous obtenons ces formules:

Cas de $n=3$ **ou** 4.

$$\int_{-h}^{-\sqrt{\frac{1}{2}}h} F(x)dx - \int_{-\sqrt{\frac{1}{2}}h}^{0} F(x)dx + \int_{0}^{\sqrt{\frac{1}{2}}h} F(x)dx - \int_{\sqrt{\frac{1}{2}}h}^{h} F(x)dx = -\tfrac{1}{4}h^4 A_3.$$

Cas de $n=5$.

$$\int_{-h}^{-0,89945h} F(x)dx - \int_{-0,89945h}^{-0,55589h} F(x)dx + \int_{-0,55589h}^{0} F(x)dx$$

$$-\int_{0}^{0,55589h} F(x)dx + \int_{0,55589h}^{0,89945h} F(x)dx - \int_{0,89945h}^{h} F(x)dx = 0,05901h^4 A_3.$$

Le cas de $l=4$ et $n=4$ ou 5 nous fournit l'équation

$$\int_{-h}^{-\frac{\sqrt{5}+1}{4}h} F(x)dx - \int_{-\frac{\sqrt{5}+1}{4}h}^{-\frac{\sqrt{5}-1}{4}h} F(x)dx + \int_{-\frac{\sqrt{5}-1}{4}h}^{\frac{\sqrt{5}-1}{4}h} F(x)dx - \int_{\frac{\sqrt{5}-1}{4}h}^{\frac{\sqrt{5}+1}{4}h} F(x)dx + \int_{\frac{\sqrt{5}+1}{4}h}^{h} F(x)dx = \tfrac{1}{8}h^5 A_4.$$

Enfin, pour le cas de $l=5$ et $n=5$, on obtient cette formule:

$$\int_{-h}^{-\frac{\sqrt{3}}{2}h} F(x)dx - \int_{-\frac{\sqrt{3}}{2}h}^{-\frac{1}{2}h} F(x)dx + \int_{-\frac{1}{2}h}^{0} F(x)dx - \int_{0}^{\frac{1}{2}h} F(x)dx + \int_{\frac{1}{2}h}^{\frac{\sqrt{3}}{2}h} F(x)dx - \int_{\frac{\sqrt{3}}{2}h}^{h} F(x)dx = -\tfrac{1}{16}h^6 A_5.$$

§ 13. D'après ce que nous venons de trouver il est facile de composer la table des valeurs de ν, s, η_1, η_2, η_3, η_4, ... dans les cas de

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ 5,$$

et en prenant pour limites de x les valeurs $-h$ et $+h$. Une telle table se trouve à la fin de notre Mémoire, et on verra dans la *Section* IV le parti qu'on peut en tirer pour l'interpolation. Il est désirable que cette table soit prolongée jusqu'à des valeurs de n plus considérables.

III.

§ 14. Dans les paragraphes précédents nous avons donné la méthode générale pour trouver, suivant notre problème, les quantités

$$s,\ \eta_1,\ \eta_2,\ \eta_3, \ldots\ldots$$

dans la formule

$$\int_a^{\eta_1} F(x)dx - \int_{\eta_1}^{\eta_2} F(x)dx + \int_{\eta_2}^{\eta_3} F(x)dx - \ldots\ldots\ldots + (-1)^\nu \int_{\eta_\nu}^{b} F(x)dx = sA_l,$$

quel que soit A_l, coefficient de la fonction

$$F(x) = A_0 + A_1 x + \ldots\ldots + A_n x^n$$

que l'on cherche à déterminer. Nous allons montrer maintenant que cette méthode est susceptible d'une simplification notable dans le cas particulier, où il s'agit de la détermination de A_n, dernier coefficient de la fonction

$$F(x) = A_0 + A_1 x + \ldots\ldots + A_n x^n.$$

Nous verrons que dans ce cas il est aisé de trouver directement les valeurs de

$$s,\ \eta_1,\ \eta_2,\ \eta_3 \ldots\ldots,$$

quel que soit le nombre n, et nous montrerons plus tard comment on peut en tirer une nouvelle formule d'interpolation.

Nous supposerons toujours, pour simplifier nos formules,

$$a = -h,\ b = h,$$

et nous commencerons par le cas de n impair.

Cas de n impair.

§ 15. En faisant dans les formules du § 6

$$a = -h,\ b = +h,$$

nous trouvons que, dans le cas de n impair et $l = n$, les quantités

$$\eta_1,\ \eta_3, \ldots\ldots,$$
$$\eta_2,\ \eta_4, \ldots\ldots$$

se déterminent par les équations

$$Q_{\frac{n+1}{2}} = 0,\ P_{\frac{n+1}{2}} = 0,$$

où $P_{\frac{n+1}{2}}$, $Q_{\frac{n+1}{2}}$ sont les termes de la dernière réduite de

$$\frac{e^{\frac{s}{2x^{n+1}}}}{\sqrt{x^2-h^2}},$$

dont le dénominateur $Q_{\frac{n+1}{2}}$ n'est pas de degré plus élevé que $\frac{n+1}{2}$. D'autre part, comme les expressions

$$\frac{e^{\frac{s}{2x^{n+1}}}}{\sqrt{x^2-h^2}},\quad \frac{1}{\sqrt{x^2-h^2}}$$

ne diffèrent entre elles que par les termes de l'ordre $\frac{1}{x^{n+2}}$ ou moins élevés, et que des termes de ces ordres n'ont aucune influence sur les fractions réduites avec les dénominateurs de degrés inférieurs à $\frac{n+2}{2}$, il est clair que dans la détermination de

$$\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}},$$

suivant la méthode mentionnée, on peut prendre l'expression

$$\frac{1}{\sqrt{x^2-h^2}}$$

au lieu de

$$\frac{e^{\frac{s}{2x^{n+1}}}}{\sqrt{x^2-h^2}}.$$

Or, d'après cela il est aisé de trouver l'expression générale des fonctions $P_{\frac{n+1}{2}}$ et $Q_{\frac{n+1}{2}}$,

et pour y parvenir nous allons chercher la loi de composition de la fraction continue qui résulte du développement de l'expression

$$\frac{1}{\sqrt{x^2-h^2}}.$$

§ 16. En remarquant que le produit des valeurs

$$x-\sqrt{x^2-h^2},\quad x+\sqrt{x^2-h^2}$$

se réduit à h^2, nous concluons qu'on aura

$$x-\sqrt{x^2-h^2}=\frac{h^2}{x+\sqrt{x^2-h^2}};$$

ou, ce qui revient au même,

$$\sqrt{x^2-h^2}-x=-\frac{h^2}{2x+(\sqrt{x^2-h^2}-x)},$$

et par là, au moyen des substitutions successives de

$$-\frac{h^2}{2x+(\sqrt{x^2-h^2}-x)}$$

à la place de

$$\sqrt{x^2-h^2}-x,$$

on trouve

$$\sqrt{x^2-h^2}-x=-\cfrac{h^2}{2x-\cfrac{h^2}{2x+(\sqrt{x^2-h^2}-x)}}$$

$$=-\cfrac{h^2}{2x-\cfrac{h^2}{2x-\cfrac{h^2}{2x-\cfrac{h^2}{2x+(\sqrt{x^2-h^2}-x)}}}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$=-\cfrac{h^2}{2x-\cfrac{h^2}{2x-\cdots\cdots-\cfrac{h^2}{2x-\cfrac{h^2}{2x+(\sqrt{x^2-h^2}-x)}}}}.$$

D'où résulte ce développement de $\frac{1}{\sqrt{x^2-h^2}}$ en fraction continue:

$$\frac{1}{\sqrt{x^2-h^2}}=\cfrac{1}{x-\cfrac{h^2}{2x-\cfrac{h^2}{2x-\cfrac{h^2}{2x-\cdots\cdots}}}}$$

Pour trouver la loi de composition des réduites de cette fraction continue, que nous désignerons par

$$\frac{P^{(0)}}{Q^{(0)}},\ \frac{P^{(1)}}{Q^{(1)}},\ \frac{P^{(2)}}{Q^{(2)}},\ldots\ldots\ldots,$$

remarquons que leurs termes sont liés entre eux par les équations

$$P^{(m+2)}=2x\,P^{(m+1)}-h^2P^{(m)},$$
$$Q^{(m+2}=2x\,P^{(m+1)}-h^2Q^{(m)}.$$

Mais en traitant ces formules, comme les équations aux *différences finies*, on en tire

$$P^{(m)}=C\left(x+\sqrt{x^2-h^2}\right)^m+C_1\left(x-\sqrt{x^2-h^2}\right)^m,$$
$$Q^{(m)}=C_2\left(x+\sqrt{x^2-h^2}\right)^m+C_3\left(x-\sqrt{x^2-h^2}\right)^m,$$

où

$$C,\ C_1,\ C_2,\ C_3$$

sont des valeurs indépendantes du nombre m. Pour déterminer ces quantités nous remarquerons que les valeurs précédentes de $P^{(m)}$, $Q^{(m)}$, pour $m=0$, $m=1$, donnent

$$P^{(0)}=C+C_1,\quad P^{(1)}=C\left(x+\sqrt{x^2-h^2}\right)+C_1\left(x-\sqrt{x^2-h^2}\right),$$
$$Q^{(0)}=C_2+C_3,\quad Q^{(1)}=C_2\left(x+\sqrt{x^2-h^2}\right)+C_3\left(x-\sqrt{x^2-h^2}\right),$$

et comme d'autre part, d'après le développement de $\frac{1}{\sqrt{x^2-h^2}}$ en fraction continue

$$\cfrac{1}{x-\cfrac{h^2}{2x-\cfrac{h^2}{2x-\cdots}}}\ldots,$$

on trouve

$$\frac{P^{(0)}}{Q^{(0)}}=\frac{0}{1},\quad \frac{P^{(1)}}{Q^{(1)}}=\frac{1}{x},$$

il en résulte les équations

$$C+C_1=0,$$
$$C\left(x+\sqrt{x^2-h^2}\right)+C_1\left(x-\sqrt{x^2-h^2}\right)=1,$$
$$C_2+C_3=1,$$
$$C_2\left(x+\sqrt{x^2-h^2}\right)+C_3\left(x-\sqrt{x^2-h^2}\right)=x.$$

Ces équations, étant résolues relativement à C, C_1, C_2, C_3, nous donnent

$$C=\frac{1}{2\sqrt{x^2-h^2}},\quad C_1=-\frac{1}{\sqrt{x^2-h^2}},$$
$$C_2=\tfrac{1}{2},\quad C_3=\tfrac{1}{2},$$

et en portant ces valeurs de C, C_1, C_2, C_3 dans les expressions de $P^{(m)}$, $Q^{(m)}$, on trouve définitivement.

$$P^{(m)}=\frac{\left(x+\sqrt{x^2-h^2}\right)^m-\left(x-\sqrt{x^2-h^2}\right)^m}{2\sqrt{x^2-h^2}}$$

$$Q^{(m)}=\frac{\left(x+\sqrt{x^2-h^2}\right)^m+\left(x-\sqrt{x^2-h^2}\right)^m}{2}.$$

Telles sont les valeurs des termes dans les réduites

$$\frac{P^{(0)}}{Q^{(0)}},\ \frac{P^{(1)}}{Q^{(1)}},\ \frac{P^{(2)}}{Q^{(2)}},\ldots\ldots\ldots\frac{P^{(m)}}{Q^{(m)}},\ldots\ldots\ldots$$

de la fraction continue

$$\cfrac{1}{x-\cfrac{h^2}{2x-\cfrac{h^2}{2x-\ldots}}}$$

qui résulte du développement de $\frac{1}{\sqrt{x^2-h^2}}$.

§ 17. D'après cela on voit que la dernière de ces réduites, dont le dénominateur n'est pas de degré plus élevé que $\frac{n+1}{2}$, a pour termes les fonctions

$$\frac{\left(x+\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}-\left(x-\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}}{2\sqrt{x^2-h^2}},$$

$$\frac{\left(x+\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}+\left(x-\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}}{2}.$$

D'où, en vertu de ce que nous avons vu sur la détermination de

$$\eta_1,\ \eta_3,\ldots\ldots,$$
$$\eta_2,\ \eta_4,\ldots\ldots,$$

il suit que ces quantités sont les racines des équations

$$(12)\ldots\ldots\ldots\ldots\ldots\left\{\begin{array}{l}\dfrac{\left(x+\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}+\left(x-\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}}{2}=0,\\[2ex] \dfrac{\left(x+\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}-\left(x-\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}}{2\sqrt{x^2-h^2}}=0.\end{array}\right.$$

Quant à la solution de ces équations, on y parvient très aisément, en remarquant que si l'on fait

$$\frac{x}{h} = \cos\varphi\,,$$

elles deviennent

$$\cos\frac{n+1}{2}\varphi = 0, \quad \frac{\sin\frac{n+1}{2}\varphi}{\sin\varphi} = 0\,,$$

ce qu'on vérifie en général, en prenant

$$\varphi = \frac{(2k+1)\pi}{n+1}\,, \quad \varphi = \frac{2l\pi}{n+1}\,,$$

où k et l sont des nombres entiers, dont le dernier ne doit pas être divisible par $\frac{n+1}{2}$. D'après cela, en faisant successivement

$$k = \frac{n-1}{2},\ k = \frac{n-3}{2}, \ldots\ldots\ldots k = 2,\ k = 1,\ k = 0,$$

$$l = \frac{n-1}{2},\ l = \frac{n-3}{2}, \ldots\ldots\ldots l = 2,\ l = 1,$$

on trouve pour les racines des équations (12), disposées suivant leur grandeur, et, par conséquent, pour les quantités cherchées

$$\eta_1,\ \eta_3, \ldots\ldots,$$

$$\eta_2,\ \eta_4, \ldots\ldots,$$

les expressions suivantes:

$$\eta_1 = h\cos\frac{n}{n+1}\pi,\ \eta_3 = h\cos\frac{n-2}{n+1}\pi, \ldots\ldots\ldots \eta_{n-2} = h\cos\frac{3\pi}{n+1},\ \eta_n = h\cos\frac{\pi}{n+1},$$

$$\eta_2 = h\cos\frac{n-1}{n+1}\pi,\ \eta_4 = h\cos\frac{n-3}{n+1}\pi, \ldots\ldots\ldots \eta_{n-1} = h\cos\frac{2\pi}{n+1}.$$

§ 18. En passant à la détermination de la quantité s, remarquons que, d'après le § 6, on doit chercher sa valeur parmi celles avec lesquelles l'expression

$$\frac{e^{\frac{s}{2x^{n+1}}}}{\sqrt{x^2-h^2}}$$

ne diffère de la réduite

$$\frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}},$$

que par les termes de l'ordre inférieur à $-(n+2)$, ce qui suppose l'équation

$$\lim. \left[\left(\frac{e^{\frac{s}{2x^{n+1}}}}{\sqrt{x^2-h^2}} - \frac{P_{\frac{n+1}{2}}}{Q_{\frac{n+1}{2}}} \right) x^{n+2} \right]_{x=\infty} = 0.$$

Mais comme nous avons trouvé

$$P_{\frac{n+1}{2}} = \frac{(x+\sqrt{x^2-h^2})^{\frac{n+1}{2}} - (x-\sqrt{x^2-h^2})^{\frac{n+1}{2}}}{2\sqrt{x^2-h^2}},$$

$$Q_{\frac{n+1}{2}} = \frac{(x+\sqrt{x^2-h^2})^{\frac{n+1}{2}} + (x-\sqrt{x^2-h^2})^{\frac{n+1}{2}}}{2},$$

et que

$$e^{\frac{s}{2x^{n+1}}} = 1 + \frac{s}{2x^{n+1}} + \frac{s^2}{8x^{2n+2}} + \ldots\ldots\ldots,$$

cette équation devient

$$\lim. \left[\left(1 + \frac{s}{2x^{n+1}} + \frac{s^2}{8x^{2n+2}} + \ldots - \frac{(x+\sqrt{x^2-h^2})^{\frac{n+1}{2}} - (x-\sqrt{x^2-h^2})^{\frac{n+1}{2}}}{(x+\sqrt{x^2-h^2})^{\frac{n+1}{2}} + (x-\sqrt{x^2-h^2})^{\frac{n+1}{2}}} \right) \frac{x^{n+2}}{\sqrt{x^2-h^2}} \right]_{x=\infty} = 0.$$

D'où, en remarquant que

$$1 - \frac{(x+\sqrt{x^2-h^2})^{\frac{n+1}{2}} - (x-\sqrt{x^2-h^2})^{\frac{n+1}{2}}}{(x+\sqrt{x^2-h^2})^{\frac{n+1}{2}} + (x-\sqrt{x^2-h^2})^{\frac{n+1}{2}}} = \frac{2h^{n+1}}{(x+\sqrt{x^2-h^2})^{n+1} + h^{n+1}},$$

$$\lim. \left[\left(\frac{s^2}{8x^{2n+2}} + \ldots\ldots \right) \frac{x^{n+2}}{\sqrt{x^2-h^2}} \right]_{x=\infty} = 0,$$

$$\lim. \left[\frac{s}{2x^{n+1}} \; \frac{x^{n+2}}{\sqrt{x^2-h^2}} \right]_{x=\infty} = \frac{s}{2},$$

on obtient

$$\frac{s}{2} + \lim. \left(\frac{2h^{n+1}}{(x+\sqrt{x^2-h^2})^{n+1} + h^{n+1}} \cdot \frac{x^{n+2}}{\sqrt{x^2-h^2}} \right)_{x=\infty} = 0,$$

et comme l'expression

$$\frac{2h^{n+1}}{(x+\sqrt{x^2-h^2})^{n+1} + h^{n+1}} \cdot \frac{x^{n+2}}{\sqrt{x^2-h^2}},$$

pour $x=\infty$, se réduit à $\frac{h^{n+1}}{2^n}$, il en résulte cette valeur de s:

$$s = -\frac{h^{n+1}}{2^{n-1}}.$$

Ainsi dans le cas de n impair et en prenant

$$a = -h,\ b = h,\ l = n,$$

on parvient directement aux valeurs des quantités

$$s,\ \eta_1,\ \eta_2, \ldots\ldots \eta_n,$$

qui, d'après la formule (1), nous donnent

$$\int_{-h}^{\eta_1} F(x)dx - \int_{\eta_1}^{\eta_2} F(x)dx + \ldots\ldots\ldots + (-1)^n \int_{\eta_n}^{h} F(x)dx = sA_n.$$

Cas de n pair.

§ 19. Dans ce cas, en cherchant la valeur de

$$\frac{P_{\frac{n}{2}}}{Q_{\frac{n}{2}}},$$

dernière réduite de la fraction continue, résultant de $\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x^{n+1}}}$, avec le dénominateur $Q_{\frac{n}{2}}$ de degré inférieur à $\frac{n}{2}+1$, on peut prendre l'expression

$$\sqrt{\frac{x-h}{x+h}}$$

au lieu de

$$\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x^{n+1}}}$$

qui n'en diffère que par les puissances de x, inférieures à $\frac{1}{x^n}$, et comme les termes des réduites de la fraction continue, résultant du développement de

$$\sqrt{\frac{x-h}{x+h}},$$

s'expriment*) par les formules

$$\frac{\left(\sqrt{\frac{x+h}{2}}+\sqrt{\frac{x-h}{2}}\right)^{2\lambda+1}+\left(\sqrt{\frac{x+h}{2}}-\sqrt{\frac{x-h}{2}}\right)^{2\lambda+1}}{2\sqrt{\frac{x+h}{2}}},$$

$$\frac{\left(\sqrt{\frac{x+h}{2}}+\sqrt{\frac{x-h}{2}}\right)^{2\lambda+1}-\left(\sqrt{\frac{x+h}{2}}-\sqrt{\frac{x-h}{2}}\right)^{2\lambda+1}}{2\sqrt{\frac{x-h}{2}}},$$

*) Voyez notre Mémoire *Sur les questions de minima qui se rattachent à la représentation approximative des fonctions* (§ 57).

il en résulte pour $P_{\frac{n}{2}}$, $Q_{\frac{n}{2}}$ les valeurs suivantes:

$$P_{\frac{n}{2}}=\frac{\left(\sqrt{\frac{x+h}{2}}+\sqrt{\frac{x-h}{2}}\right)^{n+1}+\left(\sqrt{\frac{x+h}{2}}-\sqrt{\frac{x-h}{2}}\right)^{n+1}}{2\sqrt{\frac{x+h}{2}}},$$

$$Q_{\frac{n}{2}}=\frac{\left(\sqrt{\frac{x+h}{2}}+\sqrt{\frac{x-h}{2}}\right)^{n+1}-\left(\sqrt{\frac{x+h}{2}}-\sqrt{\frac{x-h}{2}}\right)^{n+1}}{2\sqrt{\frac{x-h}{2}}}.$$

En vertu de ces valeurs de $P_{\frac{n}{2}}$, $Q_{\frac{n}{2}}$, nous concluons, suivant le § 4, que les quantités cherchées

$$\eta_1,\ \eta_3, \ldots\ldots,$$
$$\eta_2,\ \eta_4, \ldots\ldots$$

sont les racines des équations

$$\frac{\left(\sqrt{\frac{x+h}{2}}+\sqrt{\frac{x-h}{2}}\right)^{n+1}-\left(\sqrt{\frac{x+h}{2}}-\sqrt{\frac{x-h}{2}}\right)^{n+1}}{2\sqrt{\frac{x-h}{2}}}=0,$$

$$\frac{\left(\sqrt{\frac{x+h}{2}}+\sqrt{\frac{x-h}{2}}\right)^{n+1}+\left(\sqrt{\frac{x+h}{2}}-\sqrt{\frac{x-h}{2}}\right)^{n+1}}{2\sqrt{\frac{x+h}{2}}}=0.$$

Pour résoudre ces équations, on fera, comme dans le cas précédent,

$$\frac{x}{h}=\cos\varphi,$$

d'après quoi elles deviennent

$$\frac{\sin\frac{n+1}{2}\varphi}{\sqrt{1-\cos\varphi}}=0,\quad \frac{\cos\frac{n+1}{2}\varphi}{\sqrt{1+\cos\varphi}}=0,$$

ce qu'on vérifie, en prenant respectivement

$$\varphi=\frac{2l\pi}{n+1},\quad \varphi=\frac{(2k+1)\pi}{n+1},$$

où les nombres entiers l, $2k+1$ ne doivent pas être divisibles par $n+1$.

Les racines des équations que nous avons obtenues pour la détermination des quantités

$$\eta_1,\ \eta_3, \ldots\ldots,$$
$$\eta_2,\ \eta_4, \ldots\ldots,$$

s'expriment donc ainsi:

$$h\cos\frac{n\pi}{n+1},\ h\cos\frac{(n-2)\pi}{n+1}, \ldots\ldots, h\cos\frac{2\pi}{n+1},$$
$$h\cos\frac{(n-1)\pi}{n+1},\ h\cos\frac{(n-3)\pi}{n+1}, \ldots\ldots, h\cos\frac{\pi}{n+1},$$

et comme ces racines sont disposées par ordre de grandeur, elles sont respectivement égales à

$$\eta_1,\ \eta_3, \ldots\ldots,$$
$$\eta_2,\ \eta_4 \ldots\ldots$$

D'où l'on voit que les quantités

$$\eta_1,\ \eta_2,\ \eta_3,\ \eta_4, \ldots\ldots$$

s'expriment par les mêmes formules que dans le cas de n impair.

§ 20. Pour trouver la quantité s, nous remarquerons que, d'après le § 5, elle doit remplir cette condition

$$lim. \left[\left(\sqrt{\frac{x-h}{x+h}}\, e^{\frac{s}{2x^{n+1}}} - \frac{P_{\frac{n}{2}}}{Q_{\frac{n}{2}}} \right) x^{n+1} \right]_{x=\infty} = 0.$$

Or, en substituant les valeurs trouvées de $P_{\frac{n}{2}}$, $Q_{\frac{n}{2}}$, et en développant $e^{\frac{s}{2x^{n+1}}}$ en série, on a

$$lim. \left[\sqrt{\frac{x-h}{x+h}} \left\{ \begin{array}{l} 1 + \frac{s}{2x^{n+1}} + \frac{s^2}{8x^{2n+2}} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ - \dfrac{\left(\sqrt{\frac{x+h}{2}} + \sqrt{\frac{x-h}{2}}\right)^{n+1} + \left(\sqrt{\frac{x+h}{2}} - \sqrt{\frac{x-h}{2}}\right)^{n+1}}{\left(\sqrt{\frac{x+h}{2}} + \sqrt{\frac{x-h}{2}}\right)^{n+1} - \left(\sqrt{\frac{x+h}{2}} - \sqrt{\frac{x-h}{2}}\right)^{n+1}} \end{array} \right\} x^{n+1} \right]_{x=\infty} = 0,$$

et comme

$$lim. \left[\sqrt{\frac{x-h}{x+h}} \left(1 - \frac{\left(\sqrt{\frac{x+h}{2}} + \sqrt{\frac{x-h}{2}}\right)^{n+1} + \left(\sqrt{\frac{x+h}{2}} - \sqrt{\frac{x-h}{2}}\right)^{n+1}}{\left(\sqrt{\frac{x+h}{2}} + \sqrt{\frac{x-h}{2}}\right)^{n+1} - \left(\sqrt{\frac{x+h}{2}} - \sqrt{\frac{x-h}{2}}\right)^{n+1}} \right) x^{n+1} \right]_{x=\infty} = -\frac{h^{n+1}}{2^n},$$

$$lim. \left[\sqrt{\frac{x-h}{x+h}} \cdot \frac{s}{2x^{n+1}}\, x^{n+1} \right]_{x=\infty} = \frac{s}{2},$$

$$lim. \left[\sqrt{\frac{x-h}{x+h}} \left(\frac{s^2}{8x^{2n+2}} + \ldots\ldots \right) x^{n+1} \right]_{x=\infty} = 0,$$

il en résulte

$$s = \frac{h^{n+1}}{2^{n-1}}.$$

Cette expression de s ne diffère de celle du cas de n impair que par son signe. Or il est aisé de remarquer qu'on embrassera ces deux cas, en introduisant dans la

valeur de s le facteur $(-1)^n$ qui se réduit à $+1$ ou -1, suivant que n est pair ou impair. Ainsi on obtient pour s cette expression:

$$s = (-1)^n \frac{h^{n+1}}{2^{n-1}}$$

qui subsistera pour toutes les valeurs de n.

IV.

§ 21. Bien que le problème actuel ne se présente point dans la pratique, où les valeurs connues de la fonction cherchée ne sont jamais en nombre infini, les formules que nous avons trouvées, en partant de cette hypothèse, sont d'une application utile, comme nous allons le montrer.

Tant qu'on connait la fonction

$$F(x) = A_0 + A_1 x + \ldots\ldots + A_n x^n,$$

pour toutes les valeurs de x, depuis $x = x_1$ jusqu'à $x = x_l$, et qu'on les considère comme équidistantes et infiniment rapprochées, on parvient à tirer, par la seule voie d'addition et de soustraction, les valeurs des coefficients $A_1, A, \ldots\ldots A_n$, pourvus de facteurs aussi grands que possible. Ces expressions qui déterminent les coeffients

$$A_0,\ A_1, \ldots\ldots A_n$$

seront représentées, comme nous l'avons vu (§ 1), par la formule

$$\int_{x_1}^{\eta_1} F(x)dx - \int_{\eta_1}^{\eta_2} F(x)dx + \ldots\ldots + (-1)^{\upsilon} \int_{\eta_\upsilon}^{x_l} F(x)dx.$$

D'après cela toute la difficulté de la détermination des coefficients

$$A_0,\ A_1, \ldots\ldots A_n$$

se réduit à l'évaluation des intégrales

$$\int_{x_1}^{\eta_1} F(x)dx,\ \int_{\eta_1}^{\eta_2} F(x)dx, \ldots\ldots \int_{\eta_\upsilon}^{x_l} F(x)dx.$$

Or, comme ces intégrales, avec une approximation plus on moins grande, peuvent être évaluées au moyen d'un nombre limité des valeurs de $F(x)$, il est facile de comprendre qu'on peut bien profiter de ces expressions déterminant les coefficients de

$$F(x) = A_0 + A_1 x + \ldots\ldots + A_n x^n,$$

tant qu'on a un nombre suffisant de valeurs de $F(x)$, à l'aide desquelles les intégrales

$$\int_{x_1}^{\eta_1} F(x)dx, \int_{\eta_1}^{\eta_2} F(x)dx, \ldots\ldots \int_{\eta_\upsilon}^{x_i} F(x)dx$$

sont évaluables avec une approximation suffisante.

§ 22. Quant à l'évaluation des intégrales

$$\int_{x_1}^{\eta_1} F(x)dx, \int_{\eta_1}^{\eta_2} F(x)dx, \ldots\ldots \int_{\eta_\upsilon}^{x_i} F(x)dx,$$

qu'on aura à faire dans les applications de nos formules, cela ne présente aucune difficulté. Pour y parvenir plus aisément, on n'a qu'à remarquer que les intégrales

$$\int_{x_1}^{\eta_1} F(x)dx, \int_{\eta_1}^{\eta_2} F(x)dx, \ldots\ldots \int_{\eta_\upsilon}^{x_i} F(x)dx$$

désignent respectivement les aires de la courbe

$$y = F(x),$$

comprises entre $x = x_1$ et $x = \eta_1$, $x = \eta_1$ et $x = \eta_2, \ldots\ldots x = \eta_\upsilon$ et $x = x_i$, et que chacune des valeurs données de $F(x)$ détermine l'un des points de cette courbe. Ainsi l'évaluation des intégrales en question se réduit à ce problème de géométrie:

Etant donnée une suite de points, déterminer pour la courbe, passant par ces points, les aires comprises entre des limites données.

Or un tel problème est susceptible d'une solution approchée, qu'on trouve aisément. Si l'on a la représentation graphique de la courbe

$$y = F(x),$$

construite d'après les valeurs connues de $F(x)$, on trouvera ces aires directement à l'aide du planimètre. Dans le cas contraire, on pourra trouver ces aires à l'aide d'un calcul très simple, en prenant pour la courbe le polygone déterminé par les points donnés. Ainsi, en supposant que les valeurs connues de $F(x)$ sont

$$F(x_1),\ F(x_2), \ldots\ldots F(x_\sigma),\ F(x_{\sigma+1}), \ldots\ldots F(x_\tau),\ F(x_{\tau+1}), \ldots\ldots,$$

et que les quantités

$$x = x_0,\ x = X$$

sont comprises respectivement entre x_σ et $x_{\sigma+1}$, x_τ et $x_{\tau+1}$, on trouve que l'aire de la courbe

$$y = F(x),$$

entre $x = x_0$ et $x = X$, s'exprime approximativement par cette formule très simple:

$$(13)\ldots\left\{\begin{aligned} &\frac{(x_{\sigma+1}-x_0)^2F(x_\sigma)-(x_\sigma-x_0)^2F(x_{\upsilon+1})}{2(x_{\sigma+1}-x_\sigma)}+\tfrac{1}{2}(x_{\sigma+2}-x_\sigma)F(x_{\sigma+1})+\tfrac{1}{2}(x_{\sigma+3}-x_{\sigma+1})F(x_{\sigma+2})\\ &+\ldots\ldots+\tfrac{1}{2}(x_{\tau+1}-x_{\tau-1})F(x_\tau)-\frac{(x_{\tau+1}-X)^2F(x_\tau)-(x_\tau-X)^2F(x_{\tau+1})}{2(x_{\tau+1}-x_\tau)}.\end{aligned}\right.$$

Pour donner une idée nette du degré de précision de cette formule, remarquons que la différence entre l'aire de la courbe et celle du polygone, entre les limites $x = x_0$ et $x = X$, est égale à

$$(14)\ldots\ldots -\frac{N}{12}\left[\begin{aligned} &(x_{\sigma+1}-x_\sigma)^3+(x_{\sigma+2}-x_{\sigma+1})^3+\ldots\ldots+(x_\tau-x_{\tau-1})^3\\ &+(X-x_\tau)^2(3x_{\tau+1}-x_\tau-2X)-(x_0-x_\sigma)^2(3x_{\sigma+1}-x_\sigma-2x_0)\end{aligned}\right],$$

N étant une moyenne des valeurs de $\frac{d^2F(x)}{dx^2}$ entre $x = x_0$ et $x = X$.

§ 23. Avec la formule (13) que nous venons de mentionner, on trouve aisément la valeur approchée des expressions de la forme

$$\int_{x_1}^{\eta_1} F(x)dx - \int_{\eta_1}^{\eta_2} F(x)dx + \ldots\ldots + (-1)^\upsilon \int_{\eta_\upsilon}^{x_i} F(x)dx,$$

d'après les valeurs connues de $F(x)$

$$F(x_1),\ F(x_2), \ldots\ldots F(x_i).$$

Pour y parvenir on commencera par chercher dans la suite

$$x_1,\ x_2, \ldots\ldots x_i$$

les couples des termes qui sont respectivement les plus proches des quantités

$$\eta_1,\ \eta_2,\ \eta_3, \ldots\ldots \eta_{\upsilon-1},\ \eta_\upsilon.$$

En désignant ces termes par

$$x_{i'},\ x_{i'+1},\ x_{i''},\ x_{i''+1},\ x_{i'''},\ x_{i'''+1}, \ldots\ldots x_{i^{(\upsilon+1)}},\ x_{i^{(\upsilon-1)}+1},\ x_{i^{(\upsilon)}},\ x_{i^{(\upsilon)}+1},$$

on trouvera que les quantités

$$x_1,\ \eta_1,\ \eta_2, \ldots\ldots \eta_\upsilon,\ x_i$$

d'après les valeurs connues de $F(x)$

$$F(x_1),\ F(x_2),\ldots\ldots,F(x_i),$$

tous les coefficients de la fonction

$$F(x)=A_0+A_1x+\ldots\ldots+A_nx^n$$

seront donnés par la même formule

$$(16)\left\{\begin{aligned} sA_l=\tfrac{1}{2}&\left[\begin{aligned}&M_1+M_2+\ldots\ldots+M_{i'}-M_{i'+1}-M_{i'+2}-\ldots\ldots-M_{i''}+M_{i''+1}+\ldots\ldots+M_{i'''}-M_{i'''+1}-\ldots\ldots\\ &-\ldots\ldots-(-1)^{\upsilon}M_{i^{(\upsilon-1)}+1}-\ldots\ldots-(-1)^{\upsilon}M_{i^{(\upsilon)}}+(-1)^{\upsilon}M_{i^{(\upsilon)}+1}+\ldots\ldots+(-1)^{\upsilon}M_i\end{aligned}\right]\\ &-\frac{(x_{i'+1}-\eta_1)^2F(x_{i'})-(x_{i'}-\eta_1)^2F(x_{i'+1})}{x_{i'+1}-x_{i'}}+\frac{(x_{i''+1}-\eta_2)^2F(x_{i''})-(x_{i''}-\eta_2)^2F(x_{i''+1})}{x_{i''+1}-x_{i''}}\\ &-\ldots\ldots\ldots\ldots\ldots\ldots+(-1)^{\upsilon}\frac{(x_{i^{(\upsilon)}+1}-\eta_\upsilon)^2F(x_{i^{(\upsilon)}})-(x_{i^{(\upsilon)}}-\eta_\upsilon)^2F(x_{i^{(\upsilon)}+1})}{x_{i^{(\upsilon)}+1}-x_{i^{(\upsilon)}}},\end{aligned}\right.$$

en prenant pour

$$s,\ \eta_1,\ \eta_2,\ldots\ldots\eta_\upsilon$$

les valeurs qu'on obtient dans les suppositions de

$$l=0,\ 1,\ 2,\ldots\ldots n,$$

et pour

$$x_{i'},\ x_{i'+1},\ x_{i''},\ x_{i'''},\ldots\ldots x_{i^{(\upsilon)}},\ x_{i^{(\upsilon)}+1}$$

les termes de la suite

$$x_1,\ x_2,\ x_3,\ldots\ldots x_i$$

les plus proches respectivement de

$$\eta_1,\ \eta_2,\ldots\ldots\eta_\upsilon.$$

§ 25. A l'aide de la méthode, donnée dans les §§ 4, 5, 6, on trouvera toujours les quantités

$$s,\ \eta_1,\ \eta_2,\ldots\ldots\eta_\upsilon$$

qui entrent dans la formule (16).

Mais dans les cas ordinaires de la pratique, où la fonction

$$F(x)=A_0+A_1x+\ldots\ldots+A_nx^n$$

reste de degré inférieur à 6, on peut, avec le secours de la table, jointe à notre Mémoire, s'épargner la peine de chercher ces quantités.

Cette table contient les solutions de notre problème dans les cas de

$$n=0,\ 1,\ 2,\ 3,\ 4,\ 5,$$

et en prenant pour limites de x les valeurs

$$-h,\ +h.$$

Donc, toutes les fois que, dans la suite des valeurs données de $F(x)$

$$F(x_1),\ F(x_2), \ldots \ldots F(x_i),$$

on aura

$$x_1 = -x_i,$$

et que n, dans l'expression de

$$F(x) = A_0 + A_1 x + \ldots \ldots + A_n x^n,$$

ne surpassera pas 5, les quantités

$$s,\ \eta_1,\ \eta_2, \ldots \ldots \eta_\nu$$

pourront être déterminées par notre table, en prenant

$$h = x_i.$$

De plus, il n'est pas difficile de remarquer que si l'on cherche, au moyen de notre formule (16), les coefficients du développement de $F(x)$ suivant les puissances de

$$x - \frac{x_1 + x_i}{2},$$

quellesque soient les valeurs de x_i et x_1, les quantités

$$s,\ \eta_1,\ \eta_2, \ldots \ldots \eta_\nu,$$

dans les cas de

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ 5,$$

seront aussi données par cette table, et que pour cela on doit prendre

$$h = \frac{x_i - x_1}{2}.$$

En effet, si l'on pose

$$x - \frac{x_1 + x_i}{2} = X,$$

et que l'on cherche, d'après (16), les coefficients de la fonction

$$F(x) = F\left(\frac{x_1 + x_i}{2} + X\right) = K_\nu + K_1 X + \ldots \ldots + K_n X^n,$$

en supposant connuées ses valeurs pour

$$X = X_1,\ X_2, \ldots \ldots X_i,$$

correspondantes à celles de

$$x = x_1,\ x_2 \ldots \ldots x_i,$$

on trouve, pour la détermination des coefficients

$$K_0,\ K_1, \ldots \ldots K_n,$$

ce système de formules:

$$(17)\ldots sK_l=\tfrac{1}{2}\left\{\begin{array}{l} M_1+M_2+\ldots+M_{i'}-M_{i'+1}-M_{i'+2}-\ldots\ldots\ldots\ldots\ldots\ldots-M_{i''} \\ +M_{i''+1}+\ldots+M_{i'''}-M_{i'''+1}-\ldots+(-1)^{\nu}M_{i^{(\nu)}+1}+\ldots+(-1)^{\nu}M_i \end{array}\right\}$$

$$-\frac{\left(X_{i'+1}-\eta_1\right)^2F\left(\frac{x_1+x_i}{2}+X_{i'}\right)-\left(X_{i'}-\eta_1\right)^2F\left(\frac{x_1+x_i}{2}+X_{i'+1}\right)}{X_{i'+1}-X_{i'}}$$

$$+\frac{\left(X_{i''+1}-\eta_2\right)^2F\left(\frac{x_1+x_i}{2}+X_{i''}\right)-\left(X_{i''}-\eta_2\right)^2F\left(\frac{x_1+x_i}{2}+X_{i''+1}\right)}{X_{i''+1}-X_{i''}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$-(-1)^{\nu}\frac{\left(X_{i^{(\nu)}+1}-\eta_\nu\right)^2F\left(\frac{x_1+x_i}{2}+X_{i^{(\nu)}}\right)-\left(X_{i^{(\nu)}}-\eta_\nu\right)^2F\left(\frac{x_1+x_i}{2}+X_{i^{(\nu)}+1}\right)}{X_{i^{(\nu)}+1}-X_{i^{(\nu)}}},$$

$$M_1=\left(X_2-X_1\right)F\left(\frac{x_1+x_i}{2}+X_1\right),\ M_2=\left(X_3-X_1\right)F\left(\frac{x_1+x_i}{2}+X_2\right),\ldots\ldots\ldots$$

$$M_{i-1}=\left(X_i-X_{i-2}\right)F\left(\frac{x_1+x_i}{2}+X_{i-1}\right),\ M_i=\left(X_i-X_{i-1}\right)F\left(\frac{x_1+x_i}{2}+X_i\right),$$

où les quantités

$$s,\ \eta_1,\ \eta_2,\ldots\ldots\eta_\nu$$

seront déterminées, en prenant pour les valeurs limites de X celles-ci:

$$X=X_1,\ X=X_i.$$

Or, comme ces quantités, en vertu de l'équation

$$X=x-\frac{x_1+x_i}{2},$$

se réduisent, au signe près, à

$$\frac{x_i-x_1}{2},$$

on voit que, dans le cas de

$$n=0,\ 1,\ 2,\ 3,\ 4,\ 5,$$

ces solutions seront données par notre table, en prenant

$$h=\frac{x_i-x_1}{2}.$$

Quant au cas de $n>5$, suivant ce que nous venons de voir, on trouvera, pour ces valeurs de n, les quantités

$$s,\ \eta_1,\ \eta_2,\ldots\ldots\eta_\nu$$

de la formule (1), à l'aide de la méthode genérale, en prenant les limites de x au signe près égales, ce qui entraine beaucoup de simplification dans la recherche de ces quantités.

Remarquons encore que si l'on remplace la variable

$$X$$

par la valeur

$$x-\frac{x_1+x_i}{2},$$

les formules que nous venons de trouver pour la détermination des coefficients $K_0, K_1, \ldots K_n$, dans le développement de $F(x)$ suivant les puissances de

$$X=x-\frac{x_1+x_i}{2},$$

deviennent

$$(18)\ldots\ldots sK_l=\frac{1}{2}\left\{\begin{array}{l} M_1+M_2+\ldots+M_{i'}-M_{i'+1}-M_{i'+2}-\ldots\ldots\ldots\ldots\ldots\ldots-M_{i''} \\ +M_{i''+1}+\ldots+M_{i'''}-M_{i'''+1}-\ldots+(-1)^{\nu}M_{i^{(\nu)}+1}+\ldots+(-1)^{\nu}M_i \end{array}\right\}$$

$$-\frac{\left(x_{i'+1}-\eta_1-\frac{x_1+x_i}{2}\right)^2F(x_{i'})-\left(x_{i'}-\eta_1-\frac{x_1+x_i}{2}\right)^2F(x_{i'+1})}{x_{i'+1}-x_{i'}}$$

$$+\frac{\left(x_{i''+1}-\eta_2-\frac{x_1+x_i}{2}\right)^2F(x_{i''})-\left(x_{i''}-\eta_2-\frac{x_1+x_i}{2}\right)^2F(x_{i''+1})}{x_{i''+1}-x_{i''}}$$

$$-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(-1)^{\nu}\frac{\left(x_{i^{(\nu)}+1}-\eta_\nu-\frac{x_1+x_i}{2}\right)^2F(x_{i^{(\nu)}})-\left(x_{i^{(\nu)}}-\eta_\nu-\frac{x_1+x_i}{2}\right)^2F(x_{i^{(\nu)}+1})}{x_{i^{(\nu)}+1}-x_{i^{(\nu)}}},$$

$$M_1=(x_2-x_1)F(x_1),\ M_2=(x_3-x_1)F(x_2),\ldots\ldots\ldots\ldots,$$

$$M_{i-1}=(x_i-x_{i-2})F(x_{i-1}),\ M_i=(x_i-x_{i-1})F(x_i).$$

Dans la formule (17) les quantités

$$X_{i'},\ X_{i'+1},\ X_{i''},\ X_{i''+1},\ldots\ldots\ldots X_{i^{(\nu)}},\ X_{i^{(\nu)}+1}$$

étant celles de la suite

$$X_1,\ X_1,\ldots\ldots\ldots X_i$$

qui s'approchent le plus de

$$\eta_1,\ \eta_2,\ldots\ldots\eta_\nu,$$

on voit, d'aprés l'équation

$$X=x-\frac{x_1+x_i}{2},$$

qu'on trouvera les quantités

$$x_{i'},\ x_{i'+1},\ x_{i''},\ x_{i''+1},\ldots\ldots x_{i^{(\nu)}},\ x_{i^{(\nu)}+1}$$

de la formule (18), en cherchant dans la suite

$$x_1,\ x_2,\ldots\ldots x_i$$

les couples des termes respectivement les plus proches de

$$\eta_1+\frac{x_1+x_i}{2},\ \eta_2+\frac{x_1+x_i}{2},\ldots\ldots\eta_\upsilon+\frac{x_1+x_i}{2}.$$

§ 26. Comme les quantités

$$s,\ \eta_1,\ \eta_2,\ldots\ldots\eta_\upsilon,$$

comprises dans la formule (18), pour les cas les plus ordinaires de le pratique

$$n=0,\ 1,\ 2,\ 3,\ 4,\ 5,$$

se trouvent immédiatement par notre table, en prenant $h=\frac{x_i-x_1}{2}$, et que cette table, à l'aide de la méthode donnée dans les §§ 4, 5, 6, peut être facilement étendue jusqu'à la limite plus considérable et au delà des valeurs de n dont la pratique a besoin, la formule (18), déteminant les coefficients du développement de la fonction

$$F(x)=A_0+A_1x+\ldots\ldots+A_nx^n,$$

suivant les puissances de

$$x-\frac{x_1+x_i}{2},$$

est très commode pour la recherche de son expression d'après ses valeurs connues

$$F(x_1),\ F(x_2),\ldots\ldots F(x_i).$$

Cette formule ne donne l'expression de la fonction $F(x)$ qu'approximativement, à cause des erreurs qu'on commet dans les recherches des intégrales, en remplaçant la courbe par un polygone. Mais ces erreurs, à mesure que le nombre des valeurs données de $F(x)$ augmente, convergent très rapidement vers zéro, et d'après ce que nous avons vu (§ 22), on pourra, dans chaque particulier, assigner leur limite. Tant que les valeurs connues de $F(x)$ seront en nombre considérable et qu'elles sont déterminées par les observations, le plus souvent ces erreurs seront au dessous de celles qui sont dues aux observations elles mêmes. Dans ces cas notre formule, sans contredit, est très propre à la recherche de l'expression approchée de la fonction $F(x)$, vu qu'elle détermine séparément tous les coefficients du développement de $F(x)$ suivant les puissances de

$$x-\frac{x_1+x_i}{2},$$

et n'exige que des calculs très simples.

L'usage de cette formule est d'autant plus expéditif que la plupart de ses termes et les seuls dont le nombre croisse avec celui des valeurs données de $F(x)$, savoir:

$$M_1,\ M_2, \ldots\ldots M_i,$$

restent, au signe près, les mêmes dans la détermination de tous les coefficients cherchés

$$K_0,\ K_1, \ldots\ldots K_n,$$

et que les autres termes ne sont jamais en nombre supérieur au degré de $F(x)$, ordinairement peu élevé. Quant à l'évaluation de tous ces termes, sous le rapport de la simplicité, elle ne laisse rien à désirer. Mais, comme nous l'avons remarqué plus haut (§ 22), on pourra, à l'aide du planimètre, s'épargner tout-à-fait la peine de faire ces calculs, tant qu'on aura la représentation graphique de la fonction cherchée, et alors, d'après nos formules, on trouvera son expression sous la forme

$$A_0 + A_1 x + \ldots\ldots + A_n x^n$$

avec une extrême facilité.

V.

§ 27. Pour montrer sur un exemple l'application de la formule (18), nous chercherons l'expression des changements de volume de l'eau à différentes temperatures, entre $t = 0°$ et $t = 25°$, d'après les observations qu'on trouve dans le Mémoire de M. Kopp (Annalen der Physik und Chemie, von J. C. Poggendorf, 20. Band, page 45) et dont les résultats peuvent etre présentés ainsi:

m	x_m	$F(x_m)$	m	x_m	$F(x_m)$
1	0	0,000000	16	13,5	0,000480
2	0,9	— 0,000022	17	13,8	0,000568
3	1,6	— 0,000098	18	15,0	0,000706
4	2,1	— 0,000077	19	15,6	0,000841
5	5,2	— 0,000115	20	16,3	0,000927
6	5,6	— 0,000135	21	17,4	0,001057
7	6,1	— 0,000094	22	18,6	0,001256
8	6,3	— 0,000101	23	18,6	0,001298
9	7,2	— 0,000047	24	19,2	0,001419
10	8,5	— 0,000006	25	19,8	0,001496
11	8,6	0,000007	26	21,2	0,001805
12	9,1	0,000081	27	21,8	0,001989
13	11,2	0,000215	28	22,2	0,002043
14	11,9	0,000317	29	24,0	0,002421
15	12,7	0,000352	30	24,5	0,002618

où par x_m nous désignons des températures et par $F(x_m)$ des changements d'une unité du volume de l'eau à différentes températures au dessus de zéro.

Nous chercherons l'expression de $F(x)$ par la formule

$$A_0+A_1x+A_2x^2+A_3x^3,$$

et pour cela nous prendrons

$$n=3.$$

D'autre part, comme dans la suite des valeurs connues de $F(x)$ les limites de x sont

$$x_1=0,\ x_{30}=24{,}5,$$

on aura, suivant notre notation,

$$x_1=0,\ x_i=24{,}5,\ i=30,$$

et par là

$$h=\frac{x_i-x_1}{2}=\frac{24{,}5-0}{2}=12{,}25,$$

$$\frac{x_1+x_i}{2}=\frac{0+24{,}5}{2}=12{,}25.$$

D'après cela, en mettant la fonction $F(x)$ sous la forme

$$K_0+K_1(x-12{,}25)+K_2(x-12{,}25)^2+K_3(x-12{,}25)^3,$$

et en cherchant ses coefficients

$$K_0,\ K_1,\ K_2,\ K_3$$

au moyen de notre formule (18), on a

$$(19)\ldots\ldots sK_l=\frac{1}{2}\left\{\begin{matrix}M_1+M_2+\ldots+M_{i'}-M_{i'+1}-M_{i'+2}-\ldots\ldots\ldots\ldots\ldots\ldots-M_{i''}\\ +M_{i''+1}+\ldots+M_{i'''}-M_{i'''+1}-\ldots+(-1)^{\nu}M_{i^{(\nu)}+1}+\ldots+(-1)^{\nu}M_{30}\end{matrix}\right\}$$
$$-\frac{(x_{i'+1}-\eta_1-12{,}25)^2F(x_{i'})-(x_{i'}-\eta_1-12{,}25)^2F(x_{i'+1})}{x_{i'+1}-x_{i'}}$$
$$+\frac{(x_{i''+1}-\eta_2-12{,}25)^2F(x_{i''})-(x_{i''}-\eta_2-12{,}25)^2F(x_{i''+1})}{x_{i''+1}-x_{i''}}$$
$$-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+(-1)^{\nu}\frac{(x_{i^{(\nu)}+1}-\eta_\nu-12{,}25)^2F(x_{i^{(\nu)}})-(x_{i^{(\nu)}}-\eta_\nu-12{,}25)^2F(x_{i^{(\nu)}+1})}{x_{i^{(\nu)}+1}-x_{i^{(\nu)}}},$$

où les quantités

$$s,\ \nu,\ \eta_1,\ \eta_2,\ldots\ldots\eta_\nu$$

se trouvent par la table, jointe à notre Mémoire, en prenant

$$n=3,\ h=12{,}25,$$

ce qui nous donne ce système des valeurs de s, ν, η_1, η_2, ... η_ν, correspondantes à $l=0,1,2,3$:

$$l=0.$$

$$\begin{aligned}
s &= -1{,}17480\,.\,12{,}25 = -14{,}3913;\\
\nu &= 2;\\
\eta_1 &= -0{,}79370\,.\,12{,}25 = -9{,}7228;\\
\eta_2 &= 0{,}79370\,.\,12{,}25 = 9{,}7228.
\end{aligned}$$

$$l=1.$$

$$\begin{aligned}
s &= 0{,}41421\,.\,12{,}25^2 = 62{,}1579;\\
\nu &= 3;\\
\eta_1 &= -0{,}84090\,.\,12{,}25 = -10{,}3010;\\
\eta_2 &= 0;\\
\eta_3 &= 0{,}84090\,.\,12{,}25 = 10{,}3010.
\end{aligned}$$

$$l=2.$$

$$\begin{aligned}
s &= 0{,}5\,.\,12{,}25^3 = 919{,}13;\\
\nu &= 2;\\
\eta_1 &= -0{,}5\,.\,12{,}25 = -6{,}125;\\
\eta_2 &= 0{,}5\,.\,12{,}25 = 6{,}125.
\end{aligned}$$

$$l=3.$$

$$\begin{aligned}
s &= -0{,}25\,.\,12{,}25^4 = -5629{,}7;\\
\nu &= 3;\\
\eta_1 &= -0{,}70711\,.\,12{,}25 = -8{,}6620;\\
\eta_2 &= 0;\\
\eta_3 &= 0{,}70711\,.\,12{,}25 = 8{,}6620.
\end{aligned}$$

En vertu de cela et en remarquant que, d'après notre notation,

$$x_{i'},\ x_{i'+1},\ x_{i''},\ x_{i''+1},\ \ldots\ldots x_{i^{(\nu)}},\ x_{i^{(\nu)}+1}$$

désignent les couples des valeurs qui, dans la suite

$$x_1,\ x_2,\ldots\ldots\ldots x_i,$$

sont respectivement les plus proches des quantités

$$\eta_1+12{,}25,\ \ \eta_2+12{,}25,\ldots\ldots\eta_\nu+12{,}25,$$

on tire aisément de la formule (19) les équations qui déterminent séparément chacun des coefficients

$$K_0,\ K_1,\ K_2,\ K_3$$

de la fonction cherchée

$$F(x) = K_0 + K_1(x-12{,}25) + K_2(x-12{,}25)^2 + K_3(x-12{,}25)^3.$$

§ 28. Pour déterminer le coefficient K_0, on prendra

$$l = 0,$$

et on aura

$$s = -14{,}3913,\ \upsilon = 2,\ \eta_1 = -9{,}7228,\ \eta_2 = 9{,}7228.$$

Comme dans la colonne des valeurs de x_m (§ 27), les plus proches de

$$\eta_1 + 12{,}25 = -9{,}7228 + 12{,}25 = 2{,}5272,$$

$$\eta_2 + 12{,}25 = 9{,}7228 + 12{,}25 = 21{,}9728$$

sont

$$x_4 = 2{,}1\,,\ x_5 = 5{,}2\,,\ x_{27} = 21{,}8\,,\ x_{28} = 22{,}2\,,$$

on prendra, conformément à notre notation,

$$x_{i'} = x_4,\ x_{i'+1} = x_5,\ x_{i''} = x_{27},\ x_{i''+1} = x_{28},$$

$$i' = 4,\ i'' = 27.$$

Avec ces valeurs de

$$s,\ \upsilon,\ \eta_1,\ \eta_2,\ i',\ i'',$$

la formule (19) nous donne

$$-14{,}3913\,K_0 = \tfrac{1}{2}\Big[M_1 + M_2 + \ldots + M_4 - M_5 - \ldots - M_{27} + M_{28} + \ldots + M_{30}\Big]$$

$$-\frac{(x_5 - 2{,}5272)^2 F(x_4) - (x_4 - 2{,}5272)^2 F(x_5)}{x_5 - x_4}$$

$$+\frac{(x_{28} - 21{,}9728)^2 F(x_{27}) - (x_{27} - 21{,}9728)^2 F(x_{28})}{x_{28} - x_{27}}.$$

En passant à la détermination du coefficient K_1, on fera

$$l = 1,$$

et comme pour cette valeur de l nous venons de trouver

$$s = 62{,}1579,\ \upsilon = 3\,,\ \eta_1 = -10{,}3010\,,\ \eta_2 = 0\,,\ \eta_3 = 10{,}3010,$$

et que dans la colonne des valeurs de x_m les plus proches de

$$\eta_1 + 12{,}25 = -10{,}3010 + 12{,}25 = 1{,}9490,$$

$$\eta_2 + 12{,}25 = 0 + 12{,}25 = 12{,}25\,,$$

$$\eta_3 + 12{,}25 = 10{,}3010 + 12{,}25 = 22{,}5510,$$

sont

$$x_3 = 1{,}6\,,\ x_4 = 2{,}1\,,\ x_{14} = 11{,}9\,,\ x_{15} = 12{,}7\,,\ x_{28} = 22{,}2\,,\ x_{29} = 24{,}0\,,$$

on aura, d'après notre notation,

$$i' = 3,\ i'' = 14,\ i''' = 28.$$

Alors, pour la détermination du coefficient K_1, la formule (19) nous fournit cette équation:

$$62{,}1579\,K_1 = \tfrac{1}{2}\Big[M_1 + M_2 + M_3 - M_4 - \ldots - M_{14} + M_{15} + \ldots + M_{28} - M_{29} - M_{30}\Big]$$
$$- \frac{(x_4 - 1{,}9490)^2 F(x_3) - (x_3 - 1{,}9490)^2 F(x_4)}{x_4 - x_3}$$
$$+ \frac{(x_{15} - 12{,}25)^2 F(x_{14}) - (x_{14} - 12{,}25)^2 F(x_{15})}{x_{15} - x_{14}}$$
$$- \frac{(x_{29} - 22{,}5510)^2 F(x_{28}) - (x_{28} - 22{,}5510)^2 F(x_{29})}{x_{29} - x_{28}}.$$

En cherchant de la même manière l'équation qui détermine le coefficient K_2, on prendra

$$l = 2,$$

et on aura

$$s = 919{,}13,\ \upsilon = 2,\ \eta_1 = -6{,}125,\ \eta_2 = 6{,}125,$$
$$i' = 7,\ i'' = 21.$$

Pour ces valeurs de

$$l,\ s,\ \upsilon,\ \eta_1,\ \eta_2,\ i',\ i'',$$

la formule (19) devient

$$919{,}13\,K_2 = \tfrac{1}{2}\Big[M_1 + M_2 + \ldots + M_7 - M_8 - \ldots - M_{21} + M_{22} + \ldots + M_{30}\Big]$$
$$- \frac{(x_8 - 6{,}125)^2 F(x_7) - (x_7 - 6{,}125)^2 F(x_8)}{x_8 - x_7}$$
$$+ \frac{(x_{22} - 18{,}375)^2 F(x_{21}) - (x_{21} - 18{,}375)^2 F(x_{22})}{x_{22} - x_{21}}.$$

Enfin, pour la détermination de K_3, on fera

$$l = 3,$$

et on trouvera

$$s = -5629{,}7,\ \upsilon = 3,\ \eta_1 = -8{,}6620,\ \eta_2 = 0,\ \eta_3 = 8{,}6620,$$
$$i' = 4,\ i'' = 14,\ i''' = 25;$$

d'après quoi la formule (19) donne

$$-5629,7K_3=\tfrac{1}{2}\Big[M_1+M_2+\ldots+M_4-M_5-\ldots-M_{14}+M_{15}+\ldots+M_{25}-M_{26}-\ldots-M_{30}\Big]$$
$$-\frac{(x_5-3,5880)^2F(x_4)-(x_4-3,5880)^2F(x_5)}{x_5-x_4}$$
$$+\frac{(x_{15}-12,25)^2F(x_{14})-(x_{14}-12,25)^2F(x_{15})}{x_{15}-x_{14}}$$
$$-\frac{(x_{26}-20,9120)^2F(x_{25})-(x_{25}-20,9120)^2F(x_{26})}{x_{26}-x_{25}}.$$

Au moyen des formules que nous venons d'obtenir, on trouve aisément les coefficients de la fonction cherchée

$$F(x)=K_0+K_1(x-12,25)+K_2(x-12,25)^2+K_3(x-12,25)^3,$$

comme nous allons le montrer.

§ 29. Pour trouver les valeurs de

$$M_1=(x_2-x_1)F(x_1),\;\; M_2=(x_3-x_1)F(x_2),\;\; M_3=(x_4-x_2)F(x_3),\ldots\ldots$$
$$\ldots\ldots\ldots\ldots M_{i-1}=(x_i-x_{i-2})F(x_{i-1}),\;\; M_i=(x_i-x_{i-1})F(x_i),$$

on cherchera les différences

$$\begin{array}{ll}
x_2-x_1=0,9-0=0,9, & x_{17}-x_{15}=13,8-12,7=1,1,\\
x_3-x_1=1,6-0=1,6, & x_{18}-x_{16}=15,0-13,5=1,5,\\
x_4-x_2=2,1-0,9=1,2, & x_{19}-x_{17}=15,6-13,8=1,8,\\
x_5-x_3=5,2-1,6=3,6, & x_{20}-x_{18}=16,3-15,0=1,3,\\
x_6-x_4=5,6-2,1=3,5, & x_{21}-x_{19}=17,4-15,6=1,8,\\
x_7-x_5=6,1-5,2=0,9, & x_{22}-x_{20}=18,6-16,3=2,3,\\
x_8-x_6=6,3-5,6=0,7, & x_{23}-x_{21}=18,6-17,4=1,2,\\
x_9-x_7=7,2-6,1=1,1, & x_{24}-x_{22}=19,2-18,6=0,6,\\
x_{10}-x_8=8,5-6,3=2,2, & x_{25}-x_{23}=19,8-18,6=1,2,\\
x_{11}-x_9=8,6-7,2=1,4, & x_{26}-x_{24}=21,2-19,2=2,0,\\
x_{12}-x_{10}=9,1-8,5=0,6, & x_{27}-x_{25}=21,8-19,8=2,0,\\
x_{13}-x_{11}=11,2-8,6=2,6, & x_{28}-x_{26}=22,2-21,2=1,0,\\
x_{14}-x_{12}=11,9-9,1=2,8, & x_{29}-x_{27}=24,0-21,8=2,2,\\
x_{15}-x_{13}=12,7-11,2=1,5, & x_{30}-x_{28}=24,5-22,2=2,3,\\
x_{16}-x_{14}=13,5-11,9=1,6, & x_{30}-x_{29}=24,5-24,0=0,5.
\end{array}$$

En multipliant ces différences par les valeurs

$$F(x_1),\ F(x_2), \ldots\ldots F(x_{30}),$$

on obtient

$$\begin{aligned}
M_1 &= 0{,}000000 . 0{,}9 = 0{,}000000,\\
M_2 &= -0{,}000022 . 1{,}6 = -0{,}000035,\\
M_3 &= -0{,}000098 . 1{,}2 = -0{,}000118,\\
M_4 &= -0{,}000077 . 3{,}6 = -0{,}000277,\\
M_5 &= -0{,}000115 . 3{,}5 = -0{,}000402,\\
M_6 &= -0{,}000135 . 0{,}9 = -0{,}000122,\\
M_7 &= -0{,}000094 . 0{,}7 = -0{,}000066,\\
M_8 &= -0{,}000101 . 1{,}1 = -0{,}000111,\\
M_9 &= -0{,}000047 . 2{,}2 = -0{,}000103,\\
M_{10} &= -0{,}000006 . 1{,}4 = -0{,}000008,\\
M_{11} &= 0{,}000007 . 0{,}6 = 0{,}000004,\\
M_{12} &= 0{,}000081 . 2{,}6 = 0{,}000210,\\
M_{13} &= 0{,}000215 . 2{,}8 = 0{,}000602,\\
M_{14} &= 0{,}000317 . 1{,}5 = 0{,}000475,\\
M_{15} &= 0{,}000352 . 1{,}6 = 0{,}000563,\\
M_{16} &= 0{,}000480 . 1{,}1 = 0{,}000528,\\
M_{17} &= 0{,}000568 . 1{,}5 = 0{,}000852,\\
M_{18} &= 0{,}000706 . 1{,}8 = 0{,}001271,\\
M_{19} &= 0{,}000841 . 1{,}3 = 0{,}001093,\\
M_{20} &= 0{,}000927 . 1{,}8 = 0{,}001668,\\
M_{21} &= 0{,}001057 . 2{,}3 = 0{,}002431,\\
M_{22} &= 0{,}001256 . 1{,}2 = 0{,}001507,\\
M_{23} &= 0{,}001298 . 0{,}6 = 0{,}000779,\\
M_{24} &= 0{,}001419 . 1{,}2 = 0{,}001703,\\
M_{25} &= 0{,}001496 . 2{,}0 = 0{,}002992,\\
M_{26} &= 0{,}001805 . 2{,}0 = 0{,}003610,\\
M_{27} &= 0{,}001989 . 1{,}0 = 0{,}001989,\\
M_{28} &= 0{,}002043 . 2{,}2 = 0{,}004495,\\
M_{29} &= 0{,}002421 . 2{,}3 = 0{,}005568,\\
M_{30} &= 0{,}002618 . 0{,}5 = 0{,}001309.
\end{aligned}$$

D'où l'on tire sur le champ les valeurs de toutes les combinaisons des quantités

$$M_1,\ M_2, \ldots\ldots M_{30}$$

que les expressions, déterminant les coefficients

$$K_0,\ K_1,\ K_2,\ K_3,$$

contiennent, savoir:

$$M_1+M_2+\ldots\ldots+M_4-M_5-\ldots\ldots-M_{27}+M_{28}+\ldots\ldots+M_{30}=-0{,}010523,$$

$$M_1+M_2+M_3-M_4-\ldots\ldots-M_{14}+M_{15}+\ldots\ldots+M_{28}-M_{29}-M_{30}=0{,}018249,$$

$$M_1+M_2+\ldots\ldots+M_7-M_8-\ldots\ldots-M_{21}+M_{22}+\ldots\ldots+M_{30}=0{,}013457,$$

$$M_1+M_2+\ldots+M_4-M_5-\ldots-M_{14}+M_{15}+\ldots+M_{25}-M_{26}-\ldots-M_{30}=-0{,}002493.$$

D'autre part, par la substitution des valeurs de

$$x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ x_{14},\ x_{15},\ x_{21},\ x_{22},\ x_{25},\ x_{26},\ x_{28},\ x_{29},$$

$$F(x_3),\ F(x_4),\ F(x_5),\ F(x_6),\ F(x_7),\ F(x_8),\ F(x_{14}),\ F(x_{21}),\ F(x_{22}),\ F(x_{25}),\ F(x_{26}),\ F(x_{28}),\ F(x_{29}),$$

on a

$$\frac{(x_5-2{,}5272)^2F(x_4)-(x_4-2{,}5272)^2F(x_5)}{x_5-x_4}=\frac{-(5{,}2-2{,}5272)^2.0{,}000077+(2{,}1-2{,}5272)^2.0{,}000115}{5{,}2-2{,}1}$$
$$=-0{,}000170,$$

$$\frac{(x_{28}-21{,}9728)^2F(x_{27})-(x_{27}-21{,}9728)^2F(x_{28})}{x_{28}-x_{27}}=\frac{(22{,}2-21{,}9728)^2.0{,}001989-(21{,}8--21{,}9728).0{,}002043}{22{,}2-21{,}8}$$
$$=0{,}000103,$$

$$\frac{(x_4-1{,}9490)^2F(x_3)-(x_3-1{,}9490)^2F(x_4)}{x_4-x_3}=\frac{-(2{,}1-1{,}9490)^2.0{,}000098+(1{,}6-1{,}9490)^2.0{,}000077}{2{,}1-1{,}6}$$
$$=0{,}000014,$$

$$\frac{(x_{15}-12{,}25)^2F(x_{14})-(x_{14}-12{,}25)^2F(x_{15})}{x_{15}-x_{14}}=\frac{(12{,}7-12{,}25)^2.0{,}000317-(11{,}9-12{,}15)^2.0{,}000352}{12{,}7-11{,}9}$$
$$=0{,}000026,$$

$$\frac{(x_{29}-22{,}5510)^2F(x_{28})-(x_{28}-22{,}5510)^2F(_{29})}{x_{29}-x_{28}}=\frac{(24-22{,}5510)^2.0{,}002043-(22{,}2-22{,}5510)^2.0{,}002421}{24-22{,}2}$$
$$=0{,}002217,$$

$$\frac{(x_8-6{,}125)^2F(x_7)-(x_7-6{,}125)^2F(x_8)}{x_8-x_7}=\frac{-(6{,}3-6{,}125)^2.0\ 000094+(6{,}1-6{,}125)^2.0.000101}{6{,}3-6{,}1}$$
$$=-0{,}000014,$$

$$\frac{(x_{22}-18{,}375)^2F(x_{21})-(x_{21}-18{,}375)^2F(x_{22})}{x_{22}-x_{21}}=\frac{(18{,}6-18{,}375)^2.0{,}001057-(17{,}4-18{,}375)^2.0{,}001256}{18{,}6-17{,}4}$$
$$=-0{,}000951,$$

$$\frac{(x_5-3,588)^2F(x_4)-(x_4-3,588)^2F(x_5)}{x_5-x_4}=\frac{-(5,2-3,588)^2.0,000077+(2,1-3,588(^2.0,000115}{5,2-2,1}$$
$$=0,000017,$$

$$\frac{(x_{26}-20,912)^2F(x_{25})-(x_{25}-20,912)^2F(x_{26})}{x_{26}-x_{25}}=\frac{(21,2-20,912)^2.0,001496-(19.8-20.912)^2.0.001805}{21,2-19.8}$$
$$=-0,001505.$$

Dès lors les équations que nous avons trouvées (§ 28) pour la détermination des coefficients

$$K_0,\ K_1,\ K_2,\ K_3$$

nous donnent

$$-14,3913\,K_0=\frac{-0,010523}{2}+0,000171+0,000103=-0,004987,$$

$$62,1579\,K_1=\frac{0,018249}{2}-0,000014+0,000026-0,002217=0,006919,$$

$$919,13\,K_2=\frac{0,013457}{2}+0,000014-0,000951=0,005791,$$

$$-5629,7\,K_3=\frac{-0,002493}{2}-0,000017+0,000026+0,001505=0,000268,$$

et par là on obtient

$$K_0=\frac{-0.004987}{-14,3913}=0,0003465,$$

$$K_1=\frac{0.006919}{62,1579}=0,00011131,$$

$$K_2=\frac{0,005791}{919,13}=0,00000630,$$

$$K_3=\frac{0,000268}{-5629,7}=-0,0000000476.$$

En portant ces valeurs de

$$K_0,\ K_1,\ K_2,\ K_3$$

dans l'expression cherchée de $F(x)$, on a

$$F(x)=0,0003465+0,00011131\,(x-12,25)+0,00000630\,(x-12,25)^2$$
$$-0,0000000476\,(x-12,25)^3,$$

ce qui présente toutes les valeurs données de $F(x)$ avec une approximation très suffisante, comme on peut le voir d'après cette table:

m	x_m	Valeurs de $F(x_m)$ observées.	Valeurs de $F(x_m)$ calculées.	Différences.
1	0	0,000000	+ 0,000016	— 0,000016
2	0,9	— 0,000022	— 0,000036	+ 0,000014
3	1,6	— 0,000098	— 0,000067	— 0,000031
4	2,1	— 0,000077	— 0,000085	+ 0,000008
5	5,2	— 0,000115	— 0,000107	— 0,000008
6	5,6	— 0,000135	— 0,000101	— 0,000034
7	6,1	— 0,000094	— 0,000089	— 0,000005
8	6,3	— 0,000101	— 0,000083	— 0,000018
9	7,2	— 0,000047	— 0,000049	+ 0,000002
10	8,5	— 0,000006	— 0,000020	+ 0,000014
11	8,6	0,000007	— 0,000007	+ 0,000014
12	9,1	0,000081	0,000061	+ 0,000020
13	11,2	0,000215	0,000236	— 0,000021
14	11,9	0,000317	0,000308	+ 0,000009
15	12,7	0,000352	0,000398	— 0,000046
16	13,5	0,000480	0,000495	— 0,000015
17	13,8	0,000568	0,000534	+ 0,000034
18	15,0	0,000706	0,000698	+ 0,000008
19	15,6	0,000841	0,000789	+ 0,000052
20	16,3	0,000927	0,000891	+ 0,000036
21	17,4	0,001057	0,001080	— 0,000023
22	18,6	0,001256	0,001295	— 0,000039
23	18,6	0,001298	0,001295	+ 0,000003
24	19,2	0,001419	0,001408	+ 0,000011
25	19,8	0,001496	0,001525	— 0,000029
26	21,2	0,001805	0,001813	— 0,000008
27	21,8	0,001989	0,001942	+ 0,000047
28	22,2	0,002043	0,002030	+ 0,000013
29	24,0	0,002421	0,002447	— 0,000026
30	24,5	0,002618	0,002568	+ 0,000050

§ 30. D'après la formule (14), et en prenant, suivant l'expression obtenue de $F(x)$,

$$F''(x) = 0,0000126 - 0,0000002856(x - 12,25),$$

on trouve que les erreurs qu'on commet dans nos valeurs des coefficients

$$K_0,\ K_1,\ K_2,\ K_3,$$

en remplaçant, comme nous l'avons fait, la courbe

$$y = F(x)$$

par un polygone, sont comprises respectivement entre les limites

$$\begin{aligned} -0,0000023 &\text{ et } -0,0000040,\\ +0,00000059 &\text{ et } +0,00000079,\\ -0,000000022 &\text{ et } -0,000000032,\\ -0,00000000059 &\text{ et } -0,00000000081. \end{aligned}$$

D'après cela on reconnait aisément que ces erreurs sont notablement au-dessous de celles dues aux observations. Ces erreurs seraient encore plus petites, si les valeurs de l'argument des observations des n^{os} 4 et 5 n'étaient pas si éloignées entre elles.

VI.

§ 31. Par la méthode exposée dans les sections précédentes, on parviendra à trouver les coefficients de la fonction

$$F(x) = A_0 + A_1 x + \ldots\ldots + A_n x^n,$$

avec une approximation plus ou moins grande, suivant le nombre de ses valeurs connues. Mais comme les quantités

$$s,\ \eta_1,\ \eta_2, \ldots\ldots \eta_\nu$$

qui entrent dans nos formules dépendent essentiellement du nombre n, on ne peut les employer à la recherche de l'expression de $F(x)$ sans fixer d'avance le nombre de ses termes conservés, et conséquemment, tant qu'on ne sait rien sur ce nombre, il est important d'examiner les différentes hypothèses qui s'y rapportent, et de chercher séparément, dans chacune d'elles, l'expression de $F(x)$, ce qui augmente considérablement les calculs. Nous allons montrer maintenant comment par notre méthode on parvient à une formule d'interpolation, qui lève complétement cette difficulté. La formule que nous donnerons à présent embrassera toutes les hypothèses possibles sur le nombre de termes dans l'expression de $F(x)$, et répondra

à chacune d'elles suivant qu'on prolonge plus ou moins la série que cette formule représente. Sous ce rapport elle ne laissera rien à désirer, seulement, comme toutes les autres formules de ce Mémoire, elle ne donnera pas de résultat avec la moindre erreur à craindre, résultat qu'on ne saurait trouver directement qu'à l'aide de notre série citée plus haut.

§ 32. Pour parvenir à la formule d'interpolation dont nous avons parlé, convenons de désigner par le symbole

$$\int_n u$$

l'expression de la forme

$$\int_{-h}^{\eta_1} u dx - \int_{\eta_1}^{\eta_2} u dx + \int_{\eta_2}^{\eta_3} u dx - \ldots\ldots + (-1)^n \int_{\eta_n}^{h} u dx,$$

où

$$\eta_1,\ \eta_3, \ldots\ldots,$$
$$\eta\ ,\ \eta_4, \ldots\ldots$$

sont les racines des équations

$$\frac{(x+\sqrt{x^2-h^2})^{\frac{n+1}{2}} + (x-\sqrt{x^2-h^2})^{\frac{n+1}{2}}}{2} = 0,$$

$$\frac{(x+\sqrt{x^2-h^2})^{\frac{n+1}{2}} - (x-\sqrt{x^2-h^2})^{\frac{n+1}{2}}}{2\sqrt{x^2-h^2}} = 0$$

dans le cas de n impair, et des équations

$$\frac{\left(\sqrt{\frac{x+h}{2}} + \sqrt{\frac{x-h}{2}}\right)^{n+1} - \left(\sqrt{\frac{x+h}{2}} - \sqrt{\frac{x-h}{2}}\right)^{n+1}}{\sqrt{\frac{x-h}{2}}} = 0,$$

$$\frac{\left(\sqrt{\frac{x+h}{2}} + \sqrt{\frac{x-h}{2}}\right)^{n+1} + \left(\sqrt{\frac{x+h}{2}} - \sqrt{\frac{x-h}{2}}\right)^{n+1}}{\sqrt{\frac{x+h}{2}}} = 0$$

dans le cas de n pair.

D'après cette notation, les formules, trouvées dans la Section III pour la détermination du dernier coefficient de la fonction

$$F(x) = A_0 + A_1 x + \ldots\ldots + A_n x^n,$$

seront représentées ainsi:

$$\int_n F(x) = sA_n.$$

D'où, en substituant la valeur de $F(x)$, nous tirons

$$\int_n (A_0 + A_1 x + \ldots\ldots + A_n x^n) = A_0 \int_n x^0 + A_1 \int_n x + \ldots\ldots + A_n \int_n x^n = sA_n,$$

ce qui suppose

$$\int_n x^0 = 0,\ \int_n x = 0, \ldots\ldots \int_n x^{n-1} = 0.$$

Soit maintenant

$$f(x) = x_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_m x^m$$

une fonction dont on cherche la valeur pour

$$x = z.$$

D'après la valeur de

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_m x^m$$

on a

$$\int_n f(x) = \int_n (a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_m x^m)$$

$$= a_0 \int_n x^0 + a_1 \int_n x + a_2 \int_n x^2 + \ldots\ldots + a_m \int_n x^m,$$

et comme nous venons de voir que

$$\int_n x^0 = 0,\ \int_n x = 0,\ \int_n x^2 = 0, \ldots\ldots \int_n x^{n-1} = 0,$$

cela nous donne, dans la supposition de $n < m - 1$,

$$\int_n f(x) = a_n \int_n x^n + a_{n+1} \int_n x^{n+1} + \ldots\ldots + a_m \int_n x^m.$$

D'où en faisant

$$n = 0,\ 1,\ 2,\ 3, \ldots\ldots,$$

nous obtenons ce système d'équations:

$$(20) \ldots\ldots\ldots \left\{ \begin{array}{l} \int_0 f(x) = a_0 \int_0 x^0 + a_1 \int_0 x + a_2 \int_0 x^2 + \ldots\ldots + a_m \int_0 x^m, \\ \int_1 f(x) = a_1 \int_1 x + a_2 \int_1 x^2 + \ldots\ldots + a_m \int_1 x^m, \\ \int_2 f(x) = a_2 \int_2 x^2 + \ldots\ldots + a_m \int_2 x^m, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \int_m f(x) = a_m \int_m x^m. \end{array} \right.$$

Ces équations déterminent tous les coefficients

$$a_0,\ a_1,\ a_2, \ldots\ldots\ldots a_m$$

de la fonction

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_m x^m,$$

d'après lesquels on trouvera aisément sa valeur pour $x = z$.

§ 33. Pour parvenir directement à la valeur de $f(z)$, nous prendrons la somme des équations (20), après les avoir multipliées respectivement par les facteurs arbitraires

$$\theta_1,\ \theta_2,\ \theta_3,\ldots\ldots\theta_{m+1}.$$

Ainsi l'on obtient

$$\begin{aligned}\theta_1\int_0 f(x)+\theta_2\int_1 f(x)+\theta_3\int_2 f(x)+\ldots+\theta_{m+1}\int_m f(x)=a_0\theta_1\int_0 x^0+a_1(\theta_1\int_0 x+\theta_2\int_1 x)\\ +a_2(\theta_1\int_0 x^2+\theta_2\int_1 x^2+\theta_3\int_2 x^2)\\ +\ldots\ldots\ldots\ldots\ldots\ldots\\ +a_m(\theta_1\int_0 x^m+\theta_2\int_1 x^m+\ldots+\theta_{m+1}\int_m x^m).\end{aligned}$$

D'où résulte cette valeur de $f(z)=a_0+a_1\,z+a_2z^2+\ldots\ldots+a_mz^m$:

$$(21)\ldots\ldots\ldots\ldots\ \theta_1\int_0 f(z)+\theta_2\int_1 f(x)+\theta_3\int_2 f(x)+\ldots\ldots+\theta_{m+1}\int_m f(x)=f(z),$$

les facteurs

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta\ldots\ldots\theta_m$$

étant choisis de manière à ce qu'on ait

$$(22)\ldots\ldots\begin{cases}\theta_1\int_0 x^0=1,\ \theta_1\int_0 x+\theta_2\int_1 x=z,\ \theta_1\int_0 x^2+\theta_2\int_1 x^2+\theta_3\int_2 x^2=z^2,\ldots\ldots\ldots\\ \theta_1\int_0 x^m+\theta_2\int_1 x^m+\ldots\ldots\ldots\ldots+\theta_{m+1}\int_m x^m=z^m.\end{cases}$$

Or d'après la forme de ces équations on voit que les facteurs

$$\theta_1,\ \theta_2,\ \theta_3,\ldots\ldots$$

qui entrent dans l'expression (21) de $f(z)$ sont les fonctions de z, respectivement des degrés

$$0,\ 1,\ 2,\ldots\ldots,$$

et que leurs valeurs ne dépendent nullement de m, nombre de termes de la fonction cherchée. Donc si l'on fait $m=\infty$, l'expression de $f(z)$, donnée par la formule (21), jouira de la propriété dont nous avons parlé dans le § 31. Pour s'en assurer on n'a qu'à remarquer que, dans le cas de $m=\infty$, la formule (21) se réduit à une série infinie

$$\theta_1\int_0 f(x)+\theta_2\int_1 f(x)+\theta_3\int_2 f(x)+\ldots\ldots\ldots\ldots,$$

et que cette série, arrêtée au terme $\theta_{m+1}\int_m f(x)$, donne la valeur de $f(z)$ qu'on trouve d'après (21) dans la supposition de

$$f(x)=a_0+a_1x+a_2x^2+\ldots\ldots+a_mx^m.$$

§ 34. Nous allons chercher maintenant la loi de la série

$$(23)\ldots\ldots\ldots\ldots f(z)=\theta_1\underset{0}{\int}f(x)+\theta_2\underset{1}{\int}f(x)+\theta_3\underset{2}{\int}f(x)+\ldots\ldots\ldots\ldots,$$

qui résulte de la formule (21) dans le cas de

$$m=\infty.$$

Les équations (22) qui déterminent les fonctions

$$\theta_1,\ \theta_2,\ \theta_3,\ldots\ldots\ldots,$$

pour $m=\infty$, deviennent

$$(24)\ldots\ldots\ldots\ldots\ldots\ldots\begin{cases}\theta_1\underset{0}{\int}x^0=1,\\ \theta_1\underset{0}{\int}x+\theta_2\underset{1}{\int}x=z,\\ \theta_1\underset{0}{\int}x^2+\theta_2\underset{1}{\int}x^2+\theta_3\underset{2}{\int}x^2=z^2,\\ \theta_1\underset{0}{\int}x^3+\theta_2\underset{1}{\int}x^3+\theta_3\underset{2}{\int}x^3+\theta_4\underset{3}{\int}x^3=z^3,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{cases}$$

Par la solution de ces équations on trouve aisément les fonctions

$$\theta_1,\ \theta_2,\ \theta_3,\ldots\ldots\ldots;$$

mais il est difficile de reconnaître leur forme générale. Nous montrerons maintenant comment on y parvient par une méthode toute particulière.

En vertu de ce que nous avons vu (§ 32) relativement aux expressions

$$\underset{n}{\int}x^0,\ \int x,\ \underset{n}{\int}x^2,\ldots\ldots\underset{n}{\int}x^{k-1},$$

on a

$$\underset{1}{\int}x^0=0,\ \underset{2}{\int}x^0=0,\ \underset{3}{\int}x^0=0,\ldots\ldots\ldots,$$
$$\underset{2}{\int}x=0,\ \underset{3}{\int}x=0,\ldots\ldots\ldots,$$
$$\underset{3}{\int}x^2=0,\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots$$

D'où il suit que, sans rien changer aux équations (24), elles peuvent être mises sous cette forme:

$$\theta_1 \int_0 x^0 + \theta_2 \int_1 x^0 + \theta_3 \int_2 x^0 + \dots\dots\dots = 1,$$

$$\theta_1 \int_0 x + \theta_2 \int_1 x + \theta_3 \int_2 x + \dots\dots\dots = z,$$

$$\theta_1 \int_0 x^2 + \theta_2 \int_1 x^2 + \theta_3 \int_2 x^2 + \dots\dots\dots = z^2,$$

$$\theta_1 \int_0 x^3 + \theta_2 \int_1 x^3 + \theta_3 \int_2 x^3 + \dots\dots\dots = z^3,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Or si l'on multiplie ces équations respectivement par

$$\frac{1}{\alpha^2}, \ \frac{2}{\alpha^3}, \ \frac{3}{\alpha^4}, \ \frac{4}{\alpha^5}, \dots\dots\dots,$$

α étant une quantité quelconque, et qu'on prenne leur somme, il en résulte

$$\theta_1 \left[\frac{1}{\alpha^2} \int_0 x^0 + \frac{2}{\alpha^3} \int_0 x + \frac{3}{\alpha^4} \int_0 x^2 + \frac{4}{\alpha^5} \int_0 x^3 + \dots\dots \right]$$

$$+ \theta_2 \left[\frac{1}{\alpha^2} \int_1 x^0 + \frac{2}{\alpha^3} \int_1 x + \frac{3}{\alpha^4} \int_1 x^2 + \frac{4}{\alpha^5} \int_1 x^3 + \dots\dots \right]$$

$$+ \theta_3 \left[\frac{1}{\alpha^2} \int_2 x^0 + \frac{2}{\alpha^3} \int_2 x + \frac{3}{\alpha^4} \int_2 x^2 + \frac{4}{\alpha^5} \int_2 x^3 + \dots\dots \right]$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$= \frac{1}{\alpha^2} + \frac{2z}{\alpha^3} + \frac{3z^2}{\alpha^4} + \frac{4z^3}{\alpha^5} + \dots\dots\dots\dots\dots,$$

et comme

$$\frac{1}{\alpha^2} \int_n x^0 + \frac{2}{\alpha^3} \int_n x + \frac{3}{\alpha^4} \int_n x^2 + \frac{4}{\alpha^5} \int_n x^3 + \dots\dots = \int_n \left(\frac{1}{\alpha^2} + \frac{2x}{\alpha^3} + \frac{3x^2}{\alpha^4} + \frac{4x^3}{\alpha^5} + \dots\dots \right) = \int_n \frac{1}{(\alpha - x)^2},$$

$$\frac{1}{\alpha^2} + \frac{2z}{\alpha^3} + \frac{3z^2}{\alpha^4} + \frac{4z^3}{\alpha^5} + \dots\dots = \frac{1}{(\alpha - z)^2},$$

cette formule se réduit à celle-ci:

$$(25) \dots\dots\dots\dots \theta_1 \int_0 \frac{1}{(\alpha - x)^2} + \theta_2 \int_1 \frac{1}{(\alpha - x)^2} + \theta_3 \int_2 \frac{1}{(\alpha - x)^2} + \dots\dots = \frac{1}{(\alpha - z)^2}.$$

Pour trouver les quantités

$$\int_0 \frac{1}{(\alpha - x)^2}, \ \int_1 \frac{1}{(\alpha - x)^2}, \ \int_2 \frac{1}{(\alpha - x)^2}, \dots\dots,$$

contenues dans cette formule, remarquons que d'après notre notation

$$\int_n \frac{1}{(\alpha-x)^2} = \int_{-h}^{\eta_1} \frac{dx}{(\alpha-x)^2} - \int_{\eta_1}^{\eta_2} \frac{dx}{(\alpha-x)^2} + \int_{\eta_2}^{\eta_3} \frac{dx}{(\alpha-x)^2} - \ldots\ldots + (-1)^n \int_{\eta_n}^{h} \frac{dx}{(\alpha-x)^2}.$$

D'où il suit

$$\int_n \frac{1}{(\alpha-x)^2} = -\frac{1}{\alpha+h} + \frac{2}{\alpha-\eta_1} - \frac{2}{\alpha-\eta_2} + \frac{2}{\alpha-\eta_3} - \ldots\ldots + (-1)^n \frac{1}{\alpha-h};$$

et comme

$$-\frac{1}{\alpha+h} + \frac{2}{\alpha-\eta_1} - \frac{2}{\alpha-\eta_2} + \frac{2}{\alpha-\eta_3} - \ldots\ldots\ldots\ldots + (-1)^n \frac{1}{\alpha-h}$$

$$= -\frac{d\log(\alpha+h)}{d\alpha} + \frac{2d\log(\alpha-\eta_1)}{d\alpha} - \frac{2d\log(\alpha-\eta_2)}{d\alpha} + \frac{2d\log(\alpha-\eta_3)}{d\alpha} - \ldots\ldots + (-)^n \frac{d\log(\alpha-h)}{d\alpha}$$

$$= \frac{d\log\dfrac{(\alpha-\eta_1)^2(\alpha-\eta_3)^2\ldots\ldots(\alpha-h)^{(-1)^n}}{(\alpha-\eta_2)^2\ldots\ldots\ldots\ldots(\alpha+h)}}{d\alpha},$$

il en résulte

$$(26)\ldots\ldots\ldots\ldots\ldots\ldots \int_n \frac{1}{(\alpha-x)^2} = \frac{d\log\dfrac{(\alpha-\eta_1)^2(\alpha-\eta_2)^2\ldots\ldots(\alpha-h)^{(-1)^n}}{(\alpha-\eta_2)^2\ldots\ldots\ldots\ldots(\alpha+h)}}{d\alpha}.$$

A l'aide de cette formule on obtient aisément la valeur définitive de

$$\int_n \frac{1}{(\alpha-x)^2}.$$

Dans le cas de n impair, les quantités

$$\eta_1,\ \eta_3, \ldots\ldots,$$
$$\eta_2,\ \eta_4, \ldots\ldots$$

sont (§ 32) les racines des équations

$$\frac{\left(x+\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}} + \left(x-\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}}{2} = 0,$$

$$\frac{\left(x+\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}} - \left(x-\sqrt{x^2-h^2}\right)^{\frac{n+1}{2}}}{2\sqrt{x^2-h^2}} = 0,$$

et par là on trouve

$$(\alpha-\eta_1)(\alpha-\eta_3),\ldots\ldots = C_0 \frac{\left(\alpha+\sqrt{\alpha^2-h^2}\right)^{\frac{n+1}{2}} + \left(\alpha-\sqrt{\alpha^2-h^2}\right)^{\frac{n+1}{2}}}{2},$$

$$(\alpha-\eta_2)(\alpha-\eta_4),\ldots\ldots = C_1 \frac{\left(\alpha+\sqrt{\alpha^2-h^2}\right)^{\frac{n+1}{2}} - \left(\alpha-\sqrt{\alpha^2-h^2}\right)^{\frac{n+1}{2}}}{2\sqrt{\alpha^2-h^2}},$$

en désignant par C_0 et C_1 des valeurs indépendantes de α. D'après quoi la formule (26), pour n impair, nous donne

$$\int_n \frac{1}{(\alpha-x)^2} = \frac{d \log \frac{C_0^2}{C_1^2}\left(\frac{(\alpha+\sqrt{\alpha^2-h^2})^{\frac{n+1}{2}} + (\alpha-\sqrt{\alpha^2-h^2})^{\frac{n+1}{2}}}{(\alpha+\sqrt{\alpha^2-h^2})^{\frac{n+1}{2}} - (\alpha-\sqrt{\alpha^2-h^2})^{\frac{n+1}{2}}}\right)^2}{d\alpha}$$

$$= -\frac{n+1}{\sqrt{\alpha^2-h^2}} \frac{4h^{n+1}}{(\alpha+\sqrt{\alpha^2-h^2})^{n+1} - (\alpha-\sqrt{\alpha^2-h^2})^{n+1}}.$$

En passant au cas de n pair, nous remarquerons que, pour ces valeurs de n, les quantités

$$\eta_1, \; \eta_3, \ldots\ldots,$$
$$\eta_2, \eta_4, \ldots\ldots$$

sont les racines des équations

$$\frac{\left(\sqrt{\frac{x+h}{2}} + \sqrt{\frac{x-h}{2}}\right)^{n+1} - \left(\sqrt{\frac{x+h}{2}} - \sqrt{\frac{x-h}{2}}\right)^{n+1}}{2\sqrt{\frac{x-h}{2}}} = 0,$$

$$\frac{\left(\sqrt{\frac{x+h}{2}} + \sqrt{\frac{x-h}{2}}\right)^{n+1} + \left(\sqrt{\frac{x+h}{2}} - \sqrt{\frac{x-h}{2}}\right)^{n+1}}{2\sqrt{\frac{x+h}{2}}} = 0,$$

et par conséquent

$$(\alpha-\eta_1)(\alpha-\eta_3)\ldots\ldots = C_0 \frac{\left(\sqrt{\frac{\alpha+h}{2}} + \sqrt{\frac{\alpha-h}{2}}\right)^{n+1} - \left(\sqrt{\frac{\alpha+h}{2}} - \sqrt{\frac{\alpha-h}{2}}\right)^{n+1}}{2\sqrt{\frac{\alpha-h}{2}}},$$

$$(\alpha-\eta_2)(\alpha-\eta_4)\ldots\ldots = C_1 \frac{\left(\sqrt{\frac{\alpha+h}{2}} + \sqrt{\frac{\alpha-h}{2}}\right)^{n+1} + \left(\sqrt{\frac{\alpha+h}{2}} - \sqrt{\frac{\alpha-h}{2}}\right)^{n+1}}{2\sqrt{\frac{\alpha+h}{2}}}.$$

D'après cela la formule (26), pour n pair, nous donne

$$\int_n \frac{1}{(\alpha-x)^2} = \frac{d \log \frac{C_0^2}{C_1^2}\left(\frac{\left(\sqrt{\frac{\alpha+h}{2}} + \sqrt{\frac{\alpha-h}{2}}\right)^{n+1} - \left(\sqrt{\frac{\alpha+h}{2}} - \sqrt{\frac{\alpha-h}{2}}\right)^{n+1}}{\left(\sqrt{\frac{\alpha+h}{2}} + \sqrt{\frac{\alpha-h}{2}}\right)^{n+1} + \left(\sqrt{\frac{\alpha+h}{2}} - \sqrt{\frac{\alpha-h}{2}}\right)^{n+1}}\right)^2}{d\alpha}$$

$$= \frac{n+1}{\sqrt{\alpha^2-h^2}} \frac{4h^{n+1}}{(\alpha+\sqrt{\alpha^2-h^2})^{n+1} - (\alpha-\sqrt{\alpha^2-h^2})^{n+1}}.$$

Par les expressions trouvées de

$$\int_n \frac{1}{(\alpha-x)^2}$$

on obtient, pour $n = 0, 1, 2, \ldots\ldots$,

$$\int_0 \frac{1}{(\alpha-x)^2} = \frac{1}{\sqrt{\alpha^2-h^2}} \frac{4h}{\alpha+\sqrt{\alpha^2-h^2} - (\alpha+\sqrt{\alpha^2-h^2})},$$

$$\int_1 \frac{1}{(\alpha-x)^2} = -\frac{2}{\sqrt{\alpha^2-h^2}} \frac{4h^2}{(\alpha+\sqrt{\alpha^2-h^2})^2-(\alpha-\sqrt{\alpha^2-h^2})^2},$$

$$\int_2 \frac{1}{(\alpha-x)^2} = \frac{3}{\sqrt{\alpha^2-h^2}} \frac{4h^3}{(\alpha+\sqrt{\alpha^2-h^2})^3-(\alpha-\sqrt{\alpha^2-h^2})^3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

en vertu de quoi la formule (25) devient

$$\frac{4}{\sqrt{\alpha^2-h^2}}\left[\frac{h_1\theta_1}{\alpha+\sqrt{\alpha^2-h^2}-(\alpha-\sqrt{\alpha^2-h^2})} - \frac{2h^2\theta_2}{(\alpha+\sqrt{\alpha^2-h^2})^2-(\alpha-\sqrt{\alpha^2-h^2})^2} \right.$$
$$\left. + \frac{3h^3\theta_3}{(\alpha+\sqrt{\alpha^2-h^2})^3-(\alpha-\sqrt{\alpha^2-h^2})^3} - \ldots\right] = \frac{1}{(\alpha-z)^2}.$$

Pour simplifier cette formule nous poserons

$$\frac{\alpha+\sqrt{\alpha^2-h^2}}{h} = u,$$

ce qui nous donne

$$\frac{\alpha-\sqrt{\alpha^2-h^2}}{h} = \frac{1}{u}, \quad \frac{2\sqrt{\alpha^2-h^2}}{h} = u - \frac{1}{u}, \quad \frac{2\alpha}{h} = u + \frac{1}{u},$$

et d'après cela notre formule se change dans celle-ci:

$$\frac{8}{h\left(u-\frac{1}{u}\right)}\left[\frac{\theta_1}{u-\frac{1}{u}} - \frac{2\theta_2}{u^2-\frac{1}{u^2}} + \frac{3\theta_3}{u^3-\frac{1}{u^3}} - \ldots\ldots\right] = \frac{1}{\left[\frac{h}{2}\left(u+\frac{1}{u}\right)-z\right]^2},$$

ou

$$\frac{\theta_1}{u-\frac{1}{u}} - \frac{2\theta_2}{u^2-\frac{1}{u^2}} + \frac{3\theta_3}{u^3-\frac{1}{u^3}} - \ldots = \frac{u^2\left(u-\frac{1}{u}\right)}{2h\left(u^2-\frac{2zu}{h}+1\right)^2}.$$

§ 35. Les deux parties de cette équation se transforment dans des sommes très simples. En effet, comme

$$\frac{\theta_1}{u-\frac{1}{u}} - \frac{2\theta_2}{u^2-\frac{1}{u^2}} + \frac{3\theta_3}{u^3-\frac{1}{u^3}} - \ldots = \sum_{\lambda=1}^{\lambda=\infty} \frac{(-1)^{\lambda}\lambda\theta_\lambda}{u^\lambda-\frac{1}{u^\lambda}},$$

$$\frac{1}{u^\lambda-\frac{1}{u^\lambda}} = \frac{1}{u^\lambda} + \frac{1}{u^{3\lambda}} + \frac{1}{u^{5\lambda}} + \ldots = \sum_{\mu=1}^{\mu=\infty} \frac{1}{u^{(2\mu-1)\lambda}},$$

on trouve

$$\frac{\theta_1}{u-\frac{1}{u}}-\frac{2\theta_2}{u^2-\frac{1}{u^2}}+\frac{3\theta_3}{u^3-\frac{1}{u^3}}-\ldots\ldots=\sum_{\lambda=1}^{\lambda=\infty}\sum_{\mu=1}^{\mu=\infty}\frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{u^{(2\mu-1)\lambda}}.$$

D'autre part, à l'aide de la décomposition en fractions simples, on obtient

$$\frac{u^2\left(u-\frac{1}{u}\right)}{\left(u^2-\frac{2zu}{h}+1\right)^2}=\frac{h}{2\sqrt{z^2-h^2}}\left[\frac{(z+\sqrt{z^2-h^2})^2}{(hu-z-\sqrt{z^2-h^2})^2}+\frac{z+\sqrt{z^2-h^2}}{hu-z-\sqrt{z^2-h^2}}-\frac{(z-\sqrt{z^2-h^2})^2}{(hu-z+\sqrt{z^2-h^2})^2}-\frac{z-\sqrt{z^2-h^2}}{hu-z+\sqrt{z^2-h^2}}\right],$$

et par là

$$\frac{u^2\left(u-\frac{1}{u}\right)}{\left(u^2-\frac{2zu}{h}+1\right)^2}=\frac{h}{2\sqrt{z^2-h^2}}\left[\frac{h(z+\sqrt{z^2-h^2})u}{(hu-z-\sqrt{z^2-h^2})^2}-\frac{h(z-\sqrt{z^2-h^2})u}{(hu-z+\sqrt{z^2-h^2})^2}\right];$$

d'où résulte

$$\frac{1}{2h}\frac{u^2\left(u-\frac{1}{u}\right)}{\left(u^2-\frac{2zu}{h}+1\right)^2}=\frac{1}{4\sqrt{z^2-h^2}}\left\{\begin{array}{l}\frac{z+\sqrt{z^2-h^2}}{hu}+2\frac{(z+\sqrt{z^2-h^2})^2}{h^2u^2}+3\frac{(z+\sqrt{z^2-h^2})^3}{h^3u^3}+\ldots\ldots\\ -\frac{z-\sqrt{z^2-h^2}}{hu}-2\frac{(z-\sqrt{z^2-h^2})^2}{h^2u^2}-3\frac{(z-\sqrt{z^2-h^2})^3}{h^3u^3}-\ldots\ldots\end{array}\right\}$$

$$=\frac{1}{4\sqrt{z^2-h^2}}\sum_{\tau=1}^{\tau=\infty}\tau\frac{(z+\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})^\tau}{h^\tau u^\tau}.$$

D'après les transformations que nous venons de faire, notre formule devient

$$\sum_{\lambda=1}^{\lambda=\infty}\sum_{\mu=1}^{\mu=\infty}\frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{u^{(2\mu-1)\lambda}}=\frac{1}{4\sqrt{z^2-h^2}}\sum_{\tau=1}^{\tau=\infty}\tau\frac{(z+\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})^\tau}{h^\tau u^\tau}.$$

§ 36. Pour tirer de cette formule celle qui nous conduira aux valeurs cherchées des fonctions

$$\theta_1,\ \theta_2,\ \theta_3,\ldots\ldots,$$

nous l'intégrerons depuis $u=0$, jusqu'à $u=1$, après l'avoir multipliée par

$$\log^{\rho-1}\left(\frac{1}{u}\right)\frac{du}{u},$$

où ρ est un nombre arbitraire. Ainsi l'on obtient

$$\sum_{\lambda=1}^{\lambda=\infty}\sum_{\mu=1}^{\mu=\infty}(-1)^{\lambda-1}\lambda\theta_\lambda\int_0^1\frac{\log^{\rho-1}(\frac{1}{u})du}{u^{(2\mu-1)\lambda+1}}=\sum_{\tau=1}^{\tau=\infty}\tau\frac{(z+\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})^\tau}{4h^\tau\sqrt{z^2-h^2}}\int_0^1\frac{\log^{\rho-1}(\frac{1}{u})du}{u^{\tau+1}},$$

et comme les intégrales

$$\int_0^1 \frac{\log^{\rho-1}\left(\frac{1}{u}\right)du}{u^{(2\mu-1)\lambda+1}},\quad \int_0^1 \frac{\log^{\rho-1}\left(\frac{1}{u}\right)}{u^{\tau+1}}\,du$$

se réduisent à

$$\frac{1}{\lambda^\rho(2\mu-1)^\rho}\int_1^\infty \log^{\rho-1}x\,dx,\quad \frac{1}{\tau^\rho}\int_1^\infty \log^{\rho-1}x\,dx,$$

il en résulte, après la supression du facteur commun $\int_1^\infty \log^{\rho-1}x\,dx$,

$$\sum_{\lambda=1}^{\lambda=\infty}\sum_{\mu=1}^{\mu=\infty}\frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{\lambda^\rho(2\mu-1)^\rho}=\sum_{\tau=1}^{\tau=\infty}\tau\frac{(z+\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})^\tau}{4h^\tau\tau^\rho\sqrt{z^2-h^2}.},$$

ou, ce qui revient au même,

$$\sum_{\lambda=1}^{\lambda=\infty}\frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{\lambda^\rho}\sum_{\mu=1}^{\mu=\infty}\frac{1}{(2\mu-1)^\rho}=\sum_{\tau=1}^{\tau=\infty}\tau\frac{(z-\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})^\tau}{4h^\tau\tau^\rho\sqrt{z^2-h^2}.}.$$

De plus, en remarquant que

$$\sum_{\mu=1}^{\mu=\infty}\frac{1}{(2\mu-1)^\rho}=\frac{1}{1^\rho}+\frac{1}{3^\rho}+\frac{1}{5^\rho}+\frac{1}{7^\rho}+\frac{1}{9^\rho}+\ldots,$$

$$=\frac{1}{\left(1-\frac{1}{3^\rho}\right)\left(1-\frac{1}{5^\rho}\right)\left(1-\frac{1}{7^\rho}\right)\ldots},$$

on trouve que cette formule peut être mise sous la forme

$$\sum_{\lambda=1}^{\lambda=\infty}\frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{\lambda^\rho}\cdot\frac{1}{\left(1-\frac{1}{3^\rho}\right)\left(1-\frac{1}{5^\rho}\right)\left(1-\frac{1}{7^\rho}\right)\ldots}=\sum_{\tau=1}^{\tau=\infty}\tau\frac{(z+\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})}{4h^\tau\tau^\rho\sqrt{z^2-h^2}},$$

et par là on obtient

$$(26)\ldots\ldots\sum_{\lambda=1}^{\lambda=\infty}\frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{\lambda^\rho}=\sum_{\tau=1}^{\tau=\infty}\tau\frac{(z+\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})^\tau}{4h^\tau\tau^\rho\sqrt{z^2-h^2}}\left(1-\frac{1}{3^\rho}\right)\left(1-\frac{1}{5^\rho}\right)\left(1-\frac{1}{7^\rho}\right)\ldots\ldots$$

C'est au moyen de cette formule que nous trouverons l'expression générale des fonctions

$$\theta_1,\ \theta_2,\ \theta_3,\ldots\ldots\ldots$$

§ 37. Pour trouver la valeur de θ_λ remarquons que la formule (26), indépendamment du nombre ρ, ne peut avoir lieu, à moins que ses parties

$$\sum_{\lambda=1}^{\lambda=\infty} \frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{\lambda^\rho},$$

$$\sum_{\tau=1}^{\tau=\infty} \tau \frac{(z+\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})^\tau}{4h^\tau\tau^\rho\sqrt{z^2-h^2}}\left(1-\frac{1}{3^\rho}\right)\left(1-\frac{1}{5^\rho}\right)\left(1-\frac{1}{7^\rho}\right)\ldots\ldots\ldots$$

ne contiennent les termes avec $\frac{1}{\lambda^\rho}$ égaux entre eux. Or dans la première partie on trouve que ce terme est

$$\frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{\lambda^\rho}.$$

En passant à la recherche des termes correspondants dans la seconde partie, nous trouvons que le produit

$$\left(1-\frac{1}{3^\rho}\right)\left(1-\frac{1}{5^\rho}\right)\left(1-\frac{1}{7^\rho}\right)\ldots\ldots\ldots$$

se réduit à

$$1-\frac{1}{3^\rho}-\frac{1}{5^\rho}-\frac{1}{7^\rho}-\frac{1}{11^\rho}+\frac{1}{15^\rho}-\ldots\ldots,$$

qu'on peut mettre sous la forme

$$\sum_{p=1}^{p=\infty} \pm\frac{1}{d_p},$$

en désignant par

$$d_1,\ d_2,\ d_3,\ldots\ldots\ldots$$

des nombres impairs

$$1,\ 3,\ 5,\ 7,\ 11,\ 15,\ 17,\ 19,\ldots\ldots\ldots,$$

sans facteur carré, et en supposant qu'on prenne le terme

$$\frac{1}{d_p}$$

avec le signe + ou —, suivant que les diviseurs premiers de d_p sont en nombre pair ou impair. La seconde partie de notre formule se réduit par conséquent à

$$\sum_{\tau=1}^{\tau=\infty}\sum_{p=1}^{p=\infty} \pm\tau\frac{(z+\sqrt{z^2-h^2})^\tau-(z-\sqrt{z^2-h^2})^\tau}{4h^\tau\sqrt{z^2-h^2}}\,\frac{1}{(\tau d_p)^\rho},$$

et par là on trouve que les termes avec $\frac{1}{\lambda^\rho}$ sont

$$\lambda\frac{(z+\sqrt{z^2-h^2})^\lambda-(z-\sqrt{z^2-h^2})^\lambda}{4h^\lambda\sqrt{z^2-h^2}}\,\frac{1}{\lambda^\rho}\pm\frac{\lambda}{d_1}\,\frac{(z+\sqrt{z^2-h^2})^{\frac{\lambda}{d_1}}-(z-\sqrt{z^2-h^2})^{\frac{\lambda}{d_1}}}{4h^{\frac{\lambda}{d_1}}\sqrt{z^2-h^2}}\,\frac{1}{\lambda^\rho}$$

$$\pm\frac{\lambda}{d_2}\,\frac{(z+\sqrt{z^2-h^2})^{\frac{\lambda}{d_2}}-(z-\sqrt{z^2-h^2})^{\frac{\lambda}{d_2}}}{4h^{\frac{\lambda}{d_2}}\sqrt{z^2-h^2}}\,\frac{1}{\lambda^\rho}\pm\ldots.,$$

en désignant par

$$d_1,\ d_2,\ldots\ldots,$$

ceux des diviseurs impairs du nombre λ qui n'ont aucun facteur carré. Quant aux signes de ces termes, conformément à ce que nous avons vu, on prendra en général

$$\frac{\lambda}{d_p}\,\frac{(z+\sqrt{z^2-h^2})^{\frac{\lambda}{d_p}}-(z-\sqrt{z^2-h^2})^{\frac{\lambda}{d_p}}}{4h^{\frac{1}{d_p}}\sqrt{z^2-h^2}}$$

avec le signe $+$ ou $-$, suivant que les diviseurs premiers de d_p sont en nombre pair ou impair.

En égalant entre eux les termes avec $\frac{1}{\lambda^\rho}$ que nous venons de trouver dans les deux parties de la formule (26), nous obtenons cette équation:

$$\frac{(-1)^{\lambda-1}\lambda\theta_\lambda}{\lambda^\rho}=\lambda\frac{(z+\sqrt{z^2-h^2})^\lambda-(z-\sqrt{z^2-h^2})^\lambda}{4h^\lambda\sqrt{z^2-h^2}}\,\frac{1}{\lambda^\rho}\pm\frac{\lambda}{d_1}\,\frac{(z+\sqrt{z^2-h^2})^{\frac{\lambda}{d_1}}-(z-\sqrt{z^2-h^2})^{\frac{\lambda}{d_2}}}{4h^{\frac{\lambda}{d_1}}}\,\frac{1}{\lambda^\rho}$$

$$\pm\frac{\lambda}{d_2}\,\frac{(z+\sqrt{z^2-h^2})^{\frac{\lambda}{d_2}}-(z-\sqrt{z^2-h^2})^{\frac{\lambda}{d_2}}}{4h^{\frac{\lambda}{d^2}}\sqrt{z^2-h^2}}\,\frac{1}{\lambda^\rho}\pm\ldots..,$$

ce qui donne pour la valeur cherchée de la fonction θ_λ

$$\theta_\lambda=(-1)^{\lambda-1}\frac{(z+\sqrt{z^2-h^2})^\lambda-(z-\sqrt{z^2-h^2})^\lambda}{4h^\lambda\sqrt{z^2-h^2}}\pm\frac{(-1)^{\lambda-1}}{d_1}\,\frac{(z+\sqrt{z^2-h^2})^{\frac{\lambda}{d_1}}-(z-\sqrt{z^2-h^2})^{\frac{\lambda}{d_1}}}{4h^{\frac{\lambda}{d_1}}\sqrt{z^2-h^2}}$$

$$\pm\frac{(-1)^{\lambda-1}}{d_2}\,\frac{(z+\sqrt{z^2-h^2})^{\frac{\lambda}{d_2}}-(z-\sqrt{z^2-h^2})^{\frac{\lambda}{d_2}}}{4h^{\frac{\lambda}{d_2}}\sqrt{z^2-h^2}}\pm\ldots\ldots\ldots.$$

§ 38. Par l'expression trouvée de θ_λ on obtient aisément toutes les fonctions

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ldots\ldots..,$$

suivant lesquelles est ordonnée notre formule (23). Ainsi, en remarquant que dans les cas de

$$\lambda = 1,\ 2,\ 4,$$

le nombre λ n'a point de diviseur qui soit impair et en même temps sans facteur carré, nous trouvons pour θ_1, θ_2, θ_4 ces valeurs:

$$\theta_1 = \frac{z+\sqrt{z^2-h^2} - z + \sqrt{z^2-h^2}}{4h\sqrt{z^2-h^2}} = \frac{1}{2h},$$

$$\theta_2 = -\frac{(z+\sqrt{z^2-h^2})^2 - (z-\sqrt{z^2-h^2})^2}{4h^2\sqrt{z^2-h^2}} = -\frac{z}{h^2},$$

$$\theta_4 = -\frac{(z+\sqrt{z^2-h^2})^4 - (z-\sqrt{z^2-h^2})^4}{4h^4\sqrt{z^2-h^2}} = -\frac{4z^3-2h^2z}{h^4}.$$

Dans les cas de

$$\lambda = 3,\ 5,\ 6,$$

les diviseurs impairs de λ et sans facteurs carrés étant

$$d_1 = 3,\ \ d_1 = 5,\ \ d_1 = 3,$$

l'expression trouvée de θ_λ nous donne

$$\theta_3 = \frac{(z+\sqrt{z^2-h^2})^3 - (z-\sqrt{z^2-h^2})^3}{4h^3\sqrt{z^2-h^2}} - \frac{1}{3}\,\frac{z+\sqrt{z^2-h^2} - z + \sqrt{z^2-h^2}}{4h\sqrt{z^2-h^2}} = \frac{2z^2 - \frac{2}{3}h^2}{h^3},$$

$$\theta_5 = \frac{(z+\sqrt{z^2-h^2})^5 - (z-\sqrt{z^2-h^2})^5}{4h^5\sqrt{z^2-h^2}} - \frac{1}{5}\,\frac{z+\sqrt{z^2-h^2} - z + \sqrt{z^2-h^2}}{4h\sqrt{z^2-h^2}} = \frac{8z^4 - 6h^2z^2 + \frac{2}{5}h^4}{h^6},$$

$$\theta_6 = -\frac{(z+\sqrt{z^2-h^2})^6 - (z-\sqrt{z^2-h^2})^6}{4h^6\sqrt{z^2-h^2}} + \frac{1}{3}\,\frac{(z+\sqrt{z^2-h^2})^2 - (z-\sqrt{z^2-h^2})^2}{4h^2\sqrt{z^2-h^2}} = -\frac{16z^5 - 16h^2z^3 + \frac{8}{3}h^4z}{h^5}.$$

Quant aux quantités

$$\int_0,\ \int_1,\ \int_2,\ \int_3,\ \ldots\ldots,$$

contenues dans notre formule (23), remarquons que, d'après le § 32,

$$\int_n f(x) = \int_{-h}^{\eta_1} f(x)dx - \int_{\eta_1}^{\eta_2} f(x)dx + \int_{\eta_2}^{\eta_3} f(x)dx - \ldots\ldots + (-1)^n \int_{\eta_n}^{h} f(x)dx,$$

et d'après les §§ 17, 19,

$$\eta_1 = h\cos\frac{n\pi}{n+1},\qquad \eta_3 = h\cos\frac{(n-2)\pi}{n+1},\ldots\ldots\ldots,$$

$$\eta_2 = h\cos\frac{(n-1)\pi}{n+1},\quad \eta_4 = h\cos\frac{(n-3)\pi}{n+1},\ldots\ldots\ldots,$$

ce qui nous donne

$$\int\limits_n f(x) = \int_{-h}^{h\cos\frac{n\pi}{n+1}} f(x)dx - \int_{h\cos\frac{n\pi}{n+1}}^{h\cos\frac{(n-1)\pi}{n+1}} f(x)dx + \int_{h\cos\frac{(n-1)\pi}{n+1}}^{h\cos\frac{(n-2)\pi}{n+1}} f(x)dx - \ldots\ldots + (-1)^n \int_{h\cos\frac{\pi}{n+1}}^{h} f(x)dx.$$

D'où, en faisant

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ldots\ldots,$$

on obtient

$$\int\limits_0 f(x) = \int_{-h}^{h} f(x)dx,$$

$$\int\limits_1 f(x) = \int_{-h}^{0} f(x)dx - \int_{0}^{h} f(x)dx,$$

$$\int\limits_2 f(x) = \int_{-h}^{-\frac{1}{2}h} f(x)dx - \int_{-\frac{1}{2}h}^{\frac{1}{2}h} f(x)dx + \int_{\frac{1}{2}h}^{h} f(x)dx,$$

$$\int\limits_3 f(x) = \int_{-h}^{-\frac{1}{\sqrt{2}}h} f(x)dx - \int_{-\frac{1}{\sqrt{2}}h}^{0} f(x)dx + \int_{0}^{\frac{1}{\sqrt{2}}h} f(x)dx - \int_{\frac{1}{\sqrt{2}}h}^{h} f(x)dx,$$

$$\int\limits_4 f(x) = \int_{h}^{-\frac{\sqrt{5}+1}{4}h} f(x)dx - \int_{-\frac{\sqrt{5}+1}{4}h}^{-\frac{\sqrt{5}-1}{4}h} f(x)dx + \int_{-\frac{\sqrt{5}-1}{4}h}^{\frac{\sqrt{5}-1}{4}h} f(x)dx - \int_{\frac{\sqrt{5}-1}{4}h}^{\frac{\sqrt{5}+1}{4}h} f(x)dx + \int_{\frac{\sqrt{5}+1}{4}h}^{h} f(x)dx,$$

$$\int\limits_5 f(x) = \int_{-h}^{-\frac{\sqrt{3}}{2}h} f(x)dx - \int_{-\frac{\sqrt{3}}{2}h}^{-\frac{1}{2}h} f(x)dx + \int_{-\frac{1}{2}h}^{0} f(x)dx - \int_{0}^{\frac{1}{2}h} f(x)dx + \int_{\frac{\sqrt{3}}{2}h}^{\frac{1}{2}h} f(x)dx - \int_{\frac{1}{2}h}^{h} f(x)dx,$$

. . . .

En vertu des valeurs trouvées de

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ldots\ldots\ldots,$$

$$\int\limits_0,\ \int\limits_1,\ \int\limits_2,\ \int\limits_3,\ldots\ldots\ldots\ldots,$$

notre formule (23) se réduit à ce développement de la fonction $f(z)$:

$$(27)\ldots\ldots f(z)=\frac{1}{2h}\int_{-h}^{h}f(x)dx-\left[\int_{-h}^{0}f(x)dx-\int_{0}^{h}f(x)dx\right]\frac{z}{h^2}+\left[\int_{h}^{-\frac{1}{2}h}f(x)dx-\int_{-\frac{1}{2}h}^{\frac{1}{2}h}f(x)dx+\int_{\frac{1}{2}h}^{h}f(x)dx\right]\frac{2z^2-\frac{2}{3}h^2}{h^3}$$

$$-\left[\int_{-h}^{-\frac{1}{\sqrt{2}}h}f(x)dx-\int_{-\frac{1}{\sqrt{2}}h}^{0}f(x)dx+\int_{0}^{\frac{1}{\sqrt{2}}h}f(x)dx-\int_{\frac{1}{\sqrt{2}}h}^{h}f(x)dx\right]\frac{4z^3-2h^2z}{h^4}$$

$$+\left[\int_{-h}^{-\frac{\sqrt{5}+1}{4}h}f(x)dx-\int_{-\frac{\sqrt{5}+1}{4}h}^{-\frac{\sqrt{5}-1}{4}h}f(x)dx+\int_{-\frac{\sqrt{5}-1}{4}h}^{\frac{\sqrt{5}-1}{4}h}f(x)dx-\int_{\frac{\sqrt{5}-1}{4}h}^{\frac{\sqrt{5}+1}{4}h}f(x)dx+\int_{\frac{\sqrt{5}+1}{4}h}^{h}f(x)dx\right]\frac{8z^4-6h^2z^2+\frac{2}{5}h^2}{h^5}$$

$$-\left[\int_{-h}^{-\frac{\sqrt{3}}{2}h}f(x)dx-\int_{-\frac{\sqrt{3}}{2}h}^{-\frac{1}{2}h}f(x)dx+\int_{-\frac{1}{2}h}^{0}f(x)dx-\int_{0}^{\frac{1}{2}h}f(x)dx+\int_{\frac{1}{2}h}^{\frac{\sqrt{3}}{2}h}f(x)dx-\int_{\frac{\sqrt{3}}{2}h}^{h}f(x)dx\right]\frac{16z^5-16h^2z^3+\frac{8}{3}h^4z}{h^6}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

qui, suivant le nombre de termes qu'on y conserve, donne l'expression $f(z)$ sous la forme d'un polynome de degré plus ou moins élevé.

Quant à l'évaluation approximative des expressions

$$\int_{-h}^{h}f(x)dx,$$

$$\int_{-h}^{0}f(x)dx-\int_{0}^{h}f(x)dx,$$

$$\int_{-h}^{-\frac{1}{2}h}f(x)dx-\int_{-\frac{1}{2}h}^{\frac{1}{2}h}f(x)dx+\int_{\frac{1}{2}h}^{h}f(x)dx,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

d'après les valeurs données de $f(z)$

$$f(z_1),\ f(z_2),\ f(z_3),\ldots\ldots\ldots f(z_i),$$

on y parviendra, comme nous l'avons montré dans le § 22, à l'aide du planimètre, si l'on a la représentation graphique de la courbe

$$y=f(x);$$

dans le cas contraire on les cherchera à l'aide de la formule (15) qui, par le changement de x en z, $F(x)$ en $f(z)$, devient

$$(28)\ldots\ldots\int_{z_1}^{\eta_1} f(z)dz - \int_{\eta_1}^{\eta_2} f(z)dz + \int_{\eta_2}^{\eta_3} f(z)dz - \ldots\ldots + (-1)^{\upsilon}\int_{\eta_\upsilon}^{z_i} f(z)dz =$$

$$\frac{1}{2}\left[\begin{array}{l} M_1 + M_2 + \ldots + M_{i'} - M_{i'+1} - M_{i'+2} - \ldots - M_{i''} + M_{i''+1} + \ldots + M_{i'''} - M_{i'''+1} \\ - M_{i'''+2} - \ldots - (-1)^{\upsilon} M_{i^{(\upsilon)}+1} - \ldots - (-1)^{\upsilon} M_{i^{(\upsilon)}} + (-1)^{\upsilon} M_{i^{(\upsilon)}+1} + \ldots + (-1)^{\upsilon} M_i \end{array}\right]$$

$$-\frac{(z_{i'+1}-\eta_1)^2 f(z_{i'}) - (z_{i'}-\eta_1)^2 f(z_{i'+1})}{z_{i'+1}-z_{i'}} + \frac{(z_{i''+1}-\eta_2)^2 f(z_{i''}) - (z_{i''}-\eta_2)^2 f(z_{i''+1})}{z_{i''+1}-z_{i''}}$$

$$-\ldots\ldots\ldots\ldots\ldots + (-1)^{\upsilon}\frac{(z_{i^{(\upsilon)}+1}-\eta_\upsilon)^2 f(z_{i^{(\upsilon)}}) - (z_{i^{(\upsilon)}}-\eta_\upsilon)^2 f(z_{i^{(\upsilon)}+1})}{z_{i^{(\upsilon)}+1}-z_{i^{(\upsilon)}}},$$

où

$$\begin{array}{ll} z_{i'}, & z_{i'+1}, \\ z_{i''}, & z_{i''+1}, \\ \ldots & \ldots \\ z_{i^{(\upsilon)}}, & z_{i^{(\upsilon)}+1} \end{array}$$

sont les couples de termes dans la suite

$$z_1,\ z_2,\ z_3, \ldots\ldots\ldots z_i$$

respectivement les plus proches des quantites

$$\eta_1,\ \eta_2, \ldots\ldots\ldots \eta_\upsilon,$$

et

$$M_1 = (z_2-z_1)f(z_1),\quad M_2 = (z_3-z_1)f(z_2), \ldots\ldots,$$
$$M_{i-1} = (z_i - z_{i-2})f(z_{i-1}),\quad M_i = (z_i - z_{i-1})f(z_i)$$

Dans la formule (27) les limites de la variable sont $-h$ et $+h$, ce qui suppose que dans la suite des valeurs données de $f(z)$

$$f(z_1),\ f(z_2), \ldots\ldots\ldots f(z_i)$$

les quantités z_1 et z_i sont, au signe près, égales. Or, tant que cela n'a pas lieu, on y parvient très aisément par le seul changement de la variable, savoir, en prenant pour la nouvelle variable z la différence $z - \frac{z_1+z_i}{2}$.

§ 39. Pour montrer l'usage de la formule (27) nous allons l'appliquer au même exemple que nous avons traité plus haut (III). Dans cet exemple les valeurs limites de la variable sont $x_1 = 0$, $x_i = 24{,}5$. Donc, conformément à ce que nous avons dit, on prendra pour la nouvelle variable, que nous désignerons par z, la différence

$$x - \frac{0+24{,}5}{2} = x - 12{,}25.$$

D'après cela, la table des données que nous avons eue dans le § 27 se change dans celle-ci:

m	z_m	$f(z_m)$	m	z_m	$f(z_m)$
1	— 12,25	0,000000	16	1,25	0,000480
2	— 11,35	— 0,000022	17	1,55	0,000568
3	— 10,65	— 0,000098	18	2,75	0,000706
4	— 10,15	— 0,000077	19	3,35	0,000841
5	— 7,05	— 0,000115	20	4,05	0,000927
6	— 6,65	— 0,000134	21	5,15	0,001057
7	— 6,15	— 0,000094	22	6,35	0,001256
8	— 5,95	— 0,000101	23	6,35	0,001298
9	— 5,05	— 0,000047	24	6,95	0,001419
10	— 3,75	— 0,000006	25	7,55	0,001496
11	— 3,65	0,000007	26	8,95	0,001805
12	— 3,15	0,000081	27	9,55	0,001989
13	— 1,05	0,000215	28	9,95	0,002043
14	— 0,35	0,000317	29	11,75	0,002421
15	0,45	0,000352	30	12,25	0,002618

Comme dans cette table les valeurs limites de z sont

$$z_1 = -12{,}25,\quad z_{30} = 12{,}25,$$

on prendra

$$h = 12{,}25,$$

et pour cette valeur de h on aura, d'après la formule (27), ce développement de la fonction $f(z)$:

$$(29)\ldots f(z) = \frac{1}{24.5}\int_{-12,25}^{12.25} f(z)dz - \left[\int_{-12,25}^{0} f(z)dz - \int_{0}^{12,25} f(z)dz\right]\frac{z}{150{,}06}$$

$$+\left[\int_{-12,25}^{-6,125} f(z)dz - \int_{-6,125}^{6,125} f(z)dz + \int_{6,125}^{12,25} f(z)dz\right]\frac{2z^2-100{,}04}{1838{,}26}$$

$$-\left[\int_{-12,25}^{-8,662} f(z)dz - \int_{-8,662}^{0} f(z)dz + \int_{0}^{8,662} f(z)dz - \int_{8,662}^{12,25} f(z)dz\right]\frac{4z^3-300{,}13z}{22519}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

§ 40. Pour évaluer les expressions

$$\int_{-12,25}^{12,25} f(z)dz,$$

$$\int_{-12,25}^{0} f(z)dz - \int_{0}^{12,25} f(z)dz,$$

$$\int_{-12,25}^{-6,125} f(z)dz - \int_{-6,125}^{6,125} f(z)dz + \int_{6,125}^{12,25} f(z)dz,$$

$$\int_{-12,25}^{-8,662} f(z)dz - \int_{-8,662}^{0} f(z)dz + \int_{0}^{8,662} f(z)dz - \int_{8,662}^{12,25} f(z)dz,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

on cherchera préalablement les valeurs de M_1, M_2, ... M_{30}, d'après les formules

$$M_1 = (z_2 - z_1)f(z_1),\quad M_2 = (z_3 - z_1)f(z_2),\quad M_3 = (z_4 - z_2)f(z_3),$$

$$\ldots\ldots\ldots\ldots M_{29} = (z_{30} - z_{28})f(z_{29}),\quad M_{30} = (z_{30} - z_{29})f(z_{30}).$$

L'on obtient ainsi

$M_1 = -0{,}000000$,	$M_{16} = 0{,}000528$,
$M_2 = -0{,}000035$,	$M_{17} = 0{,}000852$,
$M_3 = -0{,}000118$,	$M_{18} = 0{,}001271$,
$M_4 = -0{,}000277$,	$M_{19} = 0{,}001093$,
$M_5 = -0{,}000402$,	$M_{20} = 0{,}001668$,
$M_6 = -0{,}000122$,	$M_{21} = 0{,}002431$,
$M_7 = -0{,}000066$,	$M_{22} = 0{,}001507$,
$M_8 = -0{,}000111$,	$M_{23} = 0{,}000779$,
$M_9 = -0{,}000103$,	$M_{24} = 0{,}001703$,
$M_{10} = -0{,}000008$,	$M_{25} = 0{,}002992$,
$M_{11} = 0{,}000004$,	$M_{26} = 0{,}003610$,
$M_{12} = 0{,}000210$,	$M_{27} = 0{,}001989$,
$M_{13} = 0{,}000602$,	$M_{28} = 0{,}004495$,
$M_{14} = 0{,}000475$,	$M_{29} = 0{,}005568$,
$M_{15} = 0{,}000563$,	$M_{30} = 0{,}001309$.

Ces valeurs étant déterminées, l'évaluation des expressions précédentes au moyen de la formule (28) devient très expéditive.

Pour trouver l'intégrale

$$\int_{-12,25}^{12,25} f(z)dz,$$

on prendra dans la formule (28)

$$i = 30,\ \upsilon = 0,\ z_1 = -12,25,\ z_i = 12,25,$$

et par là on aura

$$\int_{-12,25}^{12,25} f(z)dz = \tfrac{1}{2}(M_1 + M_2 + \ldots\ldots\ldots + M_{30}),$$

d'où, en vertu des valeurs trouvées de M_1, $M_2, \ldots\ldots\ldots M_{30}$, il résulte

$$\int_{-12,25}^{12,25} f(z)dz = \tfrac{1}{2} . 0,032407 = 0,016203.$$

Pour trouver la valeur de l'expression

$$\int_{-12,25}^{0} f(z)dz - \int_{0}^{12,25} f(z)dz,$$

on fera dans la formule (28)

$$i = 30,\ \upsilon = 1,\ z_1 = -12,25,\ z_i = 12,25,\ \eta_1 = 0,$$

ce qui nous donne

$$\int_{-12,25}^{0} f(z)dz - \int_{0}^{12,25} f(z)dz = \tfrac{1}{2}(M_1 + M_2 + \ldots + M_{i'} - M_{i'+1} - \ldots - M_{30}) - \frac{z^2_{i'+1} f(z_{i'}) - z^2_{i'} f(z_{i'+1})}{z_{i'+1} - z_{i'}},$$

où, suivant notre notation, $z_{i'}$, $z_{i'+1}$ désignent la couple des valeurs de z_m les plus proches de 0. — Comme dans la colonne des valeurs de z_m (§ 39) celles les plus proches de 0 sont

$$z_{14} = -0,35,\ z_{15} = 0,45,$$

et que, d'après la table des valeurs de M_1, $M_2, \ldots\ldots\ldots M_{30}$,

$$M_1 + M_2 + \ldots\ldots\ldots + M_{14} = 0,000049,$$
$$M_{15} + M_{16} + \ldots\ldots\ldots + M_{30} = 0,032358,$$

la formule précédente se réduit à celle-ci:

$$\int_{-12,25}^{0} f(z)dz - \int_{0}^{12,25} f(z)dz = \tfrac{1}{2}(0,000049 - 0,032358) - \frac{0,45^2 f(-0,35) - 0,35^2 f(0,45)}{0,45 + 0,35}.$$

D'où, en ayant égard aux valeurs de $f(-0,35)=0,000317$, $f(0,45)=0,000352$, on obtient

$$\int_{-12,25}^{0} f(z)dz - \int_{0}^{12,25} f(z)dz = \tfrac{1}{2}(0,000049-0,032358) - \frac{0,45^2\ 0,000317 - 0,35^2 . 0,000352}{0,45+0,35}$$

$$= -0,016180.$$

En posant dans la formule (28)

$$i=30,\ \nu=2,\ z_1=-12,25,\ z_i=12,25,$$

$$\eta_1=-6,125,\ \eta_2=6,125,$$

on a, pour la détermination de la valeur de

$$\int_{-12,25}^{-6,125} f(z)dz - \int_{-6,125}^{6,125} f(z)dz + \int_{6,125}^{6,125} f(z)dz,$$

$$\int_{-12,25}^{-6,125} f(z)dz - \int_{-6,125}^{6,125} f(z)dz + \int_{6,125}^{12,25} f(z)dz = \frac{1}{2}\begin{bmatrix} M_1 + M_2 + \dots\dots\dots + M_{i'} + M_{i''+1} \\ - M_{i''} + M_{i''+1} + \dots\dots\dots + M_{30} \end{bmatrix}$$

$$- \frac{(z_{i'+1}+6,125)^2 f(z_{i'}) - (z_{i'}+6,125)^2 f(z_{i'+1})}{z_{i'+1}-z_{i'}}$$

$$+ \frac{(z_{i''+1}-6,125)^2 f(z_{i''}) - (z_{i''}-6,125)^2 f(z_{i''+1})}{z_{i''+1}-z_{i''}},$$

où

$$z_{i'},\ z_{i'+1},\ z_{i''},\ z_{i''+1}$$

désignent les couples des valeurs de z_m qui sont respectivement les plus proches des quantités

$$-6,125,\quad 6,125.$$

Comme dans la suite des valeurs de z_m les termes le plus proches de

$$-6,125,\quad 6,125$$

sont

$$z_7=-6,15,\quad z_8=-5,95,$$

$$z_{21}=5,15,\qquad z_{22}=6,35,$$

cette formule nous donne

$$\int_{-12,25}^{-6,125} f(z)dz - \int_{-6,125}^{6,125} f(z)dz + \int_{6,125}^{12,25} f(z)dz = \tfrac{1}{2}[M_1 + M_2 + \ldots + M_7 - M_8 - \ldots - M_{21} + M_{22} + \ldots + M_{30}]$$
$$- \frac{(z_8 + 6,125)^2 f(z_7) - (z_7 + 6,125)^2 f(z_8)}{z_8 - z_7}$$
$$+ \frac{(z_{22} - 6,125)^2 f(z_{21}) - (z_{21} - 6,125)^2 f(z_{22})}{z_{22} - z_{21}},$$

et par là, en substituant les valeurs de

$$z_7,\ z_8,\ z_{21},\ z_{22},\ f(z_7),\ f(z_8),\ f(z_{21}),\ f(z_{22}),$$
$$M_1,\ M_2, \ldots\ldots\ldots\ldots M_{30},$$

on obtient

$$\int_{-12,25}^{-6,125} f(z)dz - \int_{-6,125}^{6,125} f(z)dz + \int_{6,125}^{12.25} f(z)dz = \tfrac{1}{2}\, 0,013457 + 0,000014 - 0,000951$$
$$= 0,005791.$$

En cherchant de la même manière la valeur de

$$\int_{-12,25}^{-8,662} f(z)dz - \int_{-8,662}^{0} f(z)dz + \int_{0}^{8,662} f(z)dz - \int_{8,662}^{12,25} f(z)dz,$$

on prendra dans la formule (28)

$$i = 30,\ z_1 = -12,25,\ z_i = 12,25,$$
$$\nu = 3,\ \eta_1 = -8,662,\ \eta_2 = 0,\ \eta_3 = 8,662,$$
$$i' = 4,\ i'' = 14,\ i''' = 25,$$

en vertu de quoi elle devient

$$\int_{-12,25}^{-8,662} f(z)dz - \int_{-8,662}^{0} f(z)dz + \int_{0}^{8,662} f(z)dz - \int_{8,662}^{12,25} f(z)dz = \tfrac{1}{2}\begin{bmatrix} M_1 + M_2 + \ldots + M_4 - M_5 - \ldots\ldots - M_{14} \\ + M_{15} + \ldots + M_{25} - M_{26} - \ldots\ldots - M_{30} \end{bmatrix}$$
$$- \frac{(z_5 + 8,662)^2 f(z_4) - (z_4 + 8,662)^2 f(z_5)}{z_5 - z_4}$$
$$+ \frac{z_{15}^2 f(z_{14}) - z_{14}^2 f(z_{15})}{z_{15} - z_{14}}$$
$$- \frac{(z_{26} - 8,662)^2 f(z_{25}) - (z_{25} - 8,662)^2 f(z_{26})}{z_{26} - z_{25}}.$$

D'où, par la substitution des valeurs de

$$f(z_4),\ f(z_5),\ f(z_{14}),\ f(z_{15}),\ f(z_{25}),\ f(z_{26}),$$
$$M_1,\ M_2, \ldots\ldots\ldots\ldots\ldots\ldots M_{30},$$
$$z_4,\ z_5,\ z_{14},\ z_{15},\ z_{25},\ z_{26},$$

on tire

$$\int_{-12,25}^{-8,662} f(z)dz - \int_{-8,662}^{0} f(z)dz + \int_{0}^{8,662} f(z)dz - \int_{8,662}^{12,25} f(z)dz = 0,000268.$$

D'après cela on trouve par la formule (29) ce développement de la fonction cherchée:

$$f(z) = \frac{0,016203}{24,5} + \frac{0,016180}{150,06} z + \frac{0,005791}{1838,26}(2z^2 - 100,04) - \frac{0,000268}{22519}(4z^3 - 300,12z) + \ldots .$$

qui donne son expression sous la forme d'un polynome de degré plus ou moins élevé, suivant le nombre de termes qu'on conserve dans cette série. Ainsi, en s'arrêtant au quatrième terme, on trouve, pour son expression sous la forme d'un polynome du troisième degré, cette formule:

$$\frac{0,016203}{24,5} + \frac{0,016180}{150,06} z + \frac{0,005791}{1838,26}(2z^2 - 100,04) - \frac{0,000268}{22519}(4z^3 - 300,12z)$$
$$= 0,0003463 + 0,00011139z + 0,00000630z^2 - 0,0000000475z^3,$$

ce qui ne diffère de l'expression, obtenue dans la section V, que par des quantités tout-à-fait négligeables.

TABLE

des solutions de l'équation

$$\int_{-h}^{\eta_1} F(x)dx - \int_{\eta_1}^{\eta_2} F(x)dx + \int_{\eta_2}^{\eta_3} F(x)dx - \dots\dots\dots + (-1)^{\upsilon} \int_{\eta_\upsilon}^{h} F(x)dx = sA_l$$

qui correspondent à la plus grande valeur du facteur s,

$F(x)$ **représentant le polynome** $A_0 + A_1 x + \dots\dots + A_n x^n$.

$n = 0.$

$l = 0.$	$\upsilon = 0.$ $s = 2h.$

$n = 1.$

$l = 0.$	$\upsilon = 0.$ $s = 2h.$
$l = 1.$	$\upsilon = 1.$ $\eta_1 = 0.$ $s = -h^2.$

$n = 2.$

$l = 0.$	$\upsilon = 2.$ $\eta_1 = -0{,}79370h,\ \ \eta_2 = 0{,}79370h.$ $s = -1{,}17480h.$
$l = 1.$	$\upsilon = 1.$ $\eta_1 = 0.$ $s = -h^2.$
$l = 2.$	$\upsilon = 2.$ $\eta_1 = -0{,}5h,\ \ \eta_2 = 0{,}5h.$ $s = 0{,}5h^3.$

$n = 3.$

$l = 0.$	$\nu = 2.$ $\eta_1 = -0{,}79370h,\ \eta_2 = 0{,}79370h.$ $s = -1{,}17480h.$
$l = 1.$	$\nu = 3.$ $\eta_1 = -0{,}84090h,\ \eta_2 = 0,\ \eta_3 = 0{,}84090h.$ $s = 0{,}41421h^2.$
$l = 2.$	$\nu = 2.$ $\eta_1 = -0{,}5h,\ \eta_2 = 0{,}5h.$ $s = 0{,}5h^3.$
$l = 3.$	$\nu = 3.$ $\eta_1 = -0{,}70711h,\ \eta_2 = 0,\ \eta_3 = 0{,}70711h.$ $s = -0{,}25h^4.$

$n = 4.$

$l = 0.$	$\nu = 4.$ $\eta_1 = -0{,}89725h,\ \eta_2 = -0{,}60587h,\ \eta_3 = 0{,}60587h,\ \eta_4 = 0{,}89725h.$ $s = 0{,}83446h.$
$l = 1.$	$\nu = 3.$ $\eta_1 = -0{,}84090h,\ \eta_2 = 0,\ \eta_3 = 0{,}84090h.$ $s = 0{,}41421h^2.$
$l = 2.$	$\nu = 4.$ $\eta_1 = -0{,}87305h,\ \eta_2 = -0{,}37305h,\ \eta_3 = 0{,}37305h,\ \eta_4 = 0{,}87305h.$ $s = -0{,}15139h^3.$
$l = 3.$	$\nu = 3.$ $\eta_1 = -0{,}70711h,\ \eta_2 = 0,\ \eta_3 = 0{,}70711h.$ $s = -0{,}25h^4.$
$l = 4.$	$\nu = 4.$ $\eta_1 = -0{,}80902h,\ \eta_2 = -0{,}30902h,\ \eta_3 = 0{,}30902h,\ \eta_4 = 0{,}80902h.$ $s = 0{,}125h^5.$

$n = 5.$

	$\upsilon = 4.$
$l = 0.$	$\eta_1 = -0,89725h,\ \eta_2 = -0,60587h,\ \eta_3 = 0,60587h,\ \eta_4 = 0,89725h.$ $s = 0,83446h.$
	$\upsilon = 5.$
$l = 1.$	$\eta_1 = -0,91682h,\ \eta_2 = -0,67418h,\ \eta_3 = 0,\ \eta_4 = 0,67418h,\ \eta_5 = 0,91682h.$ $s = -0,22772h^2.$
	$\upsilon = 4.$
$l = 2.$	$\eta_1 = -0,87305h,\ \eta_2 = -0,37305h,\ \eta_3 = 0,37305h,\ \eta_4 = 0,87305h.$ $s = -0,15139h^3.$
	$\upsilon = 5.$
$l = 3.$	$\eta_1 = -0,89945h,\ \eta_2 = -0,55589h,\ \eta_3 = 0,\ \eta_4 = 0,55589h,\ \eta_5 = 0,89945h.$ $s = 0,05901h^4.$
	$\upsilon = 4.$
$l = 4.$	$\eta_1 = -0,80902h,\ \eta_2 = -0,30902h,\ \eta_3 = 0,30902h,\ \eta_4 = 0,80902h.$ $s = 0,125h^5.$
	$\upsilon = 5.$
$l = 5.$	$\eta_1 = -0,86602h,\ \eta_2 = -0,5h,\ \eta_3 = 0,\ \eta_4 = 0,5h,\ \eta_5 = 0,86602h.$ $s = -0,0625h^6.$

- 12 -

MÉMOIRES
DE
L'ACADÉMIE IMPÉRIALE DES SCIENCES DE ST.-PÉTERSBOURG, VII^e SÉRIE.
TOME I, N° 15 ET DERNIER.

SUR L'INTERPOLATION

PAR

LA MÉTHODE DES MOINDRES CARRÉS.

PAR

P. Tchébychef,
Membre de l'Académie.

Lu le 29 avril 1859.

ST.-PÉTERSBOURG, 1859.
Commissionnaires de l'Académie Impériale des sciences:

à St.-Pétersbourg, MM. Eggers et Comp., à Riga, M. Samuel Schmidt, à Leipzig, M. Léopold Voss.

Prix: 25 Kop. arg. = 8 Ngr.

Imprimé par ordre de l'Académie.

Octobre 1859.

C. Vessélofski, Secrétaire perpétuel.

Imprimerie de l'Académie Impériale des sciences.

SUR L'INTERPOLATION

PAR LA MÉTHODE DES MOINDRES CARRÉS.

Par **P. Tchébychef.**

Dans le Mémoire *Sur les fractions continues* [1]) j'ai donné la série qui présente le résultat définitif de l'interpolation parabolique par la méthode des *moindres carrés*. Comme cette série fournit directement l'expression de la fonction interpolée sous la forme d'un polynome avec les coefficients les plus probables, et sans qu'on fixe d'avance le nombre de ses termes, on conçoit que, sous le rapport théorique, elle ne laisse rien à désirer pour l'interpolation parabolique. Mais pour rendre son usage tout-à-fait praticable, il restait à indiquer la marche commode à suivre dans l'évaluation de ses termes. C'est ce que nous avons fait pour le cas le plus simple où les valeurs de la variable, correspondantes aux valeurs connues de la fonction interpolée, sont équidistantes. En traitant ce cas particulier dans la note *Sur une nouvelle formule* [2]), nous avons indiqué une réduction de notre série à la formule que voici, très propre à l'application:

$$\begin{aligned} u = {} & \frac{1}{n}\sum u_i \cdot \varphi_0(z) + \frac{3}{n(n^2-1^2)}\sum \frac{i}{1}\,\frac{n-i}{1}\,\Delta u_i \cdot \varphi_1(z) \\ & + \frac{5}{n(n^2-1^2)(n^2-2^2)}\sum \frac{i(i+1)}{1.2}\,\frac{(n-i)(n-i-1)}{1.2}\,\Delta^2 u_i \cdot \varphi_2(z) \\ & + \frac{7}{n(n^2-1^2)(n^2-2^2)(n^2-3^2)}\sum \frac{i(i+1)(i+2)}{1.2.3}\,\frac{(n-i)(n-i-1)(n-i-2)}{1.2.3}\,\Delta^3 u_i \cdot \varphi_3(z) \\ & + \text{etc.,} \end{aligned}$$

en désignant par

$$u_1,\ u_2,\ u_3, \ldots . u_n$$

les valeurs données de u qui correspondent aux valeurs équidistantes de x

$$x = x_1,\ x_2,\ x_2, \ldots . x_n,$$

---

1) Journal de M. Liouville, T. III, 2^{me} série.

2) Mélanges mathématiques et astronomiques, T. II.

et en faisant, pour abréger,

$$z = \frac{x - \frac{1}{2}(x_1 + x_n)}{x_2 - x_1}.$$

Dans cette série les signes de sommation s'étendent à toutes les valeurs de i, depuis $i = 1$, jusqu'à $i = n$, et

$$\varphi_0(z),\ \varphi_1(z),\ \varphi_2(z),\ \varphi_3(z), \ldots\ldots$$

sont des fonctions entières de z qu'on tire de la formule

$$\Delta^l\left(z + \frac{n-1}{2}\right)\left(z + \frac{n-3}{2}\right)\ldots\left(z + \frac{n-2l+1}{2}\right)\left(z - \frac{n+1}{2}\right)\left(z - \frac{n+3}{2}\right)\ldots\left(z - \frac{n+2l-1}{2}\right),$$

en adoptant pour l les valeurs

$$0,\ 1,\ 2,\ 3, \ldots\ldots$$

Comme ces fonctions sont liées entre elles par l'équation

$$\varphi_l(z) = 2(2l-1)z\varphi_{l-1}(z) - (l-1)^2[n^2 - (l-1)^2]\varphi_{l-2}(z),$$

et que

$$\varphi_0(z) = \Delta^0 1 = 1,$$

$$\varphi_1(z) = \Delta\left(z + \frac{n-1}{2}\right)\left(z - \frac{n+1}{2}\right) = 2z,$$

on trouve sur le champ

$$\varphi_2(z) = 12z^2 - (n^2 - 1),$$

$$\varphi_3(z) = 120z^3 - 6(3n^2 - 7)z,$$

$$\varphi_4(z) = 1680z^4 - 120(3n^2 - 13)z^2 + 9(n^2 - 1)(n^2 - 9),$$

$$\varphi_5(z) = 30240z^5 - 8400(n^2 - 7)z^3 + 32(15n^4 - 230n^2 + 407)z,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Ce développement de u qui résulte de notre série, tant que les valeurs

$$x_1,\ x_2,\ x_3, \ldots\ldots x_n$$

sont équidistantes, est très commode pour l'évaluation de l'expression de u, vu que ses termes, comme ceux de la formule d'interpolation de Newton, contiennent les différences

$$\Delta u_i,\ \Delta^2 u_i,\ \Delta^3 u_i, \ldots\ldots,$$

dont les ordres vont en croissant, et que ces différences, sous les signes de sommation, ne sont accompagnées que des facteurs

$$\frac{i}{1},\ \frac{n-i}{1},$$

$$\frac{i(i+1)}{1.2},\ \frac{(n-i)(n-i-1)}{1.2},$$

$$\frac{i(i+1)(i+2)}{1.2.3},\ \frac{(n-i)(n-i-1)(n-i-2)}{1.2.3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

qui, d'après la propriété connue des nombres polygonaux, s'évaluent aisément par seule voie d'addition. Et comme cette série nous fournit l'expression de u avec les coefficients les plus probables, on conçoit qu'elle ne laisse rien à désirer pour l'interpolation dans le cas particulier où les valeurs de la variable qui correspondent aux valeurs connues de la fonction sont équidistantes.

Mais ce n'est pas le seul parti qu'on puisse tirer de notre série pour l'application; son usage est aussi très utile dans tous les autres cas d'interpolation parabolique, comme nous allons le montrer à présent, en indiquant la marche qui conduit aisément à la détermination successive de ses termes. On verra, d'après cela, que notre série procure un moyen très propre pour évaluer, terme par terme, l'expression de la fonction interpolée u, et qu'elle donne, en même temps, la somme des carrés des différences entre ses valeurs connues

$$u_1,\ u_2,\ u_3,\ldots .u_n,$$

et celles qui résultent de l'ensemble des termes trouvés pour son expression. D'après quoi on aura, sur le champ, l'erreur moyenne avec laquelle les termes trouvés de u représentent ses valeurs données, et par là on reconnaîtra tout de suite celui auquel on peut s'arrêter. Ainsi, au moyen de notre série, on trouvera tout à la fois et le nombre de termes de u qui sont importants pour l'interpolation et leurs coefficients déterminés par la méthode des moindres carrés. Pour faire comprendre la supériorité de cette méthode d'interpolation sur celles dont on se sert ordinairement, remarquons qu'elle donnera précisément, en général plus aisément, les mêmes résultats, que ceux que l'on trouve par la résolution des équations fournies par la méthode des *moindres carrés* qui suppose que le nombre des termes dans l'expression de u soit fixé d'avance. D'autre part, en déterminant et le nombre de termes de u que l'on doit calculer et leurs valeurs prescrites par la méthode des moindres carrés, elle sera, si ce n'est dans certains cas exceptionnels, plus expéditive que la méthode d'interpolation de Cauchy qui est loin de donner les résultats les plus probables découlant de la méthode des moindres carrés.

§ I.

D'après ce que nous avons montré dans le Mémoire cité plus haut, si les valeurs données de la fonction u

$$u_1,\ u_2,\ u_3,\ldots .u_n$$

qui correspondent à

$$x = x_1,\ x_2,\ x_3,\ldots .x_n$$

sont affectées d'erreurs de la même nature, et que l'on cherche son expression, par la méthode des *moindres carrés*, sous la forme d'un polynome de degré quelconque, on aura*)

$$u = K_0\psi_0(x) + K_1\psi_1(x) + K_2\psi_2(x) + \ldots\ldots,$$

*) Nous n'emprunterons de notre Mémoire antérieur que la forme de cette série; mais tout ce qui est important pour son application sera donné dans ce qui suit.

où

$$K_0,\ K_1,\ K_2,\ldots\ldots$$

sont des coefficients constants, et

$$\psi_0(x),\ \psi_1(x),\ \psi_2(x),\ldots\ldots$$

les dénominateurs des réduites de la somme

$$\sum\frac{1}{x-x_i}=\frac{1}{x-x_1}+\frac{1}{x-x_2}+\frac{1}{x-x_3}+\ldots\ldots+\frac{1}{x-x_n},$$

qu'on trouve par son développement en fraction continue

$$\cfrac{\alpha_1}{q_1+\cfrac{\alpha_2}{q_2+\cfrac{\alpha_3}{q_3+\ldots\ldots}}}$$

Dans cette fraction les constantes

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ldots\ldots$$

peuvent être choisies arbitrairement. Pour fixer les idées, nous supposerons qu'elles sont choisies de manière à ce que les coefficients de x dans les quotients

$$q_1,\ p_2,\ q_3,\ldots\ldots$$

soient égaux à 1, et nous désignerons par

$$a_1,\ -a_2,\ -a_3,\ldots\ldots$$

les valeurs de

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ldots\ldots$$

qui remplissent cette condition. D'après cela, et en remarquant que les dénominateurs

$$q_1,\ q_2,\ q_3,\ldots\ldots$$

seront des fonctions du premier degré, on aura, pour la détermination des fonctions

$$\psi_0(x),\ \psi_1(x),\ \psi_2(x),\ldots\ldots,$$

ce développement de

$$\sum\frac{1}{x-x_i}$$

en fraction continue:

$$\sum\frac{1}{x-x_i}=\cfrac{a_1}{x-b_1-\cfrac{a_2}{x-b_2-\cfrac{a_3}{x-b_3-\ldots\ldots}}}$$

D'où l'on tire, pour l'évaluation de ses réduites

$$\frac{\varphi_0(x)}{\psi_0(x)},\ \frac{\varphi_1(x)}{\psi_1(x)},\ \frac{\varphi_2(x)}{\psi_2(x)},\ldots\ldots\frac{\varphi_\lambda(x)}{\psi_\lambda(x)},\ldots\ldots,$$

les formules suivantes:

$$(1)\ldots\begin{cases}\psi_0(x)=1, & \varphi_0(x)=o,\\ \psi_1(x)=x-b_1, & \varphi_1(x)=a_1,\\ \psi_2(x)=(x-b_2)\psi_1(x)-a_2\psi_0(x), & \varphi_2(x)=(x-b_2)\varphi_1(x)-a_2\varphi_0(x),\\ \dots\dots\dots & \dots\dots\dots\\ \psi_\lambda(x)=x-b_\lambda)\psi_{\lambda-1}(x)-a_\lambda\psi_{\lambda-2}(x), & \varphi_\lambda(x)=(x-b_\lambda)\varphi_{\lambda-1}(x)-a_\lambda\varphi_{\lambda-2}(x),\end{cases}$$

et par là, en faisant

$$(2)\ldots\ldots\begin{cases}\psi_0(x)\sum\frac{1}{x-x_i}-\varphi_0(x)=R_0,\\ \psi_1(x)\sum\frac{1}{x-x_i}-\varphi_1(x)=R_1,\\ \psi_2(x)\sum\frac{1}{x-x_i}-\varphi_2(x)=R_2,\\ \dots\dots\dots\dots\\ \psi_\lambda(x)\sum\frac{1}{x-x_i}-\varphi_\lambda(x)=R_\lambda,\end{cases}$$

on obtient, relativement aux fonctions

$$R_0,\ R_1,\ R_2,\ldots\ldots R_\lambda,$$

cette suite d'équations:

$$(3)\ldots\ldots\begin{cases}R_0=\sum\frac{1}{x-x_i},\\ R_1=(x-b_1)R_0-a_1,\\ R_2=(x-b_2)R_1-a_2R_0,\\ \dots\dots\dots\dots\\ R_\lambda=(x-b_\lambda)R_{\lambda-1}-a_\lambda R_{\lambda-2}.\end{cases}$$

C'est au moyen de ces formules que nous parviendrons à déterminer toutes les quantités qui sont importantes pour l'évaluation des termes de notre série.

§ II.

Comme les réduites

$$\frac{\varphi_0(x)}{\psi_0(x)},\ \frac{\varphi_1(x)}{\psi_1(x)},\ \frac{\varphi_2(x)}{\psi_2(x)},\ldots\ldots\frac{\varphi_\mu(x)}{\psi_\mu(x)},\ \frac{\varphi_{\mu+1}(x)}{\psi_{\mu+1}(x)},\ldots\ldots$$

de la fraction continue

$$\frac{a_1}{x-b_1} - \frac{a_2}{x-b_2} - \frac{a_3}{x-b_3} - \ldots,$$

qui résulte du développement de

$$\sum \frac{1}{x-x_i}$$

ont pour dénominateurs les fonctions

$$\psi_0(x),\ \psi_1(x),\ \psi_2(x), \ldots \psi_\mu(x),\ \psi_{\mu+1}(x), \ldots,$$

respectivement des degrés

$$0, 1, 2, \ldots \mu, \mu+1, \ldots,$$

la fraction

$$\frac{\varphi_\mu(x)}{\psi_\mu(x)}$$

représentera la valeur de

$$\sum \frac{1}{x-x_i}$$

exactement jusqu'à $\frac{1}{x^{2\mu}}$, et, par consequent, la différence

$$\sum \frac{1}{x-x_i} - \frac{\varphi_\mu(x)}{\psi_\mu(x)}$$

sera de degré inférieur à -2μ. Mais la fonction $\psi_\mu(x)$ étant du degré μ, cela suppose que l'expression

$$R_\mu = \psi_\mu(x) \sum \frac{1}{x-x_i} - \varphi_\mu(x),$$

est d'un degré inférieur à $-\mu$, et de là on conclura que son développement ne peut contenir les termes avec des puissances de x supérieures à $x^{-\mu-1}$. Donc, on aura

$$R_\mu = \frac{(\mu,\mu)}{x^{\mu+1}} + \frac{(\mu,\mu+1)}{x^{\mu+2}} + \frac{(\mu,\mu+2)}{x^{\mu+3}} + \ldots\ldots,$$

en désignant par

$$(\mu,\mu),\ (\mu,\mu+1),\ (\mu,\mu+2), \ldots\ldots$$

les coefficients de

$$\frac{1}{x^{\mu+1}},\ \frac{1}{x^{\mu+2}},\ \frac{1}{x^{\mu+3}}, \ldots\ldots$$

dans le développement de R_μ.

D'après cela, en adoptant pour l'indice μ les valeurs

$$0,\ 1,\ 2,\ldots\ldots\lambda-2,\ \lambda-1,\ \lambda,$$

on trouve pour les fonctions

$$R_0,\ R_1,\ R_2,\ldots\ldots R_{\lambda-1},\ R_{\lambda-2},\ R_\lambda$$

les développements suivants:

$$(4)\ldots\ldots\ldots\ldots\left\{\begin{array}{l}
R_0 = \frac{(0,0)}{x} + \frac{(0,1)}{x^2} + \frac{(0,2)}{x^3} + \ldots\ldots, \\
R_1 = \frac{(1,1)}{x^2} + \frac{(1,2)}{x^3} + \frac{(1,3)}{x^4} + \ldots\ldots, \\
R_2 = \frac{(2,2)}{x^3} + \frac{(2,3)}{x^4} + \frac{(2,4)}{x^5} + \ldots\ldots, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
R_{\lambda-2} = \frac{(\lambda-2,\lambda-2)}{x^{\lambda-1}} + \frac{(\lambda-2,\lambda-1)}{x^{\lambda}} + \frac{(\lambda-2,\lambda)}{x^{\lambda+1}} + \ldots\ldots, \\
R_{\lambda-1} = \frac{(\lambda-1,\lambda-1)}{x^{\lambda}} + \frac{(\lambda-1,\lambda)}{x^{\lambda+1}} + \frac{(\lambda-1,\lambda+1)}{x^{\lambda+2}} + \ldots\ldots, \\
R_\lambda = \frac{(\lambda,\lambda)}{x^{\lambda+1}} + \frac{(\lambda,\lambda+1)}{x^{\lambda+2}} + \frac{(\lambda,\lambda+2)}{x^{\lambda+3}} + \ldots\ldots\ldots\ldots,
\end{array}\right.$$

où

$$\begin{array}{l}
(0,0),\ (0,1),\ (0,2),\ldots\ldots, \\
(1,1),\ (1,2),\ (1,3),\ldots\ldots, \\
(2,2),\ (2,3),\ (2,4),\ldots\ldots, \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
(\lambda-2,\lambda-2),\ (\lambda-2,\lambda-1),\ (\lambda-2,\lambda),\ldots\ldots, \\
(\lambda-1,\lambda-1),\ (\lambda-1,\lambda),\ (\lambda-1,\lambda+1),\ldots\ldots, \\
(\lambda,\lambda),\ (\lambda,\lambda+1),\ (\lambda,\lambda+2),\ldots\ldots\ldots\ldots,
\end{array}$$

sont des valeurs constantes qui se présentent comme quantités auxiliaires.

§ III.

En portant dans les formules (3) les développements de

$$R_0,\ R_1,\ R_2,\ldots R_{\lambda-2},\ R_{\lambda-1},\ R_\lambda,$$

d'après (4), on obtiendra cette suite de formules:

$$\sum\frac{1}{x-x_i}=\frac{(0,0)}{x}+\frac{(0,1)}{x^2}+\frac{(0,2)}{x^3}+\ldots\ldots,$$

$$\frac{(1,1)}{x^2}+\frac{(1,2)}{x^3}+\frac{(1,3)}{x^4}+\ldots\ldots\ldots=(x-b_1)\left[\frac{(0,0)}{x}+\frac{(0,1)}{x^2}+\frac{(0,2)}{x^3}+\ldots\ldots\right]$$
$$-a_1,$$

$$\frac{(2,2)}{x^3}+\frac{(2,3)}{x^4}+\frac{(2,4)}{x^5}+\ldots\ldots\ldots=(x-b_2)\left[\frac{(1,1)}{x^2}+\frac{(1,2)}{x^3}+\frac{(1,3)}{x^4}+\ldots\ldots\right]$$
$$-a_2\left[\frac{(0,0)}{x}+\frac{(0,1)}{x^2}+\frac{(0,2)}{x^3}+\ldots\ldots\right],$$

....

$$\frac{(\lambda,\lambda)}{x^{\lambda+1}}+\frac{(\lambda,\lambda+1)}{x^{\lambda+2}}+\frac{(\lambda,\lambda+2)}{x^{\lambda+3}}+\ldots=(x-b_\lambda)\left[\frac{(\lambda-1,\lambda-1)}{x^\lambda}+\frac{(\lambda-1,\lambda)}{x^{\lambda+1}}+\frac{(\lambda-1,\lambda+1)}{x^{\lambda+2}}+\ldots\ldots\right]$$
$$-a_\lambda\left[\frac{(\lambda-2,\lambda-2)}{x^{\lambda-1}}+\frac{(\lambda-2,\lambda-1)}{x^\lambda}+\frac{(\lambda-2,\lambda)}{x^{\lambda+1}}+\ldots\ldots\right].$$

La première de ces formules, d'après le développement de

$$\sum\frac{1}{x-x_i}$$

en série

$$\frac{\Sigma x_i^0}{x}+\frac{\Sigma x_i}{x^2}+\frac{\Sigma x_i^2}{x^3}+\ldots\ldots,$$

nous donne

$$\frac{\Sigma x_i^0}{x}+\frac{\Sigma x_i}{x^2}+\frac{\Sigma x_i^2}{x^3}+\ldots\ldots=\frac{(0,0)}{x}+\frac{(0,1)}{x^2}+\frac{(0,2)}{x^3}+\ldots\ldots.$$

D'où il suit

$$(0,0)=\Sigma x_i^0,\ (0,1)=\Sigma x_i,\ (0,2)=\Sigma x_i^2,\ldots\ldots$$

Par la seconde on obtient, en égalant entre eux les coefficients des mêmes puissances de x,

$$0=(0,0)-a_1,\ 0=(0,1)-b_1(0,0),\ (1,1)=(0,2)-b_1(0,1),$$
$$(1,2)=(0,3)-b_1(0,2),\ (1,3)=(0,4)-b_1(0,3),\ldots\ldots\ldots,$$

ce qui nous donne

$$a_1=(0,0),\ b_1=\frac{(0,1)}{(0,0)},$$
$$(1,1)=(0,2)-b_1(0,1),\ (1,2)=(0,3)-b_1(0,2),\ (1,3)=(0,4)-b_1(0,3),\ldots\ldots$$

En traitant de la même manière toutes les autres formules on reconnaitra qu'en général, dans le cas de $\lambda>1$, les quantités a_λ et b_λ se déterminent ainsi:

$$a_\lambda=\frac{(\lambda-1,\lambda-1)}{(\lambda-2,\lambda-2)},\ b_\lambda=\frac{(\lambda-1,\lambda)}{(\lambda-1,\lambda-1)}-\frac{(\lambda-2,\lambda-1)}{(\lambda-2,\lambda-2)},$$

et que toutes les quantités

$$(\lambda, \lambda), (\lambda, \lambda+1), \lambda. \lambda+2), \ldots\ldots,$$

en fonction de

$$(\lambda-2, \lambda-2), (\lambda-2, \lambda-1), (\lambda-2,\lambda)\ldots\ldots,$$
$$(\lambda-1, \lambda-1), (\lambda-1, \lambda), (\lambda-1, \lambda+1), \ldots,$$

se trouvent par cette formule:

$$(\lambda, \mu) = (\lambda-1, \mu+1) - b_\lambda(\lambda-1, \mu) - a_\lambda(\lambda-2, \mu).$$

On trouvera ainsi successivement les quantités

$$\begin{array}{c} a_1, b_1, \\ a_2, b_2, \\ a_3, b_3, \\ \ldots\ldots \\ \ldots\ldots \end{array}$$

et avec ces quantités, d'après (1), on obtiendra aisément les fonctions

$$\psi_0(x), \psi_1(x), \psi_2(x), \ldots\ldots\ldots$$

qui entrent dans la composition des termes de notre série.

§ IV.

En passant à la détermination des coefficients de notre série, nous montrerons qu'en vertu des formules (2) et (4) on aura

(5)........................ $\Sigma x_i^{\mu}\psi_\lambda(x_i) = 0,$

si $\mu < \lambda$, et

(6)........................ $\Sigma x_i^{\mu}\psi_\lambda(x_i) = (\lambda, \mu),$

si $\mu =$ ou $> \lambda$.

Pour y parvenir, remarquons que d'après (2)

$$R_\lambda = \sum \frac{\psi_\lambda'(x)}{x-x_i} - \varphi_\lambda(x),$$

et comme le reste de le division de $\psi_\lambda(x)$ par $x-x_i$ est égal à $\psi_\lambda(x_i)$, cette formule se réduit à celle-ci:

$$R_\lambda = \sum\left[F(x, x_i) + \frac{\psi_\lambda(x_i)}{x-x_i}\right] - \varphi_\lambda(x),$$

où $F(x, x_i)$ est une fonction entière qu'on trouve en quotient dans la division de $\psi_\lambda(x)$ par $x-x_i$. Or si l'on décompose la somme

$$\sum\left[F(x, x_i)+\frac{\psi(x_i)}{x-x_i}\right]$$

en deux parties

$$\sum F(x, x_i), \quad \sum\frac{\psi(x_i)}{x-x_i},$$

et que l'on développe, dans la somme

$$\sum\frac{\psi(x_i)}{x-x_i},$$

la fraction

$$\frac{1}{x-x_i}$$

en série

$$\frac{1}{x}+\frac{x_i}{x^2}+\frac{x_i^2}{x^3}+\ldots\ldots,$$

cette formule nous donnera

$$R_\lambda=\sum F(x, x_i)-\varphi_\lambda(x)+\frac{\Sigma\psi_\lambda(x_i)}{x}+\frac{\Sigma x_i\psi_\lambda(x_i)}{x^2}+\frac{\Sigma x_i^2\psi_\lambda(x_i)}{x^3}\ldots\ldots,$$

ce qui suppose, d'après (5), l'identité de ces deux suites:

$$\sum F(x, x_i)-\varphi_\lambda(x)+\frac{\Sigma\psi_\lambda(x_i)}{x}+\frac{\Sigma x_i\psi_\lambda(x_i)}{x^2}+\frac{\Sigma x_i^2\psi_\lambda(x_i)}{x^3}+\ldots\ldots,$$

$$\frac{(\lambda, \lambda)}{x^{\lambda+1}}+\frac{(\lambda, \lambda+1)}{x^{\lambda+2}}+\frac{(\lambda, \lambda+2)}{x^{\lambda+3}}+\ldots\ldots$$

Mais comme

$$\Sigma F(x, x_i), \quad \varphi_\lambda(x)$$

sont des fonctions entières, cela ne peut avoir lieu à moins que les termes avec les dénominateurs

$$x, x^2, x^3, \ldots\ldots x^\lambda, x^{\lambda+1}, x^{\lambda+2}, \ldots\ldots,$$

dans ces deux suites, ne soient respectivement égaux. Donc

$$\Sigma\psi_\lambda(x_i)=0,\ \Sigma x_i\psi_\lambda(x_i)=0,\ \Sigma x_i^2\psi_\lambda(x_i)=0, \ldots\ldots \Sigma x_i^{\lambda-1}\psi_\lambda(x_i)=0,$$
$$\Sigma x_i^\lambda\psi_\lambda(x_i)=(\lambda, \lambda),\ \Sigma x_i^{\lambda+1}\psi_\lambda(x_i)=(\lambda, \lambda+1), \ldots\ldots,$$

ce qui prouve les équations (5) et (6).

D'après cela il est aisé de déterminer les coefficients

$$K_0,\ K_1,\ K_2, \ldots\ldots$$

de la série

$$u=K_0\psi_0(x)+K_1\psi_1(x)+K_2\psi_2(x)+\ldots\ldots$$

Pour cela multiplions la série par x_i^μ, où μ est un nombre quelconque, et sommons ses termes pour toutes les valeurs de

$$x = x_1,\ x_2,\ x_3, \ldots\ldots x_n.$$

Nous obtiendrons ainsi

$$\Sigma x_i^\mu u_i = K_0 \Sigma x_i^\mu \psi_0(x_i) + K_1 \Sigma x_i^\mu \psi_1(x_i) + K_2 \Sigma x_i^\mu \psi_2(x_i) + \ldots\ldots,$$

où par u_i nous désignons la valeur de u qui correspond à $x = x_i$, et comme, en vertu de (5) et (6), on aura

$$\Sigma x_i^\mu \psi_0(x) = (0,\ \mu),\ \Sigma x_i^\mu \psi_1(x_i) = (1,\ \mu), \ldots\ldots \Sigma x_i^\mu \psi_\mu(x_i) = (\mu,\ \mu),$$
$$\Sigma x_i^\mu \psi_{\mu+1}(x_i) = 0,\ \Sigma x_i^\mu \psi_{\mu+2}(x_i) = 0,\ \Sigma x_i^{\mu+1} \psi_{\mu+3}(x_i) = 0, \ldots\ldots,$$

il en résulte

$$\Sigma x_i^\mu u_i = (0,\ \mu)\ K_0 + (1,\ \mu)\ K_1 + \ldots\ldots + (\mu - 1,\ \mu)\ K_{\mu-1} + (\mu,\ \mu)\ K_\mu.$$

D'où, pour la détermination du coefficient K_μ, en fonction des coefficients K_0, $K_1, \ldots K_{\mu-1}$, on tire cette formule très simple:

$$K_\mu = \frac{\Sigma x_i^\mu u_i - (0,\mu) K_0 - (1,\mu) K_1 - \ldots\ldots - (\mu-1,\mu) K_{\mu-1}}{(\mu,\mu)}.$$

En adoptant ici pour l'indice μ les valeurs 0, 1, 2, 3,....., on obtient, pour la détermination successive des coefficients

$$K_0,\ K_1,\ K_2,\ K_3, \ldots\ldots,$$

cette suite d'équations:

$$K_0 = \frac{\Sigma u_i}{(0,0)},$$
$$K_1 = \frac{\Sigma x_i u_i - (0,1) K_0}{(1,1)},$$
$$K_2 = \frac{\Sigma x_i^2 u_i - (0,2) K_0 - (1,2) K_1}{(2,2)},$$
$$K_3 = \frac{\Sigma x_i^3 u_i - (0,3) K_0 - (1,3) K_1 - (2,3) K_2}{(3,3)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

§ V.

Il nous reste à montrer comment on parviendra d'une manière facile à trouver la somme des carrés des différences entre les valeurs données de u

$$u_1,\ u_2,\ u_3, \ldots\ldots u_n,$$

correspondantes à

$$x = x_1,\ x_2,\ x_3, \ldots\ldots x_n,$$

et celle qui, pour les mêmes valeurs de x, résulte de notre série arrêtée au terme $K_\lambda \psi_\lambda(x)$, λ étant un nombre quelconque.

Pour y parvenir, nous allons montrer qu'on aura

$$(7) \ldots\ldots\ldots\ldots \Sigma \psi_\mu(x_i)\ \psi_\nu(x_i) = 0,$$

tant que $\nu < \mu$, et

$$(8) \ldots\ldots\ldots\ldots \Sigma \psi_\mu(x_i)\ \psi_\nu(x_i) = (\mu, \mu),$$

dans le cas de $\mu = \nu$.

En effet, d'après (1), la fonction $\psi_\nu(x)$ sera de la forme

$$x^\nu + A_1 x^{\nu-1} + A_2 x^{\nu-2} + \ldots\ldots\ldots,$$

et par conséquent on aura

$$(9) \ldots \Sigma\psi_\mu(x_i)\psi_\nu(x_i) = \Sigma x_i^\nu\, \psi_\mu(x_i) + A_1 \Sigma x_i^{\nu-1}\psi_\mu(x_i) + A_2 \Sigma x_i^{\nu-2}\psi_\mu(x_i) + \ldots\ldots$$

Mais en vertu de (5), dans le cas de $\nu < \mu$, toutes les sommes

$$\Sigma x_i^\nu \psi_\mu(x_i),\ \ \Sigma x_i^{\nu-1}\psi_\mu(x_i),\ \ \Sigma x_i^{\nu-2}\psi_\mu(x_i), \ldots\ldots$$

se réduisent à zéro, et par là, d'après la formule précédente, on trouvera

$$\Sigma\psi_\mu(x_i)\ \psi_\nu(x_i) = 0,$$

ce qui prouve l'équation (7).

De même, dans le cas de

$$\mu = \nu,$$

on trouve, d'après (5) et (6), que la somme

$$\Sigma x_i^\nu\, \psi_\mu(x_i)$$

est égale à (μ, μ), et que les sommes

$$\Sigma x_i^{\nu-1} \psi_\mu(x_i),\ \ \Sigma x_i^{\nu-2}\psi_\mu(x_i), \ldots\ldots$$

s'annulent. En vertu de quoi, pour $\mu = \nu$, la formule (9) nous donne l'équation (8)

$$\Sigma\, \psi_\mu(x_i)\psi_\nu(x_i) = (\mu, \mu).$$

Au moyen des équations (7) et (8), que nous venons de prouver, il est aisé de montrer qu'on aura toujours

$$)10) \ldots\ldots\ldots\ldots \Sigma u_i\, \psi_\mu(x_i) = (\mu, \mu)\, K_\mu.$$

Pour s'en assurer, observons que notre série

$$u = K_0\psi_0(x) + K_1\psi_1(x) + K_2\psi_2(x) + \ldots\ldots,$$

prolongée jusqu'au dernier terme, représente exactement toutes les valeurs données de u

$$u_1,\ u_2,\ u_3,\ldots\ldots u_n,$$

et par là on aura

$$\Sigma u_i\psi_\mu(x_i) = K_0\Sigma\psi_0(x_i)\psi_\mu(x_i) + K_1\Sigma\psi_1(x_i)\psi_\mu(x_i) + K_2\Sigma\psi_2(x_i)\psi_\mu(x_i) + \ldots\ldots$$

Mais d'après (7) les sommes

$$\Sigma\psi_0(x_i)\psi_\mu(x_i),\ \Sigma\psi_1(x_i)\psi_\mu(x_i),\ldots\ldots\ \Sigma\psi_{\mu-1}(x_i)\psi_\mu(x_i),\ \Sigma\psi_{\mu+1}(x_i)\psi_\mu(x_i),\ldots\ldots$$

s'annulent, et d'après (8) on trouve

$$\Sigma\psi_\mu(x_i)\psi_\mu(x_i) = (\mu,\ \mu).$$

Donc le développement précedent de $\Sigma u_i\psi(x_i)$ se réduira à un terme

$$(\mu,\ \mu)K_\mu,$$

ce qui nous donne l'équation (10).

En vertu des équations démontrées, il est aisé de trouver la somme

$$\Sigma[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - \ldots\ldots - K_\lambda\psi_\lambda(x_i)]^2,$$

où

$$u_i,$$

pour $i = 1,\ 2,\ 3,\ldots\ldots n$, désigne les valeurs données de u

$$u_1,\ u_2,\ u_3,\ldots\ldots u_n,$$

et l'expression

$$K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \ldots\ldots\ldots + K_\lambda\psi_\lambda(x_i)$$

leurs valeurs approchées, obtenues par notre série, arrêtée au terme $K_\lambda\psi_\lambda(x)$.

Pour cela mettons le carré

$$[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - K_2\psi_2(x)\ldots\ldots - K_\lambda\psi_\lambda(x_i)]^2$$

sous la forme

$$\begin{aligned}
&u_i^2 - 2u_i[K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \ldots\ldots + K_\lambda\psi_\lambda(x_i)]\\
&+ K_0\psi_0(x_i)[K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \ldots\ldots + K_\lambda\psi_\lambda(x_i)]\\
&+ K_1\psi_1(x_i)[K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \ldots\ldots + K_\lambda\psi_\lambda(x_i)]\\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+ K_\lambda\psi_\lambda(x_i)[K_0\psi_0(x_i) + K_1\psi_1(x_i) + K_2\psi_2(x_i) + \ldots\ldots + K_\lambda\psi_\lambda(x_i)],
\end{aligned}$$

ce qui nous donne

$$\Sigma[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - K_2\psi_2(x_i) - \ldots\ldots - K_\lambda\psi_\lambda(x_i)]^2$$
$$= \Sigma u_i^2 - 2K_0\Sigma u_i\psi_0(x_i) - 2K_1\Sigma u_i\psi_1(x_i) - 2K_2\Sigma u_i\psi_2(x_i) - \ldots - 2K_\lambda\Sigma u_i\psi_\lambda(x_i)$$
$$+ K_0^2\Sigma\psi_0(x_i)\psi_0(x_i) + K_0K_1\Sigma\psi_0(x_i)\psi_1(x_i) + K_0K_2\Sigma\psi_0(x_i)\psi_2(x_i) + \ldots + K_0K_\lambda\Sigma\psi_0(x_i)\psi_\lambda(x_i)$$
$$+ K_\lambda K_0\Sigma\psi_1(x_i)\psi_0(x_i) + K_1^2\Sigma\psi_1(x_i)\psi(x_i) + K_1K_2\Sigma\psi_1(x_i)\psi_2(x_i) + \ldots\ldots + K_1K_\lambda\Sigma\psi_1(x_i)\psi_\lambda(x_i)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ K_\lambda K_0\Sigma\psi_\lambda(x_i)\psi_0(x_i) + K_\lambda K_1\Sigma\psi_\lambda(x_i)\psi_1(x_i) + K_\lambda K_2\Sigma\psi_\lambda(x_i)\psi_2(x_i) + \ldots + K_\lambda^2\Sigma\psi_\lambda(x_i)\psi_\lambda(x_i).$$

Mais d'après (10) nous aurons

$$\Sigma u_i\psi_0(x_i) = (0,0)K_0,\ \Sigma u_i\psi_1(x_i) = (1,1)K_1,\ \Sigma u_i\psi_2(x_i) = (2,2)K_2, \ldots\ldots,$$

et d'après (8) et (9)

$$\Sigma\psi_0(x_i)\psi_0(x_i) = (0,0),\ \Sigma\psi_1(x_i)\psi_1(x_i) = (1,1),\ \Sigma\psi_2(x_i)\psi_2(x_i) = (2,2), \ldots\ldots,$$
$$\Sigma\psi_1(x_i)\psi_0(x_i) = 0, \quad \Sigma\psi_2(x_i)\psi_0(x_i) = 0, \ldots\ldots\ldots\ldots,$$
$$\Sigma\psi_0(x_i)\psi_1(x_i) = 0, \quad \Sigma\psi_2(x_i)\psi_0(x_i) = 0, \ldots\ldots\ldots\ldots,$$
$$\Sigma\psi_0(x_i)\psi_2(x_i) = 0, \quad \Sigma\psi_1(x_i)\psi_2(x_i) = 0, \ldots\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

En vertu de quoi la formule précedente devient

$$\Sigma[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - K_2\psi_2(x_i) - \ldots\ldots - K_\lambda\psi_\lambda(x_i)]^2$$
$$= \Sigma u_i^2 - 2(0,0)K_0^2 - 2(1,1)K_1^2 - 2(2,2)K_2^2 - \ldots - 2(\lambda,\lambda)K_\lambda^2$$
$$+ (0,0)K_0^2 + (1,1)K_1^2 + (2,2)K_2^2 + \ldots\ldots + (\lambda,\lambda)K_\lambda^2,$$

et se réduit à celle-ci:

$$\Sigma[u_i - K_0\psi_0(x_i) - K_1\psi_1(x_i) - K_2\psi_2(x_i) - \ldots\ldots - K_\lambda\psi_\lambda(x_i)]^2$$
$$= \Sigma u_i^2 - (0,0)K_0^2 - (1,1)K_1^2 - (2,2)K_2^2 - \ldots\ldots - (\lambda,\lambda)K_\lambda^2.$$

Telle est la formule donnant la somme des carrés des différences qui existent entre les valeurs données de u et leurs représentations par la série

$$u = K_0\psi_0(x) + K_1\psi_1(x) + K_2\psi_2(x) + \ldots\ldots,$$

arrêtée au terme $K_\lambda\psi_\lambda(x)$. En désignant, pour abréger, cette somme par

$$\Sigma d_\lambda^2,$$

nous aurons

$$\Sigma d_\lambda^2 = \Sigma u_i^2 - (0,0)\,K_0^2 - (1,1)\,K_1^2 - (2,2)\,K_2^2 - \ldots\ldots - (\lambda,\lambda)\,K_\lambda^2.$$

D'où, pour la détermination successive des sommes

$$\Sigma d_0^2,\ \Sigma d_1^2,\ \Sigma d_2^2, \ldots\ldots$$

qui correspondent respectivement aux cas où notre série est arrêtée aux termes 1, 2, 3,. . . ., résulte cette suite d'équations:

$$\begin{aligned}
\Sigma d_0^2 &= \Sigma u_i^2 - (0,0)\,K_0^2,\\
\Sigma d_1^2 &= \Sigma d_0^2 - (1,1)\,K_1^2,\\
\Sigma d_2^2 &= \Sigma d_1^2 - (2,2)\,K_2^2,\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

§ VI.

Nous allons maintenant résumer les formules définitives par lesquelles on parviendra à calculer, terme par terme, l'expression de u d'après la série

$$u = K_0\,\psi_0(x) + K_1\,\psi_1(x) + K_2\,\psi_2(x) + \ldots\ldots,$$

et on connaîtra, en même temps, la somme des carrés des erreurs commises dans la représentation des valeurs données de u, en s'arrêtant aux termes 1, 2, 3,. . . . . . λ.

Dans ces formules, suivant la notation employée, les valeurs données de la fonction u et de la variable x sont représentées par

$$\begin{gathered}
u_1,\ u_2,\ u_3, \ldots\ldots u_n,\\
x_1,\ x_2,\ x_3, \ldots\ldots x_n.
\end{gathered}$$

Les sommations s'étendent à toutes les valeurs de l'indice i, depuis $i = 1$, jusqu'à $i = n$, et Σd_λ^2 désigne la somme des carrés des erreurs dans la représentation des valeurs données de u par notre série, arrêtée au terme Σd_λ^2, somme d'après laquelle on trouvera l'erreur moyenne par la formule

$$E = \sqrt{\frac{1}{n}\sum d_\lambda^2}.$$

Formules relatives à la détermination du terme $K_0\,\psi_0(x)$

$$\begin{aligned}
&(0,0) = \Sigma x_i^0 = n,\\
&K_0 = \frac{\Sigma u_i}{(0,0)},\\
&\psi_0(x) = 1,\\
&\Sigma d_0^2 = \Sigma u_i^2 - (0,0)\,K_0^2.
\end{aligned}$$

Formules relatives à la détermination du terme $K_1\psi_1(x)$.

$$(0,1)=\Sigma x_i, \qquad (0,2)=\Sigma x_i^2,$$

$$a_1=(0,0),$$

$$b_1=\frac{(0,1)}{(0,0)}, \qquad (1,1)=(0,2)-b_1(0,1),$$

$$K_1=\frac{\Sigma x_i u_i-(0,1)K_0}{(1,1)},$$

$$\psi_1(x)=x-b_1,$$

$$\Sigma d_1^2=\Sigma d_0^2-(1,1)K_1^2.$$

Formules relatives à la détermination du terme $K_2\psi_2(x)$.

$$(0,3)=\Sigma x_i^3, \qquad (0,4)=\Sigma x_i^4,$$

$$(1,2)=(0,3)-b_1(0,2), \qquad (1,3)=(0,4)-b_1(0,3),$$

$$a_2=\frac{(0,1)}{(0,0)},$$

$$b_2=\frac{(1,2)}{(1,1)}-\frac{(0,1)}{(0,0)}, \qquad (2,2)=(1,3)-b_2(1,2)-a_2(0,2),$$

$$K_2=\frac{\Sigma x_i^2 u_i-(0,2)K_0-(1,2)K_1}{(2,2)},$$

$$\psi_2(x)=(x-b_2)\psi_1(x)-a_2\psi_0(x),$$

$$\Sigma d_2^2=\Sigma d_1^2-(2,2)K_2^2.$$

. . . .

. . . .

Formules relatives à la détermination du terme $K_\lambda\psi_\lambda(x)$.

$$(0,2\lambda-1)=\Sigma x_i^{2\lambda-1}, \qquad (0,2\lambda)=\Sigma x_i^{2\lambda},$$

$$(1,2\lambda-2)=(0,2\lambda-1)-b_1(0,2\lambda-2), \qquad (1,2\lambda-1)=(0,2\lambda)-b_1(0,2\lambda-1),$$

$$(2,2\lambda-3)=(1,2\lambda-2)-b_2(1,2\lambda-3)-a_2(0,2\lambda-3), \quad (2,2\lambda-2)=(1,2\lambda-1)-b_2(1,2\lambda-2)-a_2(0,2\lambda-2),$$

$$(3,2\lambda-4)=(2,2\lambda-3)-b_3(2,2\lambda-4)-a_3(1,2\lambda-4), \quad (3,2\lambda-3)=(2,2\lambda-2)-b_3(2,2\lambda-3)-a_3(1,2\lambda-3),$$

. . . .

$$(\lambda-1,\lambda)=(\lambda-2,\lambda+1)-b_{\lambda-1}(\lambda-2,\lambda)-a_{\lambda-1}(\lambda-3,\lambda),\qquad (\lambda-1,\lambda+1)=(\lambda-2,\lambda+2)-b_{\lambda-1}(\lambda-2,\lambda+1)-a_{\lambda-1}(\lambda-3,\lambda+1),$$

$$a_\lambda=\frac{(\lambda-1,\lambda-1)}{(\lambda-2,\lambda-2)},$$

$$b_\lambda=\frac{(\lambda-1,\lambda)}{(\lambda-1,\lambda-1)}-\frac{(\lambda-2,\lambda-1)}{(\lambda-2,\lambda-2)},\quad (\lambda,\lambda)=(\lambda-1,\lambda+1)+b_\lambda(\lambda-1,\lambda)-a_\lambda(\lambda-2,\lambda),$$

$$K_\lambda=\frac{\Sigma x_i^\lambda u_i-(0,\lambda)K_0-(1,\lambda)K_1-(2,\lambda)K_2-\ldots\ldots-(\lambda-1,\lambda)K_{\lambda-1}}{(\lambda,\lambda)},$$

$$\psi_\lambda(x)=(x-b_\lambda)\psi_{\lambda-1}(x)-a_\lambda\psi_{\lambda-2}(x),$$

$$\Sigma d_\lambda^2=\Sigma d^2_{\lambda-1}-(\lambda,\lambda)K_\lambda^2.$$

§ VII.

Les formules que nous venons de donner pour déterminer successivement les termes

$$K_0\psi_0(x),\ K_1\psi_1(x),\ K_2\psi_2(x),\ldots\ldots K_\lambda\psi_\lambda(x)$$

dans le développement de u d'après notre série, et pour évaluer, en même temps, la somme des carrés des erreurs avec lesquelles les termes trouvés de u représentent toutes ses valeurs données, nous fournissent une méthode d'interpolation parabolique, importante sous plus d'un rapport. En vertu de la propriété remarquable de notre série, cette méthode donne l'expression de u sous forme d'un polynome avec les coefficients les plus probables. Sans fixer d'avance le nombre de ses termes, par cette méthode, on les trouvera successivement l'un après l'autre, et on reconnaîtra tout de suite celui auquel on peut s'arrêter d'après la somme des carrés des erreurs avec lesquelles les termes trouvés de u représentent ses valeurs données, somme qui donne sur le champ l'erreur moyenne de leur représentation. De plus, il est aisé de voir par la composition de nos formules que lorsque le nombre des valeurs données de u et celui des termes de son expression sont considérables, dans notre méthode d'interpolation les calculs sont moins prolixes que dans celles maintenant en usage.

Cette prolixité des calculs est due presque entièrement aux différentes *multiplications* et *divisions* dont le nombre s'accroît plus ou moins rapidement, avec ceux des valeurs données de u et des termes dans son expression. C'est sous ce rapport que nous allons montrer l'avantage de notre méthode d'interpolation, en laissant de côté les *additions* et les *soustractions* qui, dans le travail de ces calculs, n'entrent que pour bien peu de chose, et pour lesquelles on peut aussi bien manifester l'avantage de notre méthode.

Pour trouver par nos formules l'expression de u avec $\lambda+1$ termes, on devra évaluer $3\lambda+1$ sommes

$$\Sigma x_i,\ \Sigma x_i^2,\ \Sigma x_i^3,\ldots\ldots\Sigma x_i^{2\lambda},$$
$$\Sigma u_i,\ \Sigma x_i u_i,\ \Sigma x_i^2 u_i,\ldots\ldots\Sigma x_i^\lambda u_i,$$

et au moyen de ces sommes, en cherchant les termes

$$K_0\psi_0(x),\ K_1\psi_1(x),\ K_2\psi_2(x),\ldots\ldots K_\lambda\psi_\lambda(x),$$

par ce que nous avons vu, et en les réduisant à la forme définitive

$$A + Bx + Cx^2 + \ldots\ldots,$$

on n'aura à faire des *multiplications* ou *divisions* qu'en nombre $4\lambda^2 + 2$.

Mais si l'on cherche cette expression de u, à l'ordinaire, par la méthode des *moindres carrés*, on est porté à calculer les mêmes sommes

$$\Sigma x_i,\ \Sigma x_i^2,\ \Sigma x_i^3,\ldots\ldots \Sigma x_i^{2\lambda},$$
$$\Sigma u_i,\ \Sigma x_i u_i,\ \Sigma x_i^2 u_i,\ldots\ldots \Sigma x_i^\lambda u_i$$

pour la composition des équations déterminant $\lambda + 1$ coefficients de u, et en résolvant ces équations à $\lambda + 1$ inconnues, on tombe sur les *multiplications* et les *divisions* dont le nombre, avec l'accroissement de λ, croit, comme on le sait, bien plus rapidement que $4\lambda^2 + 2$.

D'après la méthode de Cauchy, en cherchant, dans le développement de u, les termes

$$A + Bx + Cx^2 + \ldots\ldots + Hx^\lambda,$$

on doit, pour $x = x_1,\ x_2,\ x_3,\ldots\ldots,\ x_n$, évaluer plusieurs fonctions, dont les degrés montent jusqu'à λ, et composer par leur moyen les sommes qu'on nomme *subordonnées*. Or cela exige, évidemment, bien plus de *multiplications* qu'il n'en faut pour calculer les sommes

$$\Sigma x_i,\ \Sigma x_i^2,\ \Sigma x_i^3,\ldots\ldots \Sigma x_i^{2\lambda},$$
$$\Sigma u_i,\ \Sigma x_i u_i,\ \Sigma x_i u_i,\ldots\ldots \Sigma x_i^\lambda u_i,$$

qui se présentent dans l'évaluation de $\lambda + 1$ termes de notre série, et aussi pour trouver celle-ci:

$$\Sigma u_i^2$$

qui entre dans la détermination des sommes

$$\Sigma d_0^2,\ \Sigma d_1^2,\ \Sigma d_2^2,\ldots\ldots,$$

par lesquelles, dans notre méthode, on reconnaîtra le nombre des termes importants pour l'interpolation.

D'autre part, pour trouver les fonctions, comprises dans les sommes *subordonnées*, et pour évaluer par elles les coefficients A, B, $C,\ldots\ldots H$ de l'expression de

$$u = A + Bx + Cx^2 + \ldots\ldots + Hx^\lambda,$$

dans la méthode de Cauchy, il est important de faire plusieurs *multiplications* et *divisions* dont le nombre total, avec l'accroissement de λ, croît plus rapidement que $4\lambda^2 + \lambda + 3$, nombre des mêmes opérations qui se présentent quand, par notre méthode, d'après les valeurs de

$$\Sigma x_i,\ \Sigma x_i^2,\ \Sigma x_i^3, \ldots\ldots \Sigma x_i^{2\lambda},$$
$$\Sigma u_i,\ \Sigma x_i u_i,\ \Sigma x_i^2 u_i, \ldots\ldots \Sigma x_i^\lambda u_i,\ \Sigma u_i^2,$$

on cherche $\lambda + 1$ termes et on détermine successivement les sommes

$$\Sigma d_0^2,\ \Sigma d_1^2,\ \Sigma d_2^2, \ldots\ldots \Sigma d_\lambda^2.$$

Par là il est certain que, à cause du nombre de ses opérations, la méthode de Cauchy est loin d'être aussi simple que celle qui résulte de notre série. Mais comme plusieurs de ces opérations, dans la méthode de Cauchy, se simplifient de plus en plus à mesure que la convergence de la série

$$u = A + Bx + Cx^2 + \ldots\ldots + Hx^\lambda$$

s'accroît, il n'y a aucun doute qu'on ne rencontre des cas particuliers où elle devient plus expéditive que la nôtre.

§ VIII.

Pour montrer sur un exemple l'usage de notre méthode d'interpolation, nous allons l'appliquer à cette suite des valeurs de x et u*):

$x_1 = 0,15411$	$u_1 = 19,47$
$x_2 = 0,19516$	$u_2 = 21,83$
$x_3 = 0,22143$	$u_3 = 23,11$
$x_4 = 0,28802$	$u_4 = 26,11$
$x_5 = 0,32808$	$u_5 = 27,60$
$x_6 = 0,38183$	$u_6 = 28,89$
$x_7 = 0,45517$	$u_7 = 33,17$
$x_8 = 0,57012$	$u_8 = 33,38$
$x_9 = 0,75930$	$u_9 = 32,31$
$x_{10} = 0,91075$	$u_{10} = 31,88$
$x_{11} = 1,13895$	$u_{11} = 25,46.$

En cherchant à exprimer u par un seul terme

$$K_0 \psi_0(x),$$

*) Ces valeurs représentent les résultats de la première série des observations de M. Marie Davy sur la résistance au changement de conducteur qu'il donne dans son Mémoire, intitulé: *Recherches expérimentales sur l'électricité voltaïque* (Annales de chimie et de physique, série III, tome 19). — Par x nous désignons l'inverse de l'intensité du courant, réduite à sa centième partie, et par u la résistance.

on prendra

$$
\begin{aligned}
(0,0) = \Sigma x_i^0 = 11,\quad u_1 &= 19{,}47\\
u_2 &= 21{,}83\\
u_3 &= 23{,}11\\
u_4 &= 26{,}11\\
u_5 &= 27{,}60\\
u_6 &= 28{,}89\\
u_7 &= 33{,}17\\
u_8 &= 33{,}38\\
u_9 &= 32{,}31\\
u_{10} &= 31{,}88\\
u_{11} &= 25{,}46\\
\hline
\Sigma u_i &= 303{,}21\\
K_0 = \frac{\Sigma u_i}{(0,0)} &= 27{,}5645,\\
\psi_0(x) &= 1,
\end{aligned}
$$

ce qui donne, exactement jusqu'à 0,001,

$$K_0 \psi_0(x) = 27{,}564.$$

Pour trouver la somme des carrés des erreurs avec lesquelles le terme trouvé représente les valeurs données, on fera les calculs suivants:

$$
\begin{aligned}
u_1^2 &= 379{,}08\\
u_2^2 &= 476{,}55\\
u_3^2 &= 534{,}07\\
u_4^2 &= 681{,}73\\
u_5^2 &= 761{,}76\\
u_6^2 &= 834{,}73\\
u_7^2 &= 1100{,}25\\
u_8^2 &= 1114{,}22\\
u_9^2 &= 1043{,}94\\
u_{10}^2 &= 1016{,}33\\
u_{11}^2 &= 648{,}21\\
\hline
\Sigma u_i^2 &= 8590{,}77\\
-(0,0)K_0^2 &= -8357{,}84\\
\hline
\Sigma d_0^2 = \Sigma u_i^2 - (0,0)K_0^2 &= 232{,}93,
\end{aligned}
$$

ce qui donne pour l'erreur moyenne

$$E = \sqrt{\frac{1}{n}\Sigma d_0^2} = \sqrt{\frac{232{,}93}{11}} = 4{,}6.$$

En remarquant d'après cela l'insuffisance de l'expression de u par un seul terme

$$K_0 \psi_0(x),$$

on cherchera le second terme

$$K_1 \psi_1(x),$$

et pour cela on calculera successivement

$$(0,1) = \Sigma x_i,\ (0,2) = \Sigma x_i^2,$$
$$a_1 = (0,0),\ b_1 = \frac{(0,1)}{(0,0)},$$
$$(1,1) = (0,2) - b_1(0,1),$$
$$\Sigma x_i u_i,\ \Sigma x_i u_i - (0,1)K_0,$$
$$K_1 = \frac{\Sigma x_i u_i - (0,1)K_0}{(1,1)},\ \psi_1(x)$$

ainsi qu'il suit:

$x_1 = 0{,}15411$	$x_1^2 = 0{,}02375$
$x_2 = 0{,}19516$	$x_2^2 = 0{,}03809$
$x_3 = 0{,}22143$	$x_3^2 = 0{,}04903$
$x_4 = 0{,}28802$	$x_4^2 = 0{,}08295$
$x_5 = 0{,}32808$	$x_5^2 = 0{,}10764$
$x_6 = 0{,}38183$	$x_6^2 = 0{,}14579$
$x_7 = 0{,}45517$	$x_7^2 = 0{,}20718$
$x_8 = 0{,}57012$	$x_8^2 = 0{,}32504$
$x_9 = 0{,}75930$	$x_9^2 = 0{,}57654$
$x_{10} = 0{,}91075$	$x_{10}^2 = 0{,}82947$
$x_{11} = 1{,}13895$	$x_{11}^2 = 1{,}29721$
$(0,1) = \Sigma x_i = 5{,}40292$	$(0,2) = \Sigma x_i^2 = 3{,}68269$
$a_1 = (0,0) = 11$	$-b_1(0,1) = -2{,}65378$
$b_1 = \frac{(0,1)}{(0,0)} = 0{,}49117$	$(1,1) = (0,2) - b_1(0,1) = 1{,}02891$

$$\begin{aligned}
x_1 u_1 &= 3{,}00052\\
x_2 u_2 &= 4{,}26034\\
x_3 u_3 &= 0{,}11725\\
x_4 u_4 &= 7{,}52020\\
x_5 u_5 &= 9{,}05501\\
x_6 u_6 &= 11{,}03105\\
x_7 u_7 &= 15{,}09799\\
x_8 u_8 &= 19{,}03060\\
x_9 u_9 &= 24{,}53298\\
x_{10} u_{10} &= 29{,}03471\\
x_{11} u_{11} &= 28{,}99767\\
\hline
\Sigma x_i u_i &= 156{,}67832\\
-(0,1)K_0 &= -148{,}92903\\
\hline
\Sigma x_i u_i - (0,1)K_0 &= 7{,}74929\\
K_1 = \frac{\Sigma x_i u_i - (0,1)K_0}{(1,1)} &= 7{,}5320,\\
\psi_1(x) = x - b_1 &= x - 0{,}49117.
\end{aligned}$$

Donc,

$$K_1 \psi_1(x) = 7{,}5320(x - 0{,}49117) = 7{,}532\,x - 3{,}699.$$

En passant à la détermination de Σd_1^2, on prendra

$$\begin{aligned}
\Sigma d_0^2 &= 232{,}93\\
-(1,1)K_1^2 &= -\ \ 58{,}37\\
\hline
\Sigma d_1^2 = \Sigma d_0^2 - (1,1)K_1^2 &= 174{,}56,
\end{aligned}$$

d'où, pour l'erreur moyenne de la représentation des valeurs données de u par ses deux termes trouvés, résulte

$$E = \sqrt{\frac{1}{n}\Sigma d_1^2} = \sqrt{\frac{174{,}56}{11}} = 3{,}98.$$

Une erreur moyenne aussi considérable n'étant pas admissible, on cherchera le troisième terme

$$K_2 \psi_2(x),$$

et pour cela on déterminera successivement les quantités

$$(0,3) = \Sigma x_i^3,\ \ (0,4) = \Sigma x_i^4,$$

$$(1,2) = (0,3) - b_1(0,2),\ \ (1,3) = (0,4) - b_1(0,3),$$

$$a_2 = \frac{(1,1)}{(0,0)},\ \ b_2 = \frac{(1,2)}{(1,1)} - \frac{(0,1)}{(0,0)},$$

$$(2,2) = (1,3) - b_2(1,2) - a_2(0,2),$$

$$\Sigma x_i^2 u_i,\ \ \Sigma x_i^2 u_i - (0,2)K_0 - (1,2)K_1,$$

$$K_2 = \frac{\Sigma x_i^2 u_i - (0,2)K_0 - (1,2)K_1}{(2,2)}$$

et la fonction

$$\psi_2(x)$$

de la manière suivante:

$$\begin{aligned}
x_1^3 &= 0{,}00367\\
x_2^3 &= 0{,}00743\\
x_3^3 &= 0{,}01086\\
x_4^3 &= 0{,}02389\\
x_5^3 &= 0{,}03531\\
x_6^3 &= 0{,}05567\\
x_7^3 &= 0{,}09430\\
x_8^3 &= 0{,}18531\\
x_9^3 &= 0{,}43776\\
x_{10}^3 &= 0{,}75544\\
x_{11}^3 &= 1{,}47745\\
\hline
(0,3) = \Sigma x_i^3 &= 3{,}08709\\
-b_1(0,2) &= -1{,}80884\\
\hline
(1,2) = (0,3) - b_1(0,2) &= 1{,}27825\\
a_2 = \frac{(1,1)}{(0,0)} &= 0{,}09354\\
\frac{(1,2)}{(1,1)} &= 1{,}24235\\
-\frac{(0,1)}{(0,0)} &= -0{,}49117\\
\hline
b_2 = \frac{(1,2)}{(1,1)} - \frac{(0,1)}{(0,0)} &= 0{,}75118
\end{aligned}$$

$$\begin{aligned}
x_1^4 &= 0{,}00056\\
x_2^4 &= 0{,}00145\\
x_3^4 &= 0{,}00240\\
x_4^4 &= 0{,}00688\\
x_5^4 &= 0{,}01158\\
x_6^4 &= 0{,}02126\\
x_7^4 &= 0{,}04292\\
x_8^4 &= 0{,}10565\\
x_9^4 &= 0{,}33240\\
x_{10}^4 &= 0{,}68801\\
x_{11}^4 &= 1{,}68275\\
\hline
(0,4) = \Sigma x_i^4 &= 2{,}89586\\
-b_1(0,3) &= -1{,}51630\\
\hline
(1,3) = (0,4) - b_1(0,3) &= 1{,}37956\\
-b_2(1,2) &= -0{,}96020\\
-a_2(0,2) &= -0{,}34446\\
\hline
(2,2) = (1,3) - b_2(1,2) - a_2(0,2) &= 0{,}07490
\end{aligned}$$

$$\begin{aligned}
x_1^2 u_1 &= 0{,}46241\\
x_2^2 u_2 &= 0{,}83145\\
x_3^2 u_3 &= 1{,}13311\\
x_4^2 u_4 &= 2{,}16596\\
x_5^2 u_5 &= 2{,}97075\\
x_6^2 u_6 &= 4{,}21199\\
x_7^2 u_7 &= 6{,}87215\\
x_8^2 u_8 &= 10{,}84949\\
x_9^2 u_9 &= 18{,}62790\\
x_{10}^2 u_{10} &= 27{,}44337\\
x_{11}^2 u_{11} &= 33{,}02691\\
\hline
\Sigma x_i^2 u_i &= 107{,}59549\\
-(0,2)K_0 &= -101{,}51151\\
-(1,2)K_1 &= -\ \ 9{,}62778\\
\hline
\Sigma x_i^2 u_i - (0,2)K_0 - (1,2)K_1 &= -\ \ 3{,}54380
\end{aligned}$$

$$K_2 = \frac{\Sigma x_i^2 u_i - (0,2) K_0 - (1,2) K_1}{(2,2)} = -47,313,$$

$$\psi_2(x) = (x - b_2)\psi_1(x) - a_2 = (x - 0,75118)(x - 0,49117) - 0,09354$$
$$= x^2 - 1,24235\,x + 0,27628.$$

D'où il suit

$$K_2\psi_2(x) = -47,313\,(x^2 - 1,24235\,x + 0,27628)$$
$$= -47,313\,x^2 + 58,779\,x - 13,071,$$

et comme

$$\begin{aligned} \Sigma d_1^2 &= \quad 174,56, \\ -(2,2)\,K_2^2 &= -167,64, \\ \hline \Sigma d_2^2 = \Sigma d_1^2 - (2,2)\,K_0^2 &= \quad 6,92, \end{aligned}$$

on trouve pour l'erreur moyenne

$$E = \sqrt{\frac{1}{n}\Sigma d_2^2} = \sqrt{\frac{6,92}{11}} = 0,79.$$

En procédant ainsi, on obtiendra l'expression de u terme par terme, et par là l'erreur moyenne dans la représentation des valeurs données de u s'approchera de plus en plus de zéro. Mais si l'on trouve suffisant de réduire cette erreur à 0,79, on s'arrêtera aux termes trouvés

$$\begin{aligned} K_0\psi_0(x) &= \quad 27,564 \\ K_1\psi_1(x) &= \quad 7,532\,x - 3,699 \\ K_2\psi_2(x) &= -47,313\,x^2 + 58,779\,x - 13,071, \end{aligned}$$

et par là, pour l'expression cherchée de u, on aura

$$\begin{array}{l} +27,564 \\ -\ \ 3,699 + \ \ 7,532\,x \\ -13,071 + 58,779\,x - 47,313\,x^2 \\ \hline u = 10,794 + 66,311\,x - 47,313\,x^2. \end{array}$$

MÉMOIRES

DE

L'ACADÉMIE IMPÉRIALE DES SCIENCES

DE

SAINT-PÉTERSBOURG.

VII^e SÉRIE.

TOME I.

(Avec 13 Planches.)

SAINT-PÉTERSBOURG, 1859.

Commissionnaires de l'Académie Impériale des sciences:

à St.-Pétersbourg,	à Riga,	à Leipzig,
MM. Eggers et Comp.,	M. Samuel Schmidt,	M. Léopold Voss.

Prix: 6 Roubl. 35 Kop. arg. = 7 Thlr. 2 Ngr.

Imprimé par ordre de l'Académie.

Octobre 1859.

C. Vessélofski, Secrétaire perpétuel.

Imprimerie de l'Académie Impériale des sciences.

TABLE DES MATIÈRES

DU TOME I.

№ 10.

Ueber den russischen Euklas. Von **N. v. Kokscharow**, Mitgliede der Akademie. (Mit 1 Tafel.) 25 pages.

№ 11.

Ueber den russischen Zirkon. Von **N. v. Kokscharow**, Mitgliede der Akademie. (Mit 4 Tafeln.) 15 pages.

№ 12.

Euripideische Studien. Von **August Nauck**, Mitgliede der Akademie. Erster Theil. 139 pages.

№ 13.

Ueber den seitlichen Hermaphroditismus eines 22-jährigen Menschen. Von Dr. med. et chir. **Wenzel Gruber**. (Mit 3 Tafeln.) 28 pages.

№ 14.

Sur l'équation algébrique à l'aide de laquelle on détermine les oscillations très-petites d'un système de points matériels. Par **J. Somof**, Membre correspondant de l'Académie. 30 pages.

№ 15.

Sur l'interpolation par la méthode des moindres carrés. Par **P. Tchébychef**, Membre de l'Académie. 24 pages.

AVIS.

La VI^e série des Mémoires de l'Académie Impériale forme les deux recueils suivants:

Sciences mathématiques, physiques et naturelles.

Sciences politiques, histoire et philologie (Tome I—IX, 1832—1859).

Lorsque l'Académie (divisée, depuis 1841, en Classe physico-mathématique et en Classe historico-philologique) commença la publication de ces recueils, elle ne comptait dans son sein que deux naturalistes. Cette circonstance devait l'engager à réunir les mémoires de mathématiques et de physique, ainsi que ceux d'histoire naturelle dans un même recueil. Les deux premiers volumes de ce dernier ont paru, en 1831 et 1833, sous le titre unique de **Mémoires de l'Acad. Imp. des sciences, VI^e Série, Sciences mathématiques, physiques et naturelles,** et renferment des travaux de mathématiques et d'astronomie, de chimie et de physique, de botanique, de zoologie et d'anatomie. Plus tard, quand les sciences biologiques eurent à l'Académie un plus grand nombre de représentants, ce recueil fut subdivisé en deux *parties*, dont *la première* comprenait les mémoires de mathématiques et d'astronomie, de chimie et de physique, et *la seconde* ceux de sciences naturelles ou biologiques. Mais en conservant, pour l'une et l'autre publication, le titre général de *Sciences mathématiques, physiques et naturelles*, et en y ajoutant pour *la première partie* (Tome III—IX), le titre spécial:

Sciences mathématiques et physiques, Tome I—VII (1838—1859),

et pour *la seconde* (Tome III—X), le titre spécial:

Sciences naturelles, Tome I—VIII (1835—1859),

on en avait rendu les citations très compliquées. Il est bon cependant de noter que l'on peut éviter des mal-entendus, en ne citant les deux parties des Mémoires sur les sciences physico-mathématiques que *d'après les titres spéciaux*, formant la *dernière* page du titre, par ex.:

Mémoires de l'Acad. des sc. Sciences mathém. et physiques, Tome I^{er} (1838).

Mémoires de l'Acad. des sc. Sciences naturelles, Tome I^{er} (1835).

Quand en 1849 l'Académie a adopté pour principe de faire tirer de chaque mémoire des exemplaires à part, M. le directeur de l'Observatoire central de Poulkova a eu l'idée de réunir les mémoires d'astronomie dans un recueil séparé, portant le titre:

Recueil de mémoires présentés à l'Académie des sciences par les astronomes de Poulkova, ou offerts à l'Observatoire central par d'autres astronomes du pays; publié avec l'autorisation de l'Académie, par W. Struve. Deux Volumes, 1853 — 1859.

Ces volumes renferment les mémoires d'astronomie, déjà imprimés dans le Tome V^{e} et suiv. des Mémoires mathématiques et physiques.

En outre on a encore publié séparément les

Mémoires présentés à l'Académie Impériale des sciences de St.-Pétersbourg par divers savants, Tome I—IX (1831 — 1859).

Ce recueil porte aussi le titre, plus commode à citer:

Mémoires des savants étrangers (= Записки постороннихъ ученыхъ), **Tome I—IX,** St.-Pét. 1831 — 1859.

Pour simplifier le mode de publication des Mémoires et pour en accélérer l'envoi aux sociétés savantes et aux universités du pays et de l'étranger, entretenant un échange de leurs publications contre celles de l'Académie Impériale, les Classes physico-mathématique et historico-philologique ont arrêté, au mois de décembre 1858, la suppression des différents recueils précités et la publication de tous les mémoires présentés, tant par leurs membres que par des savants étrangers, sous ce titre unique:

Mémoires de l'Académie Impériale des sciences de St.-Pétersbourg, VII^{e} Série, T. I, N^{o} 1er et suiv.

Chaque mémoire sera mis en vente séparément chez MM. les commissionnaires de l'Académie, aussitôt que l'impression en sera achevée, et l'on annexera au dernier N^{o} de chaque tome un titre général et une table des matières.